University of Toronto
SOLUTIONS to **MAT335H1F TERM TEST**
Monday, October 17, 2011
Duration: 50 minutes

Only aids permitted: Scientific calculator, to be supplied by student.

General Comments about the Test:

- The point in Question 1(e) is to show that for $x_0 \notin [0,3]$, the orbit of x_0 under T eventually ends up in $[0,3]$; but you don't know if it is attracted to $x = 9/4$, or to the cycles of parts (b) and (c), as some of you said.

- A fair number of students had difficulty working with natural logs. For example, many couldn't simplify $\ln(e^{-1}) = -1$.

- Not to mention that some students couldn't solve the equations for the fixed points or the 2-cycle in Questions 2(b) and 2(c), or couldn't simplify the derivatives of F_c at the fixed points.

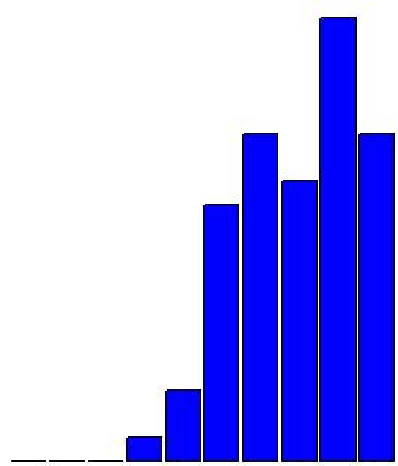
- When it comes to determining if a fixed point (or a cycle) is attracting, repelling or neutral you have to include *all* the cases , that is

$$-1 < F_c'(x) < 1,\ F_c'(x) = \pm 1,\ \text{and}\ F_c'(x) < -1 \text{ or } F_c'(x) > 1.$$

  Many students simply ignored the cases involving -1.

Breakdown of Results: 74 students wrote this test. The marks ranged from 38% to 98%, and the average was 73.7%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	18.9%
A	44.6%	80-89%	25.7%
B	16.2%	70-79%	16.2%
C	18.9%	60-69%	18.9%
D	14.9%	50-59%	14.9%
F	5.4%	40-49%	4.1%
		30-39%	1.3%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



1. [25 marks; each part is worth 5 marks.] Let $T(x) = \begin{cases} x+1, & \text{if } x \leq 2 \\ 9-3x, & \text{if } x > 2 \end{cases}$

(a) Find the fixed point of T and determine if it is attracting, repelling or neutral.

Solution: $x \leq 2 \Rightarrow T(x) = x+1 > x$, so $T(x) = x$ only if $x > 2$.

$$T(x) = x \Leftrightarrow 9 - 3x = x \Leftrightarrow 9 = 4x \Leftrightarrow x = \frac{9}{4} \text{ or } 2.25.$$

This fixed point is repelling since

$$T'(2.25) = -3 \text{ and } |-3| > 1.$$

(b) Confirm that the orbit of $x_0 = 3/4$ under T is periodic. What is its prime period?

Solution:

$$T(3/4) = \frac{7}{4};\ T(7/4) = \frac{11}{4};\ T(11/4) = 9 - \frac{33}{4} = \frac{3}{4}.$$

The prime period is 3.

(c) Confirm that the orbit of $x_0 = -1$ under T is eventually periodic.

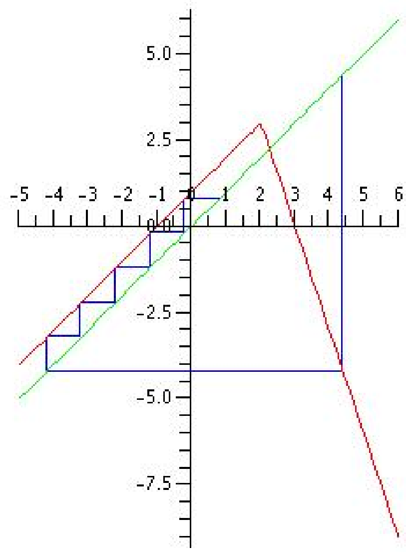
Solution: $T(-1) = 0$; $T(0) = 1$; $T(1) = 2$; $T(2) = 3$; $T(3) = 0$. So the orbit of $x_0 = -1$ under T is attracted to a 4-cycle.

(d) Confirm that $T : [0,3] \longrightarrow [0,3]$

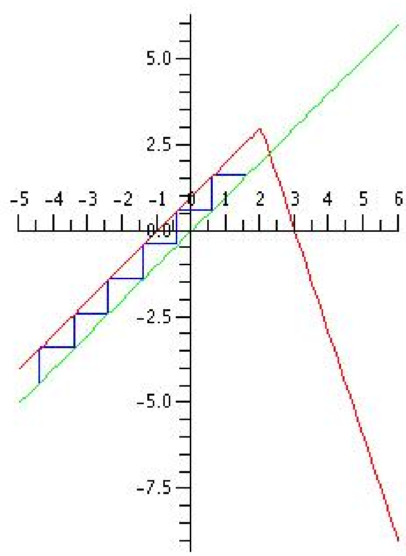
Solution: If $0 \leq x \leq 2$, then $1 \leq x+1 \leq 3$, so $T(x) \in [0,3]$:
if $2 < x \leq 3$, then $-6 > -3x \geq -9 \Rightarrow 3 > 9 - 3x \geq 0 \Rightarrow T(x) \in [0,3]$.
Or show the graph of T is contained in the square $[0,3] \times [0,3]$; see 1(e).

(e) Using the graph below explain why $\{x_0 \mid T^n(x_0) \to \pm\infty\}$ is empty.

Solution: Need only show that if $x_0 < 0$ or $x_0 > 3$ the orbits under T always end up in $[0,3]$. So $T^n(x_0) \to \pm\infty$ is not possible.



The orbit of $x_0 = 4.4$ under T ends up in $[0,3]$.



The orbit of $x_0 = -4.4$ under T goes to $[0,3]$.

2. [25 marks] Consider the family of functions $F_c(x) = x\ln(x^2+c)$, for $c > 0$.

(a) [2 marks] Show that

$$F_c'(x) = \ln(x^2+c) + \frac{2x^2}{x^2+c}.$$

Solution: use product rule.

$$F_c'(x) = 1 \cdot \ln(x^2+c) + x \cdot \frac{2x}{x^2+c} = \ln(x^2+c) + \frac{2x^2}{x^2+c}.$$

(b) [10 marks] Find the fixed points of F_c and determine for which values of c they are attracting, neutral or repelling.

Solution: $F_c(x) = x \Rightarrow x\ln(x^2+c) = x \Rightarrow x = 0$, or

$$\ln(x^2+c) = 1 \Leftrightarrow x^2 + c = e \Leftrightarrow x = \pm\sqrt{e-c}, \text{ for } 0 < c \leq e.$$

1. $F_c'(0) = \ln c$ and $|\ln c| < 1 \Leftrightarrow -1 < \ln c < 1 \Leftrightarrow 1/e < c < e$. So the fixed point $x = 0$ is attracting if $1/e < c < e$, is neutral if $c = 1/e$ or e, and is repelling if $0 < c < 1/e$ or $c > e$.
2. The fixed points $x = \pm\sqrt{e-c}$ are real only if $c \leq e$, and so $F_c'(\pm\sqrt{e-c}) = 1 + 2(e-c)/e \geq 1$. Hence these fixed points are neutral if $c = e$ and repelling if $0 < c < e$.

(c) [8 marks] Solve $F_c(x) = -x$ for x. (The solutions will give you a 2-cycle for F_c.) Determine for which values of c this 2-cycle is attracting.

Solution: $F_c(x) = -x \Rightarrow x\ln(x^2+c) = -x \Rightarrow x = 0$, which is a fixed point, or

$$\ln(x^2+c) = -1 \Leftrightarrow x^2 + c = e^{-1} \Leftrightarrow x = \pm\sqrt{e^{-1}-c}, \text{ for } 0 < c < 1/e.$$

Note: the 2-cycle becomes a single point, $x = 0$, if $c = 1/e$. Now

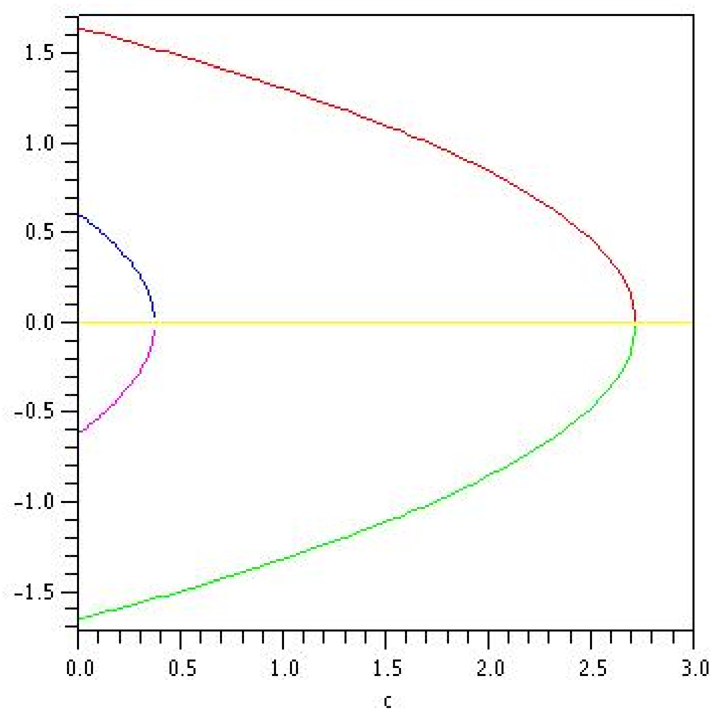
$$F_c'(\pm\sqrt{e^{-1}-c}) = -1 + 2(e^{-1}-c)/e^{-1} = 1 - 2ce.$$

So the 2-cycle $x = \pm\sqrt{e^{-1}-c}$ is attracting if and only if

$$\begin{aligned} |F_c'(\sqrt{e^{-1}-c})F_c'(-\sqrt{e^{-1}-c})| < 1 \quad &\Leftrightarrow \quad (1-2ce)^2 < 1 \\ &\Leftrightarrow \quad -1 < 1 - 2ce < 1 \\ &\Leftrightarrow \quad -2 < -2ce < 0 \\ &\Leftrightarrow \quad e^{-1} > c > 0 \end{aligned}$$

(d) [5 marks] Based on your results in parts (b) and (c) sketch the bifurcation diagram for $F_c(x)$ for $0 < c \leq 3$. Label the nodes and indicate if each node is a tangent (or saddle-node) bifurcation, a period-doubling bifurcation, or neither.

Bifurcation Diagram:



Legend:
Yellow: the fixed point $x = 0$
Red: the fixed point $x = \sqrt{e - c}$
Green: the fixed point $x = -\sqrt{e - c}$
Blue and violet: the 2-cycle $x = \pm\sqrt{e^{-1} - c}$
The node $(c, x) = (1/e, 0)$ is a period-doubling bifurcation.
The node $(c, x) = (e, 0)$ is neither.