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Chapter 16 Lecture Notes

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Chapter 16:

Second-Order Linear Homogeneous Differential Equations

Second Order Linear Homogeneous Differential Equation

$$\text{DE: } A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0$$

This equation is called second order because it includes the second derivative of y; it is called homogenous because the right side of the equation is zero; it is called linear because the set of all solutions to this DE forms a subspace of functions in the sense that if

1. if both y_1 and y_2 satisfy DE, then so does $y_1 + y_2$
2. if k is a scalar and y satisfies DE, then so does ky

In other words, the set of solutions to DE is closed under addition and scalar multiplication. This is only one connection of many between differential equations and linear algebra.

Superposition Principle

The fact that the set of solutions to DE is closed under addition and scalar multiplication can be combined into the single statement:

If k_1 and k_2 are scalars, and if y_1 and y_2 are both solutions to DE, then

$$k_1y_1 + k_2y_2$$

is also a solution to DE.

This statement is known as the superposition principle. Its proof is a simple computation:

$$\begin{aligned}
& A(x)(k_1y_1+k_2y_2)''+B(x)(k_1y_1+k_2y_2)'+C(x)(k_1y_1+k_2y_2) \\
= \; & A(x)(k_1y_1''+k_2y_2'')+B(x)(k_1y_1'+k_2y_2')+C(x)(k_1y_1+k_2y_2) \\
= \; & k_1(A(x)y_1''+B(x)y_1'+C(x)y_1)+k_2(A(x)y_2''+B(x)y_2'+C(x)y_2) \\
= \; & k_1\cdot 0+k_2\cdot 0, \text{ since both } y_1 \text{ and } y_2 \text{ satisfy DE} \\
= \; & 0+0=0
\end{aligned}$$

One thing we can't prove at the moment is that the dimension of the subspace of all solutions to DE is two. But it makes sense, since to isolate y in DE, you would expect to integrate twice, and so obtain two arbitrary constants of integration in the general solution to DE. Solving DE in general, though, is quite hard. I'll just give one example, which is essentially based on trial and error.

Example 1: $x^2y''+2x\,y'-6y=0$

Try $y=x^m$. Then $y'=mx^{m-1}$, $y''=m(m-1)x^{m-2}$, and

$$\begin{aligned}
x^2y''+2x\,y'-6y &= x^2\left(m(m-1)x^{m-2}\right)+2x\left(mx^{m-1}\right)-6x^m \\
&= m(m-1)x^m+2m\,x^m-6x^m \\
&= \left(m^2-m+2m-6\right)x^m \\
&= \left(m^2+m-6\right)x^m=0, \text{for all } x \\
\Leftrightarrow m^2+m-6 &= 0 \Leftrightarrow m_1=2 \text{ or } m_2=-3. \text{ Thus:}
\end{aligned}$$

$$y_1=x^{m_1}=x^2 \text{ and } y_2=x^{m_2}=\frac{1}{x^3}$$

are two independent solutions to Example 1, and the general solution is

$$y=k_1y_1+k_2y_2=k_1x^2+\frac{k_2}{x^3}.$$

Example 2: $x^2y'' + 2x\,y' - 6y = 8x - 6$

This is a non-homogeneous DE related to Example 1. As in linear algebra, all the solutions to the non-homogeneous equation can be found by adding all the solutions to the homogeneous equation to a single particular solution of the non-homogeneous equation. You can check that

$$y_p = -2x + 1$$

is a particular solution to Example 2. Then

$$y = k_1x^2 + \frac{k_2}{x^3} + y_p = k_1x^2 + \frac{k_2}{x^3} - 2x + 1$$

is the general solution to Example 2.

Constant Coefficients

The second order, linear homogeneous differential equation with constant coefficients is

$$ay'' + by' + cy = 0.$$

We can solve this equation completely. Again, we start with an inspired guess: let $y = e^{rx}$. Then $y' = re^{rx}, y'' = r^2e^{rx}$, and

$$\begin{aligned} ay'' + by' + cy = 0 \quad &\Leftrightarrow \quad ar^2e^{rx} + bre^{rx} + ce^{rx} = 0 \\ &\Leftrightarrow \quad (ar^2 + br + c)e^{rx} = 0 \\ &\Leftrightarrow \quad ar^2 + br + c = 0, \text{ since } e^{rx} \neq 0 \end{aligned}$$

$ar^2 + br + c = 0$ is called the auxiliary quadratic to the second order, linear homogeneous differential equation with constant coefficients.

The roots of the auxiliary quadratic, $ar^2 + br + c = 0$, are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

as you know. The different types of solutions to

$$\text{DE: } ay'' + by' + cy = 0$$

depend on the discriminant

$$\Delta = b^2 - 4ac$$

of the auxiliary quadratic. Analysis and examples of the three cases follow.

Case 1: $\Delta > 0$

In this case, the simplest case, the auxiliary quadratic has two distinct real roots, say r_1 and r_2. Then the general solution to

$$ay'' + by' + cy = 0$$

is

$$y = k_1 e^{r_1 x} + k_2 e^{r_2 x},$$

where constants k_1 and k_2 can be determined by initial conditions.

Example 3: $y'' + 2y' - 15y = 0; y = 2, y' = -2$, if $x = 0$

The auxiliary quadratic is $r^2 + 2r - 15 = 0 \Leftrightarrow r_1 = 3, r_2 = -5$. So the general solution is

$$y = k_1 e^{3x} + k_2 e^{-5x};$$

and

$$y' = 3k_1 e^{3x} - 5k_2 e^{-5x}.$$

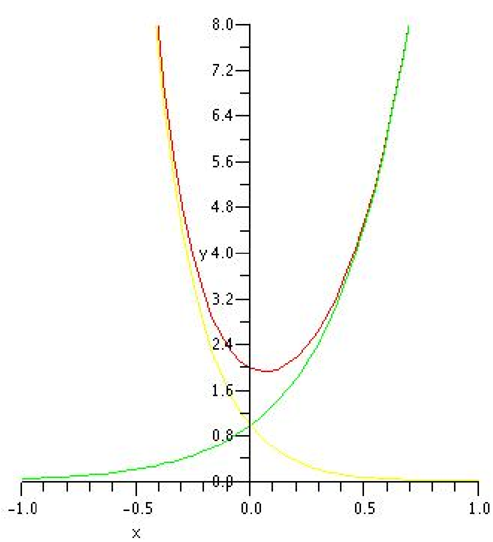
To find k_1 and k_2, we substitute the initial conditions into the equations for y and y' to obtain:

$$\left\{ \begin{array}{rcrcr} k_1 & + & k_2 & = & 2 \\ 3k_1 & - & 5k_2 & = & -2 \end{array} \right. \Leftrightarrow (k_1, k_2) = (1, 1),$$

as you may check. Thus the solution to Example 3 is:

$$y = e^{3x} + e^{-5x}.$$

Graph of Solution for Example 3



Case 2: $\Delta = 0$

In this case, there is a single, repeated solution to the auxiliary quadratic: $r = -\dfrac{b}{2a}$. So $y_1 = e^{rx}$ is one solution to DE. We claim $y_2 = xe^{rx}$ is another solution to DE:

$$\begin{aligned} & ay_2'' + by_2' + cy_2 \\ = \;& a\left(2re^{rx} + r^2xe^{rx}\right) + b\left(e^{rx} + rxe^{rx}\right) + cxe^{rx} \\ = \;& (ar^2 + br + c)xe^{rx} + (2ar + b)e^{rx} \\ = \;& 0 \text{ (why?) } + 0, \text{ since } r = -\frac{b}{2a} \\ = \;& 0 \end{aligned}$$

The general solution is $y = k_1y_1 + k_2y_2 = k_1e^{rx} + k_2xe^{rx}$.

Example 4: $9y'' + 12y' + 4y = 0; y = 2, y' = -2$, if $x = 0$

$9r^2 + 12r + 4 = (3r + 2)^2 = 0 \Leftrightarrow r = -2/3$; so the general solution is

$$y = k_1e^{-2x/3} + k_2xe^{-2x/3};$$

and

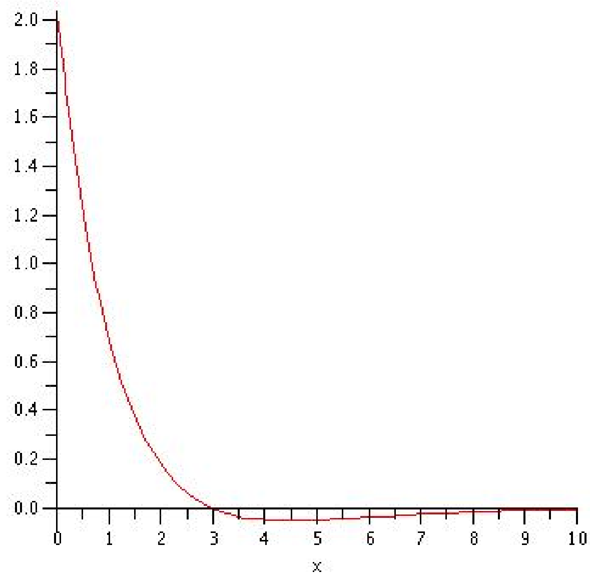
$$y' = \left(k_2 - \frac{2}{3}k_1\right)e^{-2x/3} - \frac{2}{3}k_2xe^{-2x/3}.$$

To find k_1 and k_2, we substitute the initial conditions into the equations for y and y' to obtain:

$$\left\{\begin{array}{rcrcr} k_1 & & & = & 2 \\ -\frac{2}{3}k_1 & + & k_2 & = & -2 \end{array}\right. \Leftrightarrow (k_1, k_2) = \left(2, -\frac{2}{3}\right),$$

as you may check. So the solution is $y = 2e^{-2x/3} - \frac{2}{3}xe^{-2x/3}$.

Graph of Solution for Example 4



Case 3: $\Delta < 0$

In this case the auxiliary quadratic has two complex roots, say $r = p \pm qi$, with $i^2 = -1$. Naively we could say,

$$\begin{aligned} y = k_1 e^{r_1 x} + k_2 e^{r_2 x} &= k_1 e^{(p+iq)x} + k_2 e^{(p-iq)x} \\ &= k_1 e^{px} e^{iqx} + k_2 e^{px} e^{-iqx}. \end{aligned}$$

But what meaning are we to give to $e^{\pm iqx}$? We can use Euler's Formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

which we derived in Chapter 10.

$$\begin{aligned} y &= k_1 e^{px} e^{iqx} + k_2 e^{px} e^{-iqx} \\ &= k_1 e^{px} (\cos(qx) + i\sin(qx)) + k_2 e^{px} (\cos(qx) - i\sin(qx)) \\ &= (k_1 + k_2) e^{px} \cos(qx) + (k_1 - k_2) i\, e^{px} \sin(qx) \\ &= c_1 e^{px} \cos(qx) + c_2 e^{px} \sin(qx), c_1 = k_1 + k_2, c_2 = (k_1 - k_2) i, \end{aligned}$$

where constants c_1 and c_2 can be determined by initial conditions. Case 3 is by far the most interesting mathematically. Surprisingly, even though we have used the so-called imaginary number i, we will find that Case 3 is also the most interesting physically, and is used in a major application, namely mechanical vibrations.

Example 5: $9y'' + 6y' + 145y = 0; y = 2, y' = -2$ if $x = 0$.

$9r^2 + 6r + 145 = 0 \Leftrightarrow r = \dfrac{-6 \pm \sqrt{36 - 36 \cdot 145}}{18} = -\dfrac{1}{3} \pm 4i$. So the general solution is

$$y = c_1 e^{-x/3} \cos(4x) + c_2 e^{-x/3} \sin(4x);$$

from which $y' =$

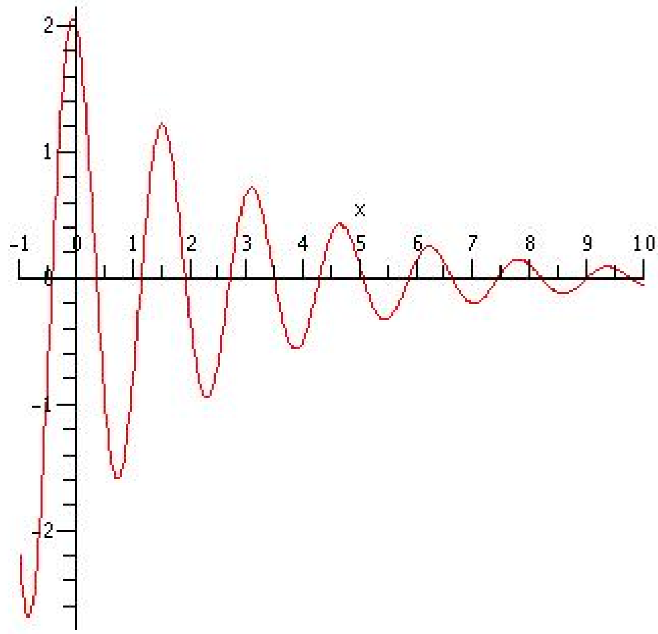
$$e^{-x/3} \left(-\frac{c_1}{3} \cos(4x) - 4c_1 \sin(4x) - \frac{1}{3} c_2 \sin(4x) + 4c_2 \cos(4x) \right).$$

Use the initial conditions to find c_1 and c_2 :

$$\left\{ \begin{array}{rcrcr} c_1 & & & = & 2 \\ -\frac{1}{3} c_1 & + & 4c_2 & = & -2 \end{array} \right. \Leftrightarrow (c_1, c_2) = \left(2, -\frac{1}{3}\right).$$

So the solution is $y = 2e^{-x/3} \cos(4x) - \frac{1}{3} e^{-x/3} \sin(4x)$.

Graph of Solution for Example 5



Special Case: $b = 0$ and $ay'' + cy = 0$

In this case, the auxiliary quadratic is $ar^2 + c = 0$. The roots will be

$$r = \pm\sqrt{-\frac{c}{a}}.$$

If the roots are real, then we are back to Case 1. But if $-c/a < 0$, then the roots will be purely imaginary:

$$r = \pm\sqrt{\frac{c}{a}}\,i,$$

and the solutions will be a special case of Case 3:

$$y = c_1 \cos\sqrt{\frac{c}{a}}\,x + c_2 \sin\sqrt{\frac{c}{a}}\,x.$$

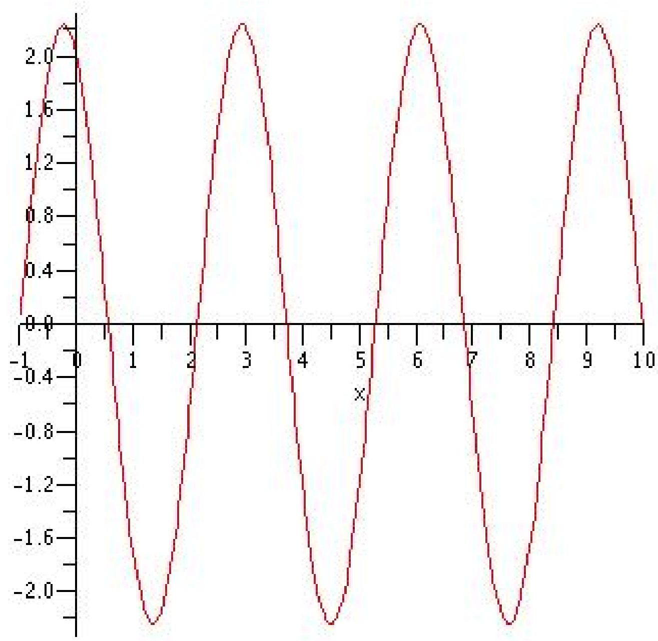
Example 6: $y'' + 4y = 0; y = 2, y' = -2$, if $x = 0$

$$
\begin{aligned}
r^2 + 4 = 0 \quad &\Leftrightarrow \quad r = \pm 2\,i \\
&\Rightarrow \quad y = c_1 \cos(2x) + c_2 \sin(2x) \\
&\Rightarrow \quad y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)
\end{aligned}
$$

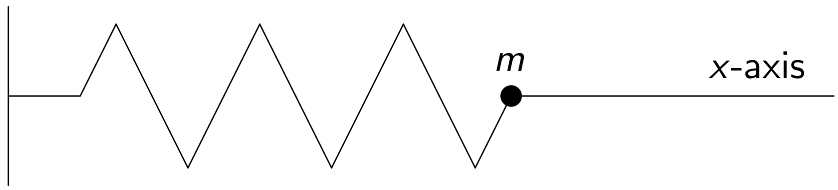
Using the initial values we find $c_1 = 2$ and $c_2 = -1$, as you can check. So the solution is

$$y = 2\cos(2x) - \sin(2x).$$

Graph of Solution for Example 6



Mass On A Spring



Suppose

1. x is the position of a mass on a spring at time t.
2. x is measured as displacement from the equilibrium position, $x = 0$.
3. m is the mass of the object on the spring.
4. c depends on the friction of the surrounding medium.
5. k is the spring constant, from Hooke's Law.

Differential Equation for a Mass on a Spring

Then x must satisfy

$$mx''(t) + cx'(t) + kx(t) = 0.$$

This differential equation is second order, linear, homogenous, with constant coefficients.

Derivation of DE

We do a force analysis, considering the resisting force of the surrounding medium, with c representing the coefficient of friction; and the restoring force of the spring, with k representing the spring constant.

$$\begin{aligned} F = ma &= F_{\text{resisting}} + F_{\text{restoring}} \\ &= -cv - kx \\ \Leftrightarrow m\frac{d^2x}{dt^2} &= -c\frac{dx}{dt} - kx \\ \Leftrightarrow m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx &= 0, \text{ or } mx''(t) + cx'(t) + kx(t) = 0 \end{aligned}$$

The Solution to This Differential Equation

The solution for x depends on the associated quadratic equation

$$mr^2 + cr + k = 0,$$

which has solutions

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}.$$

There are three cases:
Case 1: $c^2 - 4km > 0$, in which case the quadratic has two real roots.
Case 2: $c^2 - 4km = 0$, in which case the quadratic has one distinct real root, which is repeated.
Case 3: $c^2 - 4km < 0$, in which case the quadratic has two complex roots.

Case 1: Overdamped Vibrations: $c^2 > 4km$

In this case, the associated quadratic has two distinct real solutions, say r_1 and r_2. Then the solution is

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

where c_1 and c_2 are constants that depend on initial conditions. Since both $r_1 < 0$ and $r_2 < 0$,

$$\lim_{t \to \infty} x = 0.$$

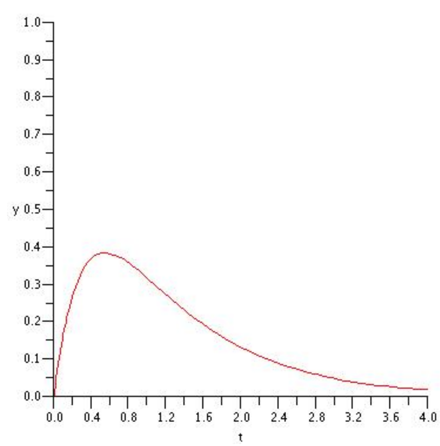
In this case the friction is so strong, that the motion dies very quickly.

Example 7: $m = 1; c = 4; k = 3; x(0) = 0; x'(0) = 2$

$r^2 + 4r + 3 = 0 \Leftrightarrow r = -3$ or $r = -1$. Then the solution is

$$x = -e^{-3t} + e^{-t}.$$

The graph of displacement, x versus time, t, is:



Case 2: Critically Damped Vibrations: $c^2 = 4km$

In this case, the associated quadratic has only one distinct real solution, say r. Then the solution is

$$x = c_1 e^{rt} + c_2 t e^{rt},$$

where c_1 and c_2 are constants that depend on initial conditions. Since $r < 0$ it is still true that

$$\lim_{t \to \infty} x = 0.$$

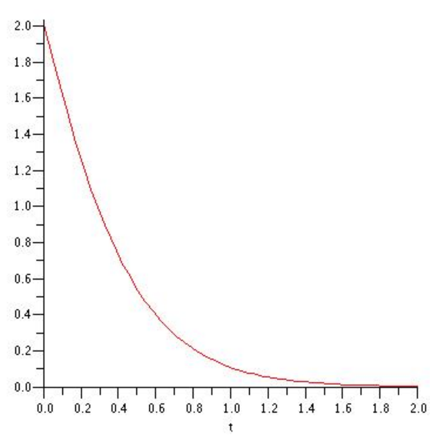
This case is very similar to the previous case.

Example 8: $m = 1/2; c = 4; k = 8; x(0) = 2; x'(0) = -4$

$\frac{1}{2}r^2 + 4r + 8 = 0 \Leftrightarrow r^2 + 8r + 16 = 0 \Leftrightarrow r = -4$. The solution is

$$x = 2e^{-4t} + 4te^{-4t}.$$

The graph is:



Case 3: Underdamped Vibrations: $c^2 < 4km$

In this case the quadratic has two complex roots, say $\alpha \pm \beta\, i$. Then the solution is

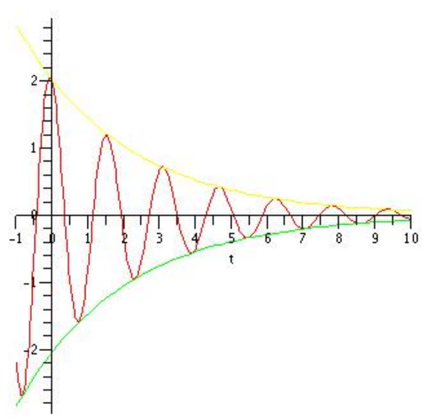
$$x = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t),$$

where c_1 and c_2 are constants that depend on initial conditions. This case is much more interesting, both mathematically and physically. Since $\alpha < 0$, $x \to 0$ still, as $t \to \infty$; but the presence of the two trigonometric terms means that the vibrations will oscillate with period $2\pi/|\beta|$, with ever decreasing amplitudes. In this case the friction is not strong enough to completely stop the vibrations.

Example 9: $m = 9; c = 6; k = 145; x(0) = 2; x'(0) = -2$

$9r^2 + 6r + 145 = 0 \Leftrightarrow r = \dfrac{-6 \pm \sqrt{36 - 36 \cdot 145}}{18} = -\dfrac{1}{3} \pm 4i$. The solution is

$$x = 2e^{-t/3}\cos(4t) - \frac{1}{3}e^{-t/3}\sin(4t).$$



The graph is to the left. We say its pseudo period is $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$ and its time varying amplitude is

$$\frac{\sqrt{37}}{3}e^{-t/3}.$$

Where do these values come from?

In General, for $x = Ae^{\alpha t}\cos(\beta t) + Be^{\alpha t}\sin(\beta t)$

The pseudo period is $T = \dfrac{2\pi}{|\beta|}$, and the time varying amplitude is

$$\sqrt{A^2 + B^2}e^{\alpha t}.$$

This is calculated by using some trigonometry:

$$\begin{aligned} A\cos(\beta t) + B\sin(\beta t) &= \sqrt{A^2+B^2}\left(\frac{A\cos(\beta t) + B\sin(\beta t)}{\sqrt{A^2+B^2}}\right) \\ &= \sqrt{A^2+B^2}\,(\cos\phi\cos(\beta t) + \sin\phi\sin(\beta t)) \\ &= \sqrt{A^2+B^2}\cos(\beta t - \phi), \text{ with } \tan\phi = \frac{B}{A} \end{aligned}$$

Thus $x = \sqrt{A^2+B^2}e^{\alpha t}\cos(\beta t - \phi)$, for some phase angle ϕ.

Pseudo Period and Time Varying Amplitude of Example 9

Our solution was

$$\begin{aligned} x &= 2e^{-t/3}\cos(4t) - \frac{1}{3}e^{-t/3}\sin(4t) \\ &= \frac{\sqrt{37}}{3}e^{-t/3}\left(\frac{6}{\sqrt{37}}\cos(4t) - \frac{1}{\sqrt{37}}\sin(4t)\right) \\ &= \frac{\sqrt{37}}{3}e^{-t/3}\left(\cos\phi\cos(4t) + \sin\phi\sin(4t)\right), \text{ for } \tan\phi = -\frac{1}{6} \\ &= \frac{\sqrt{37}}{3}e^{-t/3}\cos(4t - \phi), \text{ for phase angle } \phi = \tan^{-1}\left(-\frac{1}{6}\right). \end{aligned}$$

The pseudo period of the vibration $\frac{2\pi}{4} = \frac{\pi}{2}$; and $\frac{\sqrt{37}}{3}e^{-t/3}$ is the time-varying amplitude of the vibration.

True Periodic Vibrations: $c = 0$

In the absence of friction, if $c = 0$, then $mx''(t) + kx(t) = 0$, and the auxiliary quadratic is $mr^2 + k = 0$, which has roots

$$r = \pm\sqrt{\frac{k}{m}}i = \pm\omega_0 i, \text{ if we set } \omega_0 = \sqrt{\frac{k}{m}}.$$

Then

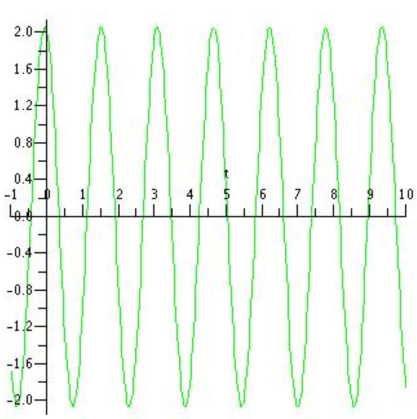
$$x = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t),$$

where c_1 and c_2 are constants that depend on initial conditions. This is true periodic motion, with period

$$T = \frac{2\pi}{\omega_0}, \text{ and amplitude } A = \sqrt{c_1^2 + c_2^2}.$$

Example 10: $m = 9; c = 0; k = 145; x(0) = 2; x'(0) = -2$

Then $\omega_0 = \dfrac{\sqrt{145}}{3}$ and $x = 2\cos(\omega_0 t) - \dfrac{2}{\omega_0}\sin(\omega_0 t)$, as you can check.



The graph is to the left. The period of the vibration is

$$T = \frac{2\pi}{\omega_0} = \frac{6\pi}{\sqrt{145}}$$

and the amplitude is

$$\sqrt{2^2 + \frac{2^2}{\omega_0^2}} = 2\sqrt{\frac{154}{145}}.$$

Comparison of Examples 9 and 10

Examples 9 and 10 differ only in the value of c.

- In Example 10, $c = 0$, and

$$T = \frac{6\pi}{\sqrt{145}} \simeq 1.565370417 \text{ sec}; \; \nu = \frac{1}{T} \simeq 0.6388264331 \text{ Hz.}.$$

- For Example 9, $c = 6$, and we found

$$T = \frac{\pi}{2} \simeq 1.570796327 \text{ sec}; \;\; \text{so } \nu = \frac{1}{T} \simeq 0.6366197723 \text{ Hz.}$$

In this comparison, the friction is so small it barely slows down the vibration. Greater friction would have greater effect.

Graphs for Examples 9 and 10