MAT186H1F LEC0103 Burbulla

Week 1 Lecture Notes

Fall 2021

A1: Exponential and Logarithmic Functions
A2: Trigonometry
A3: Inverse Functions

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Properties of Exponentials

Let a, b be positive numbers; let x, y be real numbers. Then

$$a^{x+y} = a^x a^y, \;\; a^{x-y} = \frac{a^x}{a^y}, \;\; a^{xy} = (a^x)^y, (ab)^x = a^x b^x,$$

$$a^0 = 1, \quad a^1 = a, \quad a^{-1} = \frac{1}{a}.$$

Warning:

- some exponential expressions are defined even if $a \leq 0$:

  $$(-8)^2 = 64, \;\; 0^5 = 0, \;\; (-8)^{1/3} = -2.$$

  But $(-4)^{1/2}$ is not defined, in terms of real numbers.
- The expression a^{x^y} is ambiguous since

  $$(a^x)^y \neq a^{(x^y)}.$$

Logarithms are Exponents

If $y = a^m$, then the exponent m is called the logarithm of y with respect to the base a. The notation for this is $\log_a y$, and we write

$$y = a^m \Leftrightarrow m = \log_a y.$$

For example:

- $\log_2 32 = 5$, because $2^5 = 32$
- $\log_{10} 1000 = 3$, because $10^3 = 1000$
- $\log_3 \left(\frac{1}{27}\right) = -3$, because $3^{-3} = 27^{-1}$
- $\log_2 1 = 0$, because $2^0 = 1$.

Properties of Logarithms: for $a > 0, a \neq 1$

Let x and y be positive real numbers; z any real number. Then:

$$\log_a(xy) = \log_a x + \log_a y \qquad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a\left(x^z\right) = z\log_a x \qquad \log_a 1 = 0$$

$$\log_a a = 1 \qquad \log_a\left(\frac{1}{a}\right) = -1$$

And of course:

$$\log_a(a^z) = z$$

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Exponential and Logarithmic Functions are Inverses

Let $f(x) = a^x$; let $g(x) = \log_a x$. Then f and g are inverses of each other, in the sense that for all $x > 0$

$$(f \circ g)(x) = f(g(x)) = f(\log_a x) = a^{\log_a x} = x,$$

and for all x

$$(g \circ f)(x) = g(f(x)) = g(a^x) = \log_a(a^x) = x.$$

We shall talk more about this later. In the meanwhile, memorize these two results:

$$a^{\log_a x} = x \text{ and } \log_a(a^x) = x.$$

These are both direct results of the definition of logarithms.

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Problem 1

Solve for x if

$$4^{3x-1} = 8^{3x+3}.$$

Solution: since $4 = 2^2$ and $8 = 2^3$, the original equation can be written as

$$2^{2(3x-1)} = 2^{3(3x+3)}.$$

Then

$$\begin{aligned} \log_2\left(2^{2(3x-1)}\right) = \log_2\left(2^{3(3x+3)}\right) &\Rightarrow 2(3x-1) = 3(3x+3) \\ &\Rightarrow 6x - 2 = 9x + 9 \\ &\Rightarrow -3x = 11 \\ &\Rightarrow x = -\frac{11}{3} \end{aligned}$$

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Problem 2

Solve for x if

$$\log_3 x + \log_3(x+6) = 3.$$

Solution: since logarithms of negative numbers are not defined, $x > 0$. Solving:

$$\begin{aligned} \log_3 x + \log_3(x+6) = 3 &\Rightarrow \log_3\left(x(x+6)\right) = 3 \\ &\Rightarrow x^2 + 6x = 3^3 \\ &\Rightarrow x^2 + 6x - 27 = 0 \\ &\Rightarrow (x+9)(x-3) = 0 \\ &\Rightarrow x = -9 \text{ or } x = 3 \end{aligned}$$

Since $x > 0$, the only acceptable solution is $x = 3$.

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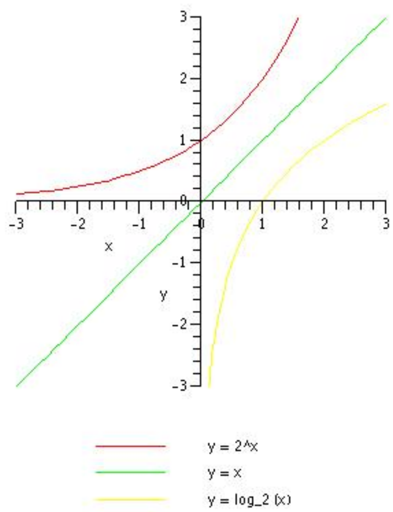
Graphs of $y = 2^x$ and $y = \log_2 x$

Below are the graphs of $y = 2^x$ and $y = \log_2(x)$. They illustrate some common features of logarithmic and exponential functions:



1. The graphs of $\log_a(x)$ and a^x are symmetric in the line $y = x$.
2. If $a > 1$ the x-axis is a HA, horizontal asymptote, to the graph of $y = a^x$: $\lim_{x \to -\infty} a^x = 0$.
3. If $a > 1$ the y-axis is a vertical asymptote, VA, to the graph of $y = \log_a(x)$:

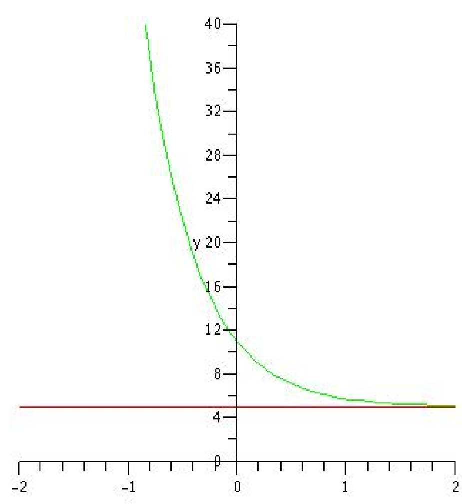
$$\lim_{x \to 0^+} \log_a(x) = -\infty.$$

Example 1

Consider the exponential function $f(x) = 6 \cdot 2^{-3x} + 5$. We can plot its graph without using a graphing calculator by observing that

1. $f(0) = 11$,
2. the graph of f is decreasing,
3. and $y = 5$ is a HA, since

$$\lim_{x \to \infty} f(x) = 6 \cdot 0 + 5 = 5.$$



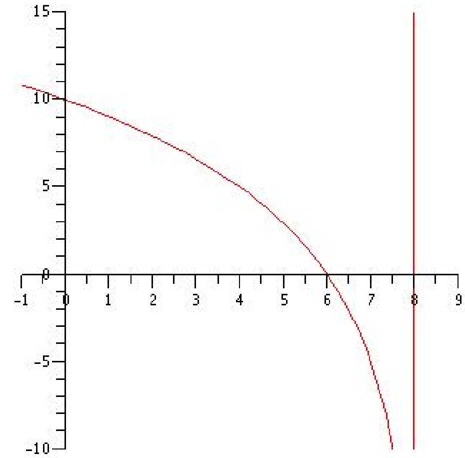
You can also think of the graph of f as a reflected, expanded, shifted version of the graph of $y = 8^x$. Why 8? $2^3 = 8$ so

$$f(x) = 6 \cdot 8^{-x} + 5.$$

Problem 3

Find values of a, b, c such that the graph of $f(x) = c\log_2(a + bx)$ is as shown, that is,

1. $f(6) = 0$,
2. $f(0) = 10$,
3. and $x = 8$ is a VA to the graph of f.



Solution: $f(6) = 0 \Rightarrow a + 6b = 1$; $x = 8$ is a VA implies $a + 8b = 0$. Solving gives $b = -1/2, a = 4$. Then $f(0) = 10 \Rightarrow 10 = c\log_2 4 \Rightarrow c = 5$. So

$$f(x) = 5\log_2(4 - x/2).$$

Log-log plots

Consider the function $y = a \cdot x^k$. Taking the logarithm of both sides, say with respect to base 10, gives

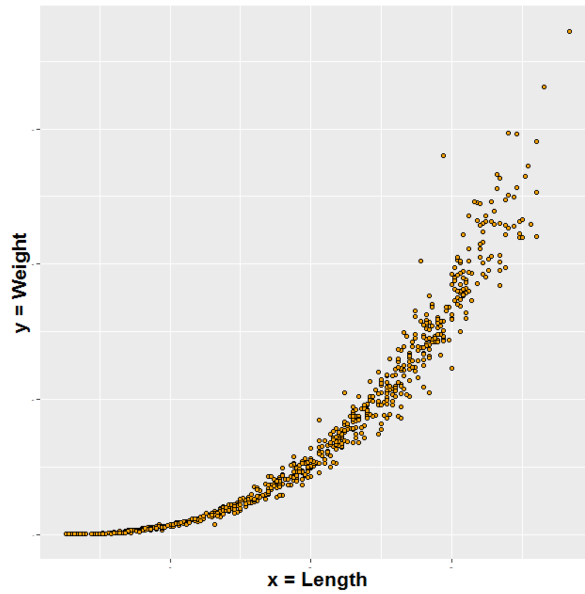
$$\log_{10}(y) = \log_{10}(a \cdot x^k) = \log_{10}(a) + k\log_{10}(x),$$

which defines a linear relationship between $\log_{10}(y)$ and $\log_{10}(x)$, where the slope k of the line is the power of x in the original formula. This observation is often used to model data points if it appears that they follow a power law.

Example 2

Scientists took samples of Eurasian Ruffe (a species of fish) from the St. Louis River Harbour in 1992 and measured their lengths and weights. To the right is a plot of the weights y in grams of the fish versus their lengths x in centimetres. (The axis labels are missing; no matter.) Try to model this data with a power-law function

$$y = a \cdot x^k.$$



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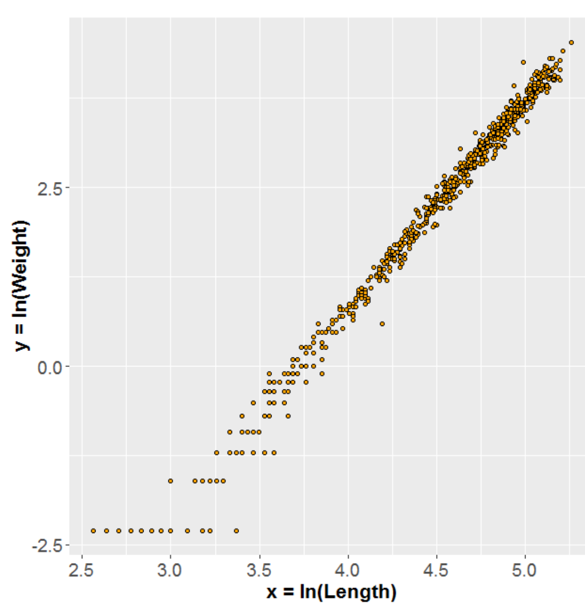
Example 2, Continued

To the right is a log-log plot of the data, where the logs are natural logs base e. The data points line up nicely along a line with approximate slope 3 and approximate vertical intercept of -2.8. Thus approximately

$$\begin{aligned} \ln(y) &= -2.8 + 3\ln(x) \\ \Rightarrow y &= e^{-2.8+3\ln x} \\ &= e^{-2.8}e^{3\ln x} \\ &= e^{-2.8}x^3 \\ &\approx 0.608\,x^3. \end{aligned}$$



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The Number e; the Natural Base

The *natural* base for exponentials and logarithms is the number

$$e = 2.718281828459045\ldots,$$

which can be defined in various ways. Here is one:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

Like the number π the number e is an irrational and transcendental number. What is "natural" about e will become clear when we calculate derivatives. In the meanwhile, we write

$$\ln(x) = \log_e(x) \text{ and } \exp(x) = e^x.$$

Change of Base Formula

All bases a, $a > 0, a \neq 1$, are equally good, in the sense that the **change of base formula** allows you to switch logarithms from one base a to another base b :

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

Your calculator should be able to calculate natural logarithms, $\ln(x)$, and logarithms base 10 which are called common logarithms, $\log(x)$.

Aside: can you prove the change of base formula?
Proof: rearranging gives $\log_a(b)\log_b(x) = \log_a(x)$, which can be verified:

$$a^{\log_a(b)\log_b(x)} = \left(a^{\log_a(b)}\right)^{\log_b(x)} = b^{\log_b(x)} = x.$$

Example 3: Exponential Growth

Any quantity $P(t)$ that increases in time according to the formula

$$P(t) = P_0 e^{kt},$$

for positive constants P_0 and k, is said to grow exponentially. Common examples are unlimited population growth and compound interest. There are two key features of exponential growth:

1. $\lim_{t\to\infty} P(t) = \infty$
2. The doubling time is independent of the initial value P_0.

For example, if an initial investment of P_0 accrues interest at an annual rate of 6% interest compounded annually, then the accumulated value in the account after t years is $P(t) = P_0 e^{.06\,t}$. To find the doubling time let $P(t) = 2P_0$ and solve for t, in yrs:

$$2P_0 = P_0 e^{.06t} \Rightarrow 2 = e^{.06t} \Rightarrow \ln 2 = .06t \Rightarrow t = \frac{\ln(2)}{.06} \approx 11.6$$

Problem 4

In a certain lab experiment it is observed that the population of bacteria with an unlimited food supply doubles every 30 min.

1. How long would it take for the population of bacteria to become 4 times as large? 8 times as large?
2. How long does it take for the population of bacteria to triple in size?

Solution:

1. two doubling periods: 60 min; three doubling periods: 90 min.
2. If $P(t) = P_0 e^{kt}$, we know the doubling time is $\ln(2)/k$. Then $30 = \ln(2)/k \Rightarrow k = \ln(2)/30 \approx 0.0231$. To find the tripling time, let $P(t) = 3P_0$ and solve for t, in minutes:

$$3P_0 = P_0 e^{kt} \Rightarrow 3 = e^{kt} \Rightarrow kt = \ln 3 \Rightarrow t = \frac{\ln 3}{k} \approx 47.55.$$

Example 4: Radioactive Decay

The mass $x(t)$ of a radioactive substance decays exponentially: $x(t) = x_0 e^{-kt}$, where t is in years, x_0 is the mass of the substance at $t = 0$, and $k > 0$ is constant. Originally there are 100 grams of a particular radioactive substance which decays according to the formula

$$x(t) = 100e^{-t/650}.$$

Find the **half-life** of this substance: the number of years it takes until the mass of the substance is half of the original amount.
Solution: let $x(t) = 50$ and solve for t :

$$\begin{aligned} 50 = 100e^{-t/650} \quad &\Rightarrow \quad e^{t/650} = 2 \\ &\Rightarrow \quad \frac{t}{650} = \ln(2) \\ &\Rightarrow \quad t = 650\ln(2) \approx 450.545667 \end{aligned}$$

So the half-life is about 450.5 years.

Problem 5: Carbon Dating

The half-life of carbon-14 is 5,700 years. If a specimen of charcoal found in Stonehenge contains only 63% of its original carbon-14, how old is Stonehenge? **Solution:** Let $x = x_0 e^{-kt}$ be the amount of carbon-14 present in the charcoal at time t, with t in years since the charcoal was created. Use the half-life to find k :

$$\frac{x_0}{2} = x_0\, e^{-5700k} \Rightarrow e^{5700k} = 2 \Rightarrow 5700k = \ln 2 \Rightarrow k = \frac{\ln 2}{5700};$$

so $k \approx 0.0001216$. Now let $x = 0.63x_0$, and solve for t :

$$0.63x_0 = x_0 e^{-kt} \Rightarrow 0.63 = e^{-kt} \Rightarrow \ln 0.63 = -kt \Rightarrow t = -\frac{\ln(0.63)}{k}.$$

So $t \approx 3\,800$ and the age of Stonehenge is approximately 3,800 yrs.

Power Law or Exponential Growth?

The functions $y = x^2, x^3, x^4$ and e^x are all increasing functions. If you are trying to model experimental data and the data points are scattered in an increasing pattern how can you distinguish a power law from exponential growth? A key observation is that for exponential growth the doubling time—if the variable is considered to be time—is constant. This is not the case for a power law. For example, if $y = t^2$, then the time to double from $y = 1$ to $y = 2$ is $t = \sqrt{2}$; but the time to double from $y = 2$ to $y = 4$ is $t = 2$. In general, the doubling time for a power law is not constant. By contrast, for *any* exponential function $f(x) = a \cdot b^x$, and *any* interval of length h, say $x_1 \leq x \leq x_1 + h$, the value of the ratio $f(x_1 + h)/f(x_1)$ is a constant:

$$\frac{f(x_1 + h)}{f(x_1)} = \frac{a \cdot b^{x_1+h}}{a \cdot b^{x_1}} = b^{x_1+h-x_1} = b^h.$$

How To Check For Exponential Models

This means that to find an exponential model for data points (x_i, y_i), for $1 \leq i \leq n$, you should calculate the ratio y_j/y_i for some points $(x_i, y_i), (x_j, y_j)$, with $x_j = x_i + h$. If the ratio is approximately constant then $f(x) = a \cdot b^x$, for some a, with

$$b^h = y_j/y_i \Leftrightarrow \ln b = \frac{\ln(y_j/y_i)}{h} \Leftrightarrow b = e^{\frac{\ln(y_j/y_i)}{h}} = \left(\frac{y_j}{y_i}\right)^{1/h}.$$

That is,

$$f(x) = a \cdot \left(\frac{y_j}{y_i}\right)^{x/h}$$

would be a good model for the data. Note this result will work if data points show an increasing trend ($y_j > y_i$) or a decreasing trend ($y_j < y_i$).

Optional. Example 5: Musical Scales

Two notes are an octave apart if the frequency of one note is twice the frequency of the other. A musical scale consists of eight notes, which you might know as Do, Re, Mi, Fa, Sol, La, Ti, Do, but which could also be represented as C, D, E, F, G, A, B, C. In each case, the scale covers one octave, but the intervals are not constant: full tone, full tone, semi-tone, full tone, full tone, full tone, semi-tone. Thus 12 semi-tones make up one octave. Let a be the ratio of one semi-tone: $a^{12} = 2 \Rightarrow a = 2^{1/12}$.

a^0	1	1.
a^1	$2^{1/12}$	1.059463094
a^2	$2^{1/6}$	1.122462048
a^3	$2^{1/4}$	1.189207115
a^4	$2^{1/3}$	1.259921050
a^5	$2^{5/12}$	1.334839854
a^6	$2^{1/2}$	1.414213562
a^7	$2^{7/12}$	1.498307077
a^8	$2^{2/3}$	1.587401052
a^9	$2^{3/4}$	1.681792831
a^{10}	$2^{5/6}$	1.781797436
a^{11}	$2^{11/12}$	1.887748625
a^{12}	2	2.

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1. Frequency ratios in the theoretical diatonic scale: T is one tone or two semi-tones; S is one semi-tone:

C	D	E	F	G	A	B	C
	T	T	S	T	T	T	S
a^0	a^2	a^4	a^5	a^7	a^9	a^{11}	a^{12}
1	1.12246	1.25992	1.3349	1.49831	1.68179	1.88775	2

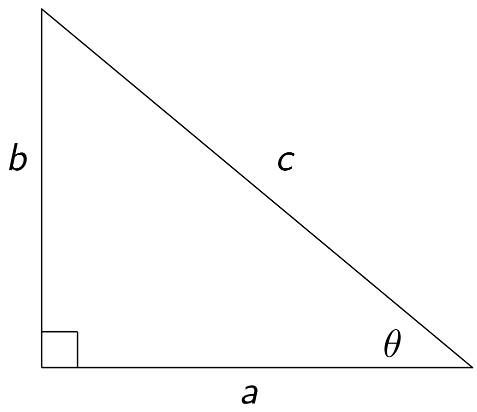
2. Diatonic scale, just intonation tuning:

		3*rd*	4*th*	5*th*	major 6*th*		
C	D	E	F	G	A	B	C
1	9/8	5/4	4/3	3/2	5/3	15/8	2
1	1.125	1.25	1.333	1.5	1.667	1.875	2

3. Diatonic scale, Pythagorean tuning:

C	D	E	F	G	A	B	C
1	9/8	81/64	4/3	3/2	27/16	243/128	2
1	1.125	1.265625	1.3333	1.5	1.6875	1.8984375	2

The Six Trigonometric Functions and Right Triangles



- $\sin\theta = \frac{b}{c}$
- $\cos\theta = \frac{a}{c}$
- $\tan\theta = \frac{b}{a} = \frac{\sin\theta}{\cos\theta}$
- $\csc\theta = \frac{c}{b} = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{c}{a} = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{a}{b} = \frac{1}{\tan\theta}$

The Pythagorean Identities:

• $\sin^2\theta + \cos^2\theta = 1$ • $\tan^2\theta + 1 = \sec^2\theta$ • $1 + \cot^2\theta = \csc^2\theta$

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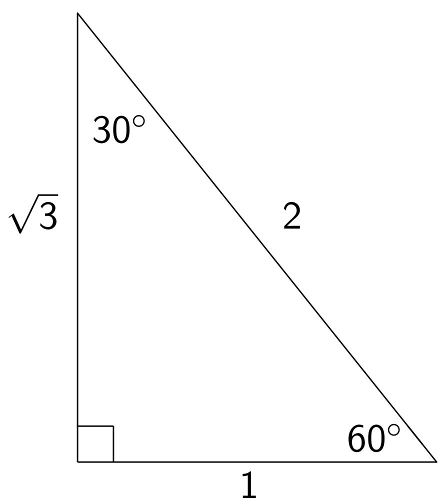
30-60-90 Triangle



- $\sin 60^\circ = \frac{\sqrt{3}}{2}$
- $\cos 60^\circ = \frac{1}{2}$
- $\tan 60^\circ = \sqrt{3}$
- $\sin 30^\circ = \frac{1}{2}$
- $\cos 30^\circ = \frac{\sqrt{3}}{2}$
- $\tan 30^\circ = \frac{1}{\sqrt{3}}$

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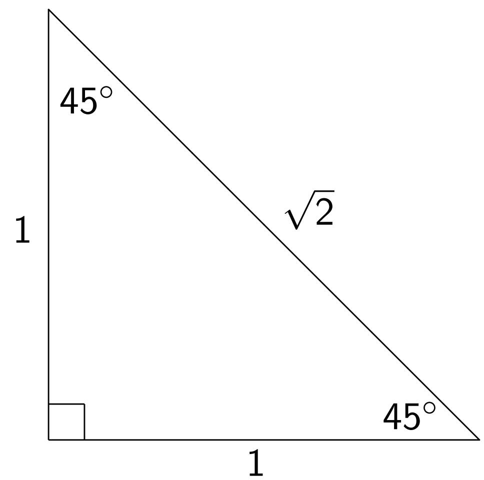
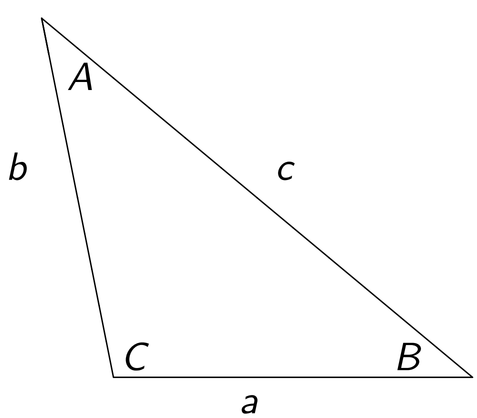
45-45-90 Triangle



- $\sin 45^\circ = \dfrac{1}{\sqrt{2}}$
- $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$
- $\tan 45^\circ = 1$

Sine Law and Cosine Law



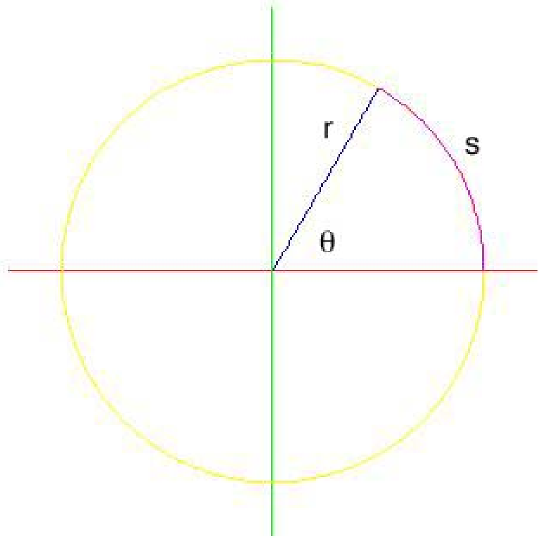
Sine Law:

- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Cosine Law:

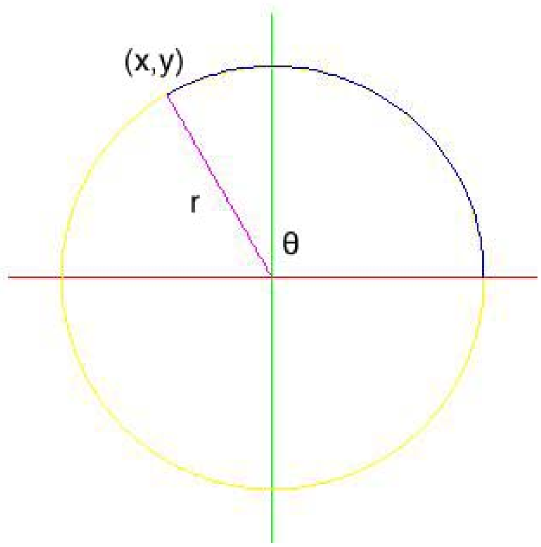
- $a^2 = b^2 + c^2 - 2bc\cos A$
- $b^2 = a^2 + c^2 - 2ac\cos B$
- $c^2 = a^2 + b^2 - 2ab\cos C$

Radian Measure and Circles



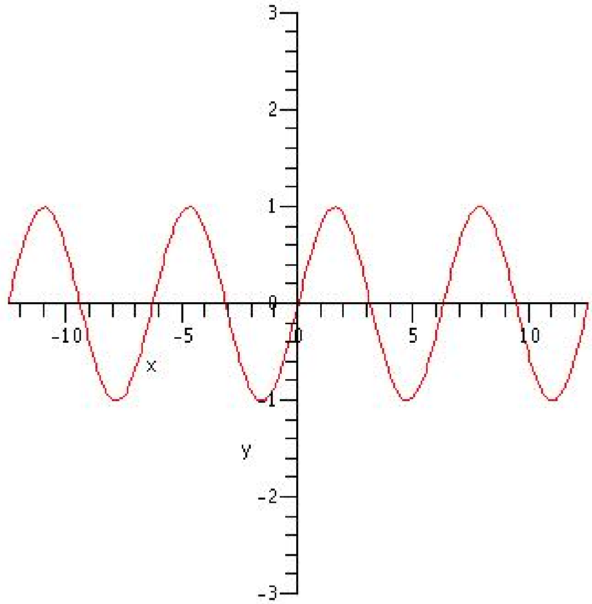
1. $\theta = \frac{s}{r}$
2. $s = r\theta$
3. π radians $= 180°$
4. Radian measure is the ratio of two lengths; so it is unit free.

The Six Trigonometric Functions; θ in radians.

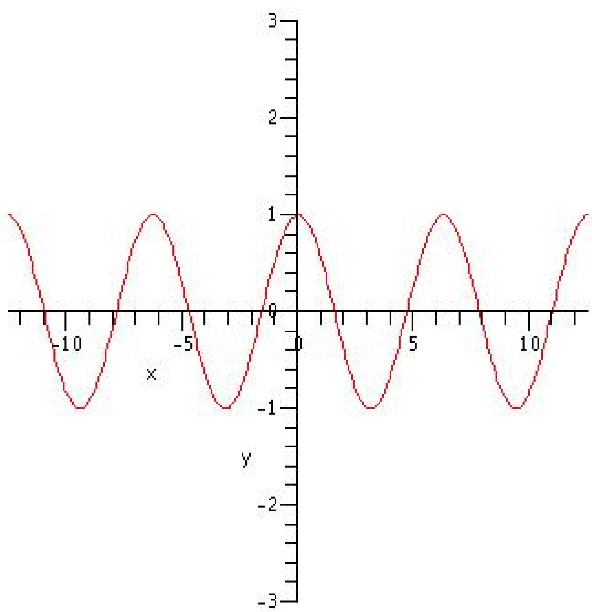


1. $\sin\theta = \frac{y}{r}$
2. $\cos\theta = \frac{x}{r}$
3. $\tan\theta = \frac{y}{x}$
4. $\csc\theta = \frac{r}{y}$
5. $\sec\theta = \frac{r}{x}$
6. $\cot\theta = \frac{x}{y}$

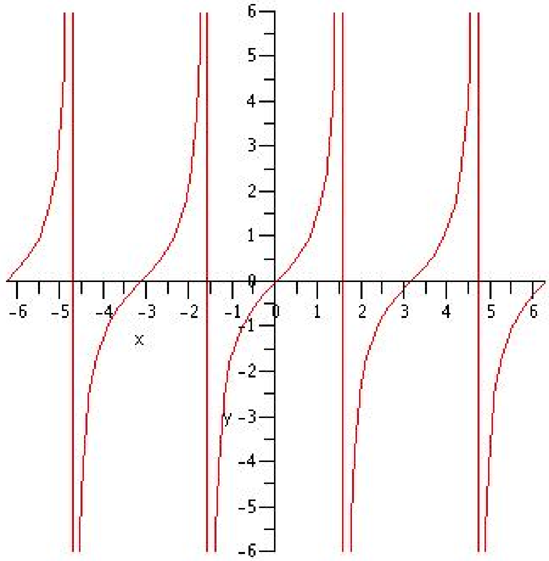
Graph of $y = \sin x$. Domain: $\mathbb{R}$; Range: $[-1, 1]$



Graph of $y = \cos x$. Domain: $\mathbb{R}$; Range: $[-1, 1]$



Graph of $y = \tan x$. Domain: $x \neq \frac{(2k+1)}{2}\pi$; Range: $\mathbb{R}$



Note: the vertical red lines represent vertical asymptotes to the graph of

$$y = \tan x$$

at

$$x = \frac{(2k+1)}{2}\pi,$$

where k is an integer.

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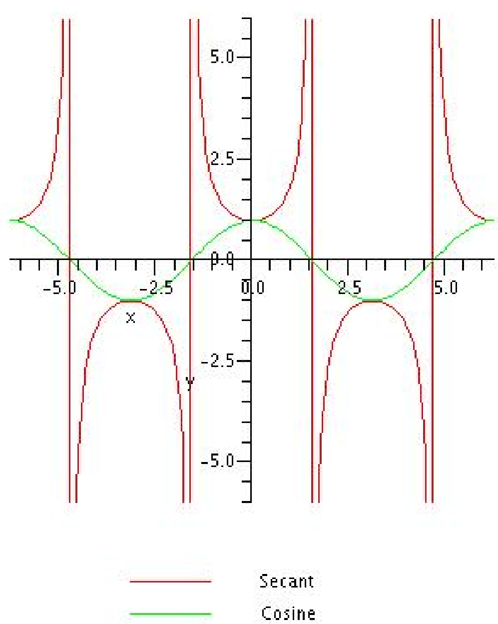
Graph of $y = \sec x$. Domain: $x \neq \frac{(2k+1)}{2}\pi$; Range: $|y| \geq 1$



Note: the vertical red lines represent vertical asymptotes to the graph of

$$y = \sec x$$

at

$$x = \frac{(2k+1)}{2}\pi,$$

where k is an integer.

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Sum and Difference Formulas

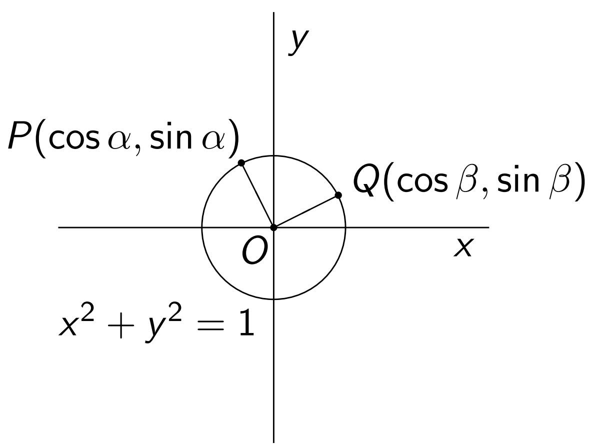
The following formulas may not have been covered in high school, but they are important!

1. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$
2. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$
3. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
4. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

We shall only prove the fourth formula, since the other three follow from it. For example:

$$\begin{aligned}\sin(\alpha+\beta) &= \cos(\pi/2-(\alpha+\beta)) = \cos(\pi/2-\alpha-\beta)\\ &= \cos(\pi/2-\alpha)\cos\beta + \sin(\pi/2-\alpha)\sin\beta\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}$$

One Proof of $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$



1. Let P have coordinates $(\cos\alpha, \sin\alpha)$.
2. Let Q have coordinates $(\cos\beta, \sin\beta)$.
3. Let $\theta = \alpha - \beta$.
4. Use the cosine law to calculate $\overline{PQ}$.

$$\begin{aligned}&\overline{PQ}^2 = \overline{OP}^2 + \overline{OQ}^2 - 2\overline{OP}\cdot\overline{OQ}\cos\theta\\ \Leftrightarrow\ &(\cos\alpha-\cos\beta)^2 + (\sin\alpha-\sin\beta)^2 = 1^2+1^2-2\cos(\alpha-\beta)\\ \Leftrightarrow\ &2\cos(\alpha-\beta) = 2-2+2\cos\alpha\cos\beta+2\sin\alpha\sin\beta\\ \Leftrightarrow\ &\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}$$

Double Angle Formulas

By taking the special case when $\alpha = \beta = \theta$, the formulas for

$$\sin(\alpha + \beta) \text{ and } \cos(\alpha + \beta)$$

can be written as double angle formulas:

1. $\sin(2\theta) = 2\sin\theta\cos\theta$

2. There are three formulas for $\cos(2\theta)$:

$$\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \text{or} &= 1 - 2\sin^2\theta \\ \text{or} &= 2\cos^2\theta - 1 \end{aligned}$$

The last two can be re-written as

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2} \text{ and } \cos^2\theta = \frac{1 + \cos(2\theta)}{2}.$$

Half-Angle Formulas

Substitute $A = 2\theta$ in the last two formulas of the previous slide, and you obtain

$$\sin^2\frac{A}{2} = \frac{1 - \cos A}{2} \text{ and } \cos^2\frac{A}{2} = \frac{1 + \cos A}{2}.$$

These formulas can be used to obtain

$$\sin\frac{A}{2} \text{ and } \cos\frac{A}{2}$$

in terms of $\cos A$, but you need to know which quadrant $\frac{A}{2}$ is in to know which square root – positive or negative – to use.

Problem 1

Find the exact value of both $\sin \frac{3}{4}\pi$ and $\cos \frac{\pi}{12}$.

Solutions:

1. $\sin \frac{3}{4}\pi = \sin\left(\pi - \frac{\pi}{4}\right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$
2. $$\begin{aligned} \cos \frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\ &= \cos \frac{\pi}{4} \cos \frac{\pi}{6} + \sin \frac{\pi}{4} \sin \frac{\pi}{6} \\ &= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{1 + \sqrt{3}}{2\sqrt{2}} \end{aligned}$$

Problem 2

Given that $\sin\theta = \frac{1}{4}$ and that $\cos\theta < 0$, find the exact values of both

$$\cos\theta \text{ and } \sin(2\theta).$$

Solutions:

1. $\cos\theta = -\sqrt{1 - \left(\frac{1}{4}\right)^2} = -\frac{\sqrt{15}}{4}$
2. $\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{1}{4}\right)\left(-\frac{\sqrt{15}}{4}\right) = -\frac{\sqrt{15}}{8}$

Problem 3

Given that $\cos\theta = \frac{2}{3}$, $\sin\theta < 0$, and that $0 < \theta < 2\pi$, find the exact values of $\cos(2\theta), \sin\theta$ and $\sin\left(\frac{\theta}{2}\right)$.

Solutions: θ is in the fourth quadrant. (Why?)

1. $\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{2}{3}\right)^2 - 1 = -\frac{1}{9}$
2. $\sin\theta = -\sqrt{1 - \left(\frac{2}{3}\right)^2} = -\frac{\sqrt{5}}{3}$
3. $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2} = \frac{1}{6} \Rightarrow \sin\left(\frac{\theta}{2}\right) = +\frac{1}{\sqrt{6}}$,

since

$$3\pi/2 < \theta < 2\pi \Rightarrow 3\pi/4 < \theta/2 < \pi.$$

Problem 4: Solve this PCE example in a different way.

Find all solutions x in the interval $-\pi/2 \le x \le \pi/2$ to the equation $8\sin^2 x\cos^2 x = 1$. **Alternate Solution:**

$$\begin{aligned}
8\sin^2 x\cos^2 x = 1 &\Rightarrow 8\sin^2 x(1-\sin^2 x) - 1 = 0\\
&\Rightarrow 8\sin^4 x - 8\sin^2 x + 1 = 0\\
&\Rightarrow \sin^2 x = \frac{8 \pm \sqrt{64-32}}{16} = \frac{1}{2} \pm \frac{\sqrt{2}}{4}\\
&\Rightarrow \frac{1-\cos(2x)}{2} = \frac{1}{2} \pm \frac{\sqrt{2}}{4}\\
&\Rightarrow \cos(2x) = \pm\frac{1}{\sqrt{2}}\\
&\Rightarrow 2x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4} \Rightarrow x = \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}
\end{aligned}$$

Problem 5

Suppose one instrument produces a note with frequency 180 Hz and another instrument produces a note with frequency 220 Hz; say the notes are described by $x(t) = \cos(360\pi\, t)$ and $y(t) = \cos(440\pi\, t)$, respectively. Show that the combined signal, $x(t) + y(t)$, can be expressed as

$$x(t) + y(t) = 2\cos(40\pi\, t)\, \cos(400\pi t).$$

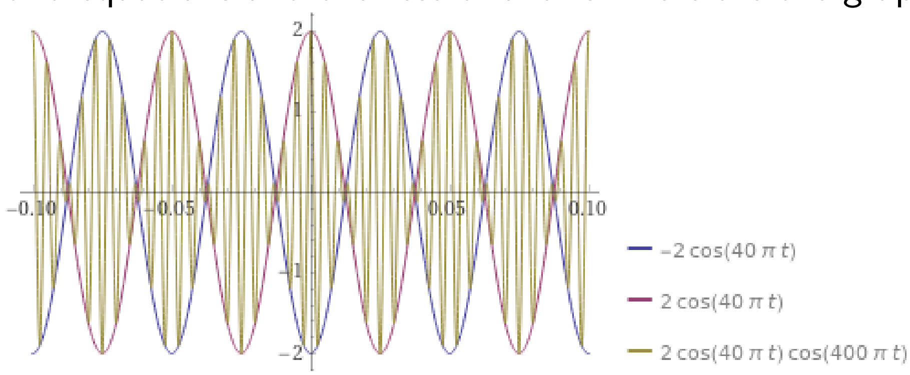
Using a graphing calculator or other tool, plot the graphs of

$$-2\cos(40\pi\, t),\ 2\cos(40\pi t) \text{ and } x(t) + y(t)$$

for $-0.1 \le t \le 0.1$. What do you notice?

Comments About Problem 5

Use $x(t) = \cos(400\pi t - 40\pi t)$ and $y(t) = \cos(400\pi t + 40\pi t)$. We have $x(t) = \cos(400\pi t)\cos(40\pi t) + \sin(400\pi t)\sin(40\pi t)$ and $y(t) = \cos(400\pi t)\cos(40\pi t) - \sin(400\pi t)\sin(40\pi t)$. Add these two equations and the result follows. Here are the graphs:



We say the amplitude of $\cos(400\pi t)$ is *modulated* by $2\cos(40\pi t)$. You would hear the $\cos(400\pi t)$ signal fade in and out as its amplitude rises and falls. This is known as *beating*.

What is an Inverse Function?

If

1. $g(f(x)) = x$ for all x in the domain of f, and
2. $f(g(x)) = x$ for all x in the domain of g,

then we say f and g are inverse functions, and we write

$$f^{-1} = g \text{ and } g^{-1} = f.$$

For example, $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverse functions, since

1. $g(f(x)) = g(x^3) = (x^3)^{1/3} = x^{3/3} = x^1 = x$, and
2. $f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x^{3/3} = x^1 = x$.

A1: Exponential and Logarithmic Functions
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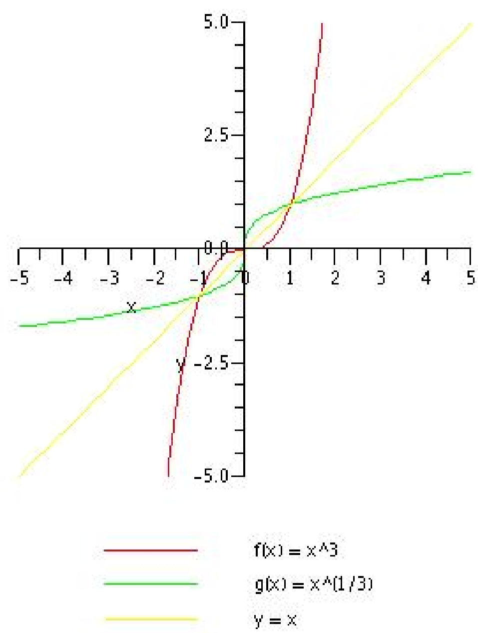
The Graphs of $f(x) = x^3$ and $g(x) = x^{1/3}$.



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How to Tell if a Function Has an Inverse?

The function f has an inverse if and only if it is one-to-one. This means:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$$

Equivalently: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$. Geometrically, a function f is one-to-one if any horizontal line that intersects the graph of f intersects it in exactly one point. **For example:**

1. $f(x) = 2x$ is one-to-one: $2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
2. $f(x) = x^2$ is not one-to-one: $x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$.
3. $f(x) = x^3$ is one-to-one:

$$\begin{aligned} x_1^3 = x_2^3 \quad &\Rightarrow \quad x_1^3 - x_2^3 = 0 \\ &\Rightarrow \quad (x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2) = 0 \\ &\Rightarrow \quad x_1 = x_2, \text{ since } x_i \text{ is real.} \end{aligned}$$

How To Find $f^{-1}(x)$?

If $y = f(x)$ is a one-to-one function, then the formula for $f^{-1}(x)$ can be found by interchanging x and y and solving for y. That is:

$$x = f(y) \Leftrightarrow y = f^{-1}(x).$$

For example, if $f(x) = x^3$ then

$$x = f(y) \Leftrightarrow x = y^3 \Leftrightarrow y = x^{1/3} \Leftrightarrow f^{-1}(x) = x^{1/3}.$$

Consequently:

1. (x, y) is on the graph of f if and only if (y, x) is on the graph of f^{-1}.
2. The graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric with respect to the line $y = x$.
3. The domain of f is the range of f^{-1}.
4. The range of f is the domain of f^{-1}.

Problem 1

Let $f(x) = \dfrac{x+1}{x-2}$. Is f one-to-one? If so, find f^{-1}.

Solution: yes.

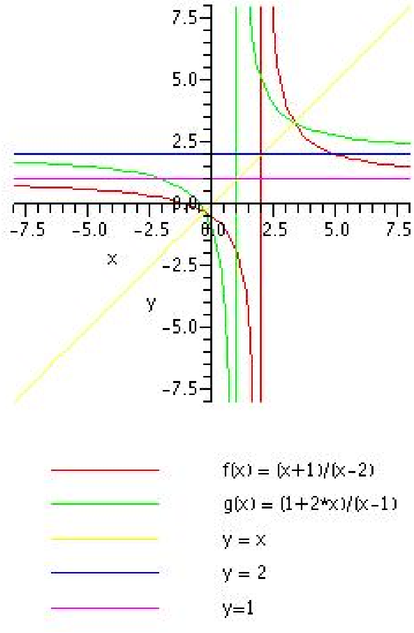
$$\begin{aligned} f(x_1) = f(x_2) \quad &\Rightarrow \quad \frac{x_1+1}{x_1-2} = \frac{x_2+1}{x_2-2} \\ &\Rightarrow \quad x_1x_2 - 2x_1 + x_2 - 2 = x_1x_2 + x_1 - 2x_2 - 2 \\ &\Rightarrow \quad 3x_2 = 3x_1 \\ &\Rightarrow \quad x_2 = x_1 \end{aligned}$$

To find $f^{-1}(x)$:

$$x = f(y) \Leftrightarrow x = \frac{y+1}{y-2} \Leftrightarrow xy - 2x = y + 1 \Leftrightarrow y = \frac{1+2x}{x-1}.$$

Hence: $$f^{-1}(x) = \frac{1+2x}{x-1}.$$

Graphs for Problem 1



Exponential and Logarithmic Functions are Inverses

Let $f(x) = a^x$; let $g(x) = \log_a x$. Then as we have seen, f and g are inverses of each other because for all $x > 0$,

$$f(g(x)) = f(\log_a x) = a^{\log_a x} = x,$$

and for all x,

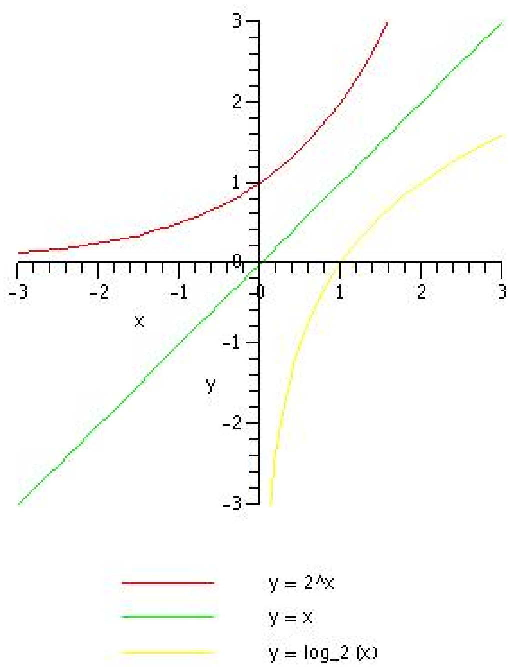
$$g(f(x)) = g(a^x) = \log_a(a^x) = x.$$

Memorize these two results:

$$a^{\log_a x} = x \text{ and } \log_a(a^x) = x.$$

They can be proven using the definition of logarithms, but they are simply consequences of f and g being inverse functions.

Graphs of $y = 2^x$ and $y = \log_2 x$



Problem 2

Which of the following functions are one-to-one? If so, can you find their inverses?

1. $$f(x) = 3x + 2|x|$$
2. $$g(x) = x + \sin(x)$$
3. $$h(x) = \ln(x^2 + 1)$$
4. $$k(x) = e^x - e^{-x}$$

Answer: $h(x) = \ln(x^2 + 1)$ is not one-to-one since $h(-x) = h(x)$; the others are one-to-one. The easiest way is to graph all four functions and see which ones pass the horizontal line test.

A1: Exponential and Logarithmic Functions
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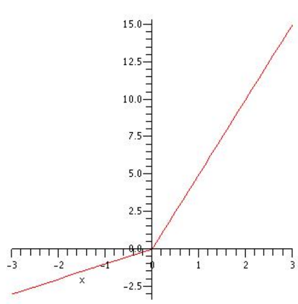
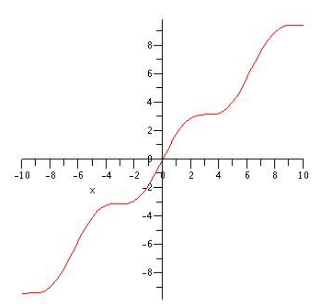
Comments About Problem 2

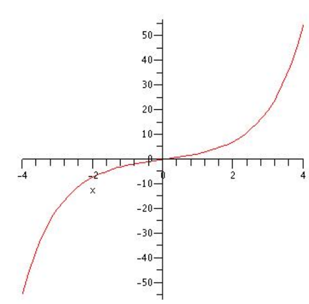
Plots of f, g and k :



$f(x) = 5x$ if $x \geq 0$;
$f(x) = x$ if $x < 0$.
$f^{-1}(x) = x/5$ if $x \geq 0$;
$f^{-1}(x) = x$ if $x < 0$.



you can't solve for g^{-1}



$k(y) = x \Rightarrow$
$e^y - e^{-y} = x \Rightarrow$
$e^{2y} - xe^y - 1 = 0 \Rightarrow$
$e^y = (x \pm \sqrt{x^2 + 4})/2$

Since $e^y > 0$ we must take $e^y = (x + \sqrt{x^2 + 4})/2$, so

$$k^{-1}(x) = \ln((x + \sqrt{x^2 + 4})/2).$$

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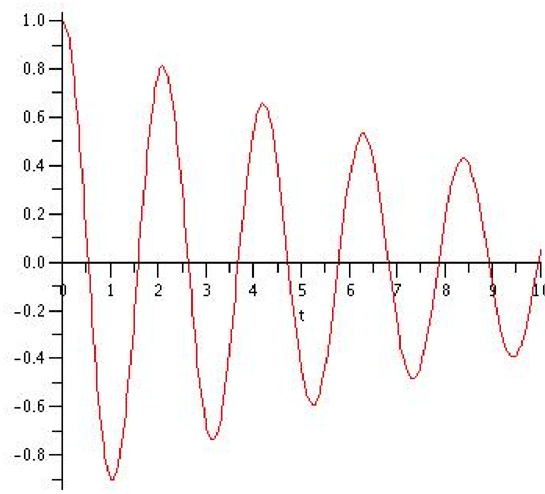
Problem 3

What is the length of the longest interval on which the function

$$f(t) = e^{-t/10}\cos(3t)$$

can have an inverse? See the graph to the right.



Comments About Problem 3

The graph of f still has cyclic behaviour, determined by period of $\cos(3t)$, although the amplitude decreases as time passes due to the factor $e^{-t/10}$. To have an inverse we have to restrict f to an interval on which the graph is one-to-one. The length of half of the period of $\cos(3t)$ is $\pi/3$. But to *prove* that this is actually the distance between consecutive critical points of f involves some calculus:

$$f'(t) = -\frac{1}{10}e^{-t/10}\cos(3t) - 3e^{-t/10}\sin(3t) = 0 \Leftrightarrow \tan(3t) = -\frac{1}{30}.$$

Thus the graph of f is one-to-one for the period of $\tan(3t)$, namely $\pi/3$.

How Can Trigonometric Functions have Inverses?

- Since all the trigonometric functions are periodic, they are not one-to-one. Properly speaking no trigonometric function can have an inverse.
- However the problem of finding angles with given trigonometric ratios is a common problem, showing up in lots and lots of applications.
- So by convention, we agree to define the inverse of trigonometric functions in particular, standard ways.
- You will find that your calculator has these standard definitions built in for $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$.
- Surprisingly, there is as yet no universally agreed upon definition of $\sec^{-1} x$; so you won't find it on your calculator.

Notation for Inverse Trigonometric Functions

- f^{-1} is the common notation for the inverse of the one-to-one function f. But with respect to trig functions it can cause confusion.
- $\sin^{-1} x$ could be misread as $(\sin x)^{-1} = \csc x$. To avoid this potential problem there is an alternate notation: $\arcsin x$.
- Thus the four inverse trig functions we shall be using, namely

$$\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \sec^{-1} x,$$

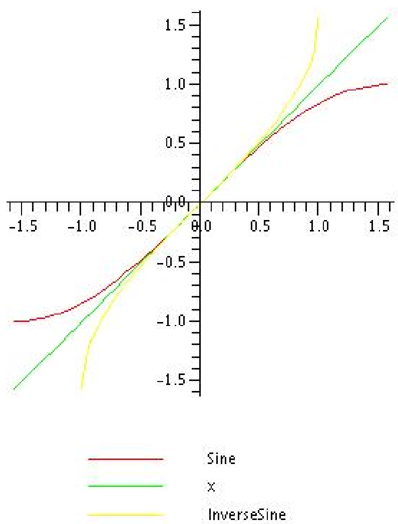
can also be written as

$$\arcsin x, \arccos x, \arctan x, \text{arcsec } x.$$

- Be familiar with each notation.

Definition of Inverse Sine

$$y = \sin^{-1} x \Leftrightarrow \sin y = x \text{ and } |y| \le \pi/2$$



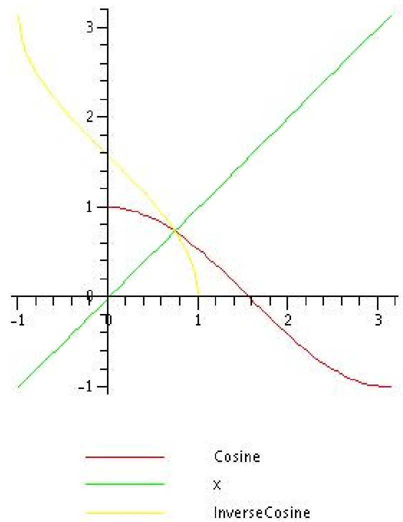
1. The domain of $y = \sin^{-1} x$ is $[-1, 1]$.
2. The range of $y = \sin^{-1} x$ is $[-\pi/2, \pi/2]$.
3. The graph of $y = \sin^{-1} x$ is always increasing.

Example 1

1. $\sin^{-1} 1 = \frac{\pi}{2}$. Not 90! Even though your calculator supplies answers in terms of radians or degrees, our definition of $\sin^{-1}$ always gives values as unit-less numbers, that is, radians.
2. $\sin^{-1}(-1) = -\frac{\pi}{2}$.
3. $\sin^{-1} 0 = 0$.
4. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.
5. $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

Definition of Inverse Cosine

$$y = \cos^{-1} x \Leftrightarrow \cos y = x \text{ and } 0 \leq y \leq \pi$$



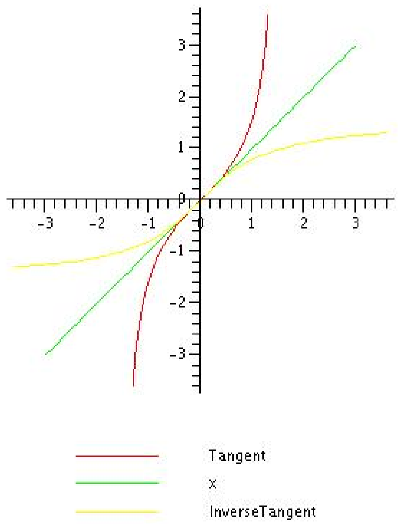
1. The domain of $y = \cos^{-1} x$ is $[-1, 1]$.
2. The range of $y = \cos^{-1} x$ is $[0, \pi]$.
3. The graph of $y = \cos^{-1} x$ is always decreasing.

Example 2

1. $\cos^{-1} 0 = \dfrac{\pi}{2}$
2. $\cos^{-1} 1 = 0$
3. $\cos^{-1}(-1) = \pi$
4. $\cos^{-1} \left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$
5. $\cos^{-1} \left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}$

Definition of Inverse Tangent:

$$y = \tan^{-1} x \Leftrightarrow \tan y = x \text{ and } |y| < \pi/2$$



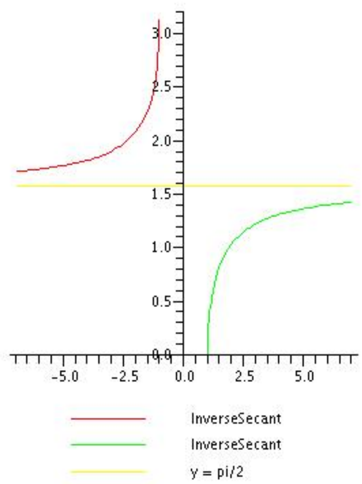
1. The domain of $y = \tan^{-1} x$ is $\mathbb{R}$.
2. The range of $y = \tan^{-1} x$ is $(-\pi/2, \pi/2)$.
3. The graph of $y = \tan^{-1} x$ is always increasing.
4. $y = \pm\dfrac{\pi}{2}$ are horizontal asymptotes to the graph of $y = \tan^{-1} x$.

Example 3

1. $\tan^{-1} 0 = 0$
2. $\tan^{-1} 1 = \dfrac{\pi}{4}$
3. $\tan^{-1}(-1) = -\dfrac{\pi}{4}$
4. $\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$
5. $\tan^{-1}\left(-\sqrt{3}\right) = -\dfrac{\pi}{3}$
6. $\lim_{x\to\infty} \tan^{-1} x = \dfrac{\pi}{2}$
7. $\lim_{x\to-\infty} \tan^{-1} x = -\dfrac{\pi}{2}$

Our Definition of Inverse Secant:

$$y = \sec^{-1} x \Leftrightarrow \sec y = x \text{ and } 0 \leq y \leq \pi,\ y \neq \frac{\pi}{2}$$



1. The domain of $y = \sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$
2. The range of $y = \sec^{-1} x$ is $[0, \pi/2) \cup (\pi/2, \pi]$.
3. The graph of $y = \sec^{-1} x$ is always increasing.
4. $y = \dfrac{\pi}{2}$ is a horizontal asymptote to the graph of $y = \sec^{-1} x$.

A1: Exponential and Logarithmic Functions
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Example 4

1. $\sec^{-1} 1 = 0$
2. $\sec^{-1} \sqrt{2} = \dfrac{\pi}{4}$
3. $\sec^{-1}(-2) = \dfrac{2\pi}{3}$
4. $\sec^{-1}(-1) = \pi$
5. $\lim_{x \to \infty} \sec^{-1} x = \dfrac{\pi}{2}$
6. $\lim_{x \to -\infty} \sec^{-1} x = \dfrac{\pi}{2}$

Warning: the definition of $\sec^{-1}$ can be quite different in different courses/books.

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More About the Definition of $\sec^{-1} x$

1. Our choice for the definition of $\sec^{-1} x$ has a disadvantage, and an advantage.
2. The disadvantage is that the formula for its derivative involves absolute value signs, as we shall see later.
3. The advantage is that $y = \sec^{-1} x$ and $y = \cos^{-1} x$ have the same range. Thus $y = \sec^{-1} x \Rightarrow x = \sec y \Rightarrow \frac{1}{x} = \cos y$

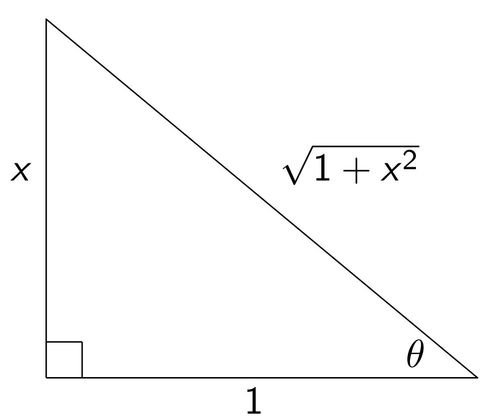
$$\Rightarrow y = \cos^{-1}\left(\frac{1}{x}\right) \Rightarrow \sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right).$$

You can use this result to evaluate $\sec^{-1} x$ with your calculator.

Example 5: Simplifying Trig Expressions with Triangles.

Problem: Evaluate $\sin(\tan^{-1}(x))$ in terms of x, if $x > 0$.

Solution: Let $\theta = \tan^{-1}(x)$. Then $\tan\theta = x$.



- In the triangle to the left, $x = \tan\theta$.
- The length of the hypotenuse is $\sqrt{1+x^2}$.
- Then

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}.$$

Problem 4

Given that $x < 0$, find the value of the expression

$$\tan\left(\cos^{-1}\left(\frac{x}{\sqrt{3+x^2}}\right)\right).$$

Solution: let

$$\theta = \cos^{-1}\left(\frac{x}{\sqrt{3+x^2}}\right).$$

Then θ is in the second quadrant, because $x < 0$, and

$$\cos\theta = \frac{x}{\sqrt{3+x^2}} = \frac{x}{r}$$

with $x^2 + y^2 = r^2 = 3 + x^2$. Take $y = \sqrt{3}$, since y is positive in the second quadrant. Then

$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{x}.$$

Problem 5

Find all $x \in \mathbb{R}$ such that $x = -\sin(\cos^{-1}(3x+1))$.
Solution: let $\theta = \cos^{-1}(3x+1)$. Then $\cos\theta = 3x + 1$ and $0 \leq \theta \leq \pi$, by definition of the inverse cosine function. Thus $\sin\theta \geq 0$ and consequently

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(3x+1)^2} = \sqrt{-9x^2-6x}.$$

Then

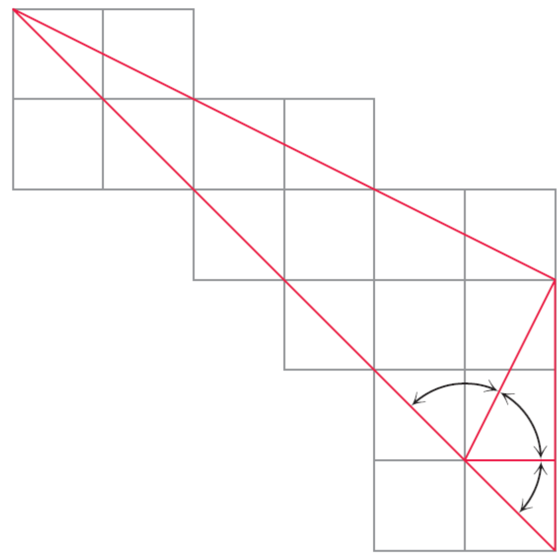
$$\begin{aligned} x = -\sqrt{-9x^2-6x} \quad &\Rightarrow \quad x^2 = -9x^2 - 6x \\ &\Rightarrow \quad 5x^2 + 3x = 0 \\ &\Rightarrow \quad x(5x+3) = 0 \\ &\Rightarrow \quad x = 0 \text{ or } x = -\frac{3}{5}. \end{aligned}$$

Problem 6

Explain how you can use the diagram at the right to find the exact value of

$\tan^{-1}(1)+\tan^{-1}(2)+\tan^{-1}(3)$.

Confirm your answer with your calculator.



A1: Exponential and Logarithmic Functions
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Comments About Problem 6

Let

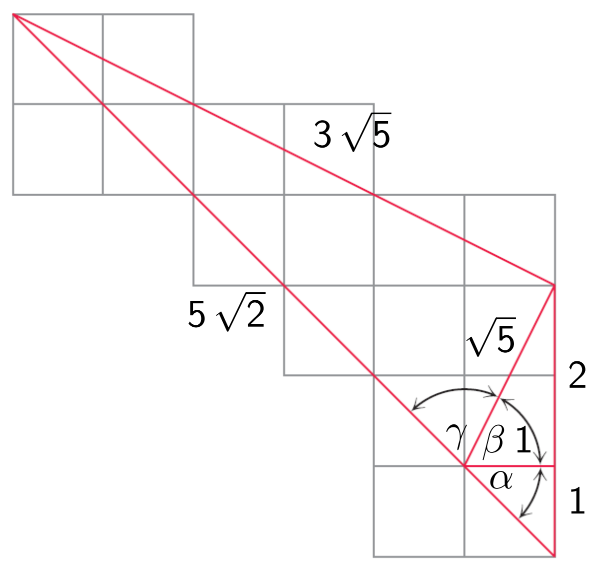
$\alpha = \tan^{-1}(1) \Leftrightarrow \tan\alpha = 1,$

$\beta = \tan^{-1}(2) \Leftrightarrow \tan\beta = 2,$

and

$\gamma = \tan^{-1}(3) \Leftrightarrow \tan\gamma = 3.$

By labelling lengths on the diagram you can see the positions of α, β, γ. Thus



$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \alpha + \beta + \gamma = \pi.$$

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