MAT187H1F Lec0101 Burbulla

Chapter 6 Lecture Notes
Review and Two New Sections

Sprint 2017

Net Distance and Total Distance Travelled

Suppose s is the position of a particle at time t for $t \in [a, b]$. Then

$$\int_a^b v\, dt = \int_a^b s'(t)\, dt = s(b) - s(a).$$

$s(b) - s(a)$ is called the displacement, or net distance travelled, by the particle over the time interval $[a, b]$. This is in contrast to the total distance travelled, which is given by the integral of speed. That is, the total distance travelled by the particle for $t \in [a, b]$, is

$$\int_a^b |v|\, dt.$$

Derivation of Total Distance

If v_i^* is the velocity of the particle at some time t_i^* in the time interval $[t_{i-1}, t_i]$, then the speed $|v_i^*|$ is approximately constant on the time interval. So on this time interval the distance travelled is approximately $|v_i^*| \times \Delta t$, since

$$\text{distance} = \text{speed} \times \text{time} .$$

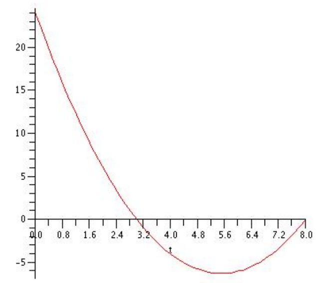
So the total distance travelled is

$$\lim_{n\to\infty} \sum_{i=1}^{n} |v_i^*| \times \Delta t = \int_a^b |v|\, dt.$$

Example 1

Suppose $v = t^2 - 11t + 24$ for $t \in [0, 8]$. Then

$$\begin{aligned}\int_0^8 v\,dt &= \int_0^8 \left(t^2 - 11t + 24\right)\,dt \\ &= \left[\frac{t^3}{3} - \frac{11}{2}t^2 + 24t\right]_0^8 \\ &= \frac{32}{3}\end{aligned}$$



This means that after 8 seconds the particle is $32/3$ units to the right of where it started from.

Chapter 6: Applications of Integration

6.1 Velocity and Net Change
6.2 Regions Between Curves
6.3 Volume by Slicing
6.4 Volume by Shells
6.5 Length of Curves
6.6 Surface Area
6.7 Physical Applications
6.9 Exponential Models
6.10 Hyperbolic Functions: An Overview

The total distance is a much more complicated calculation because $v > 0$ for $t < 3$; but $v < 0$ for $t > 3$. Thus the total distance travelled is

$$\begin{aligned}\int_0^8 |v|\,dt &= \int_0^3 v\,dt - \int_3^8 v\,dt \\ &= \left[\frac{t^3}{3} - \frac{11}{2}t^2 + 24t\right]_0^3 - \left[\frac{t^3}{3} - \frac{11}{2}t^2 + 24t\right]_3^8 \\ &= \frac{63}{2} - \left(-\frac{125}{6}\right) \\ &= \frac{157}{3}\end{aligned}$$

Chapter 6 Lecture Notes Review and Two New Sections MAT187H1F Lec0101 Burbulla

Integral Formulas

- Often to find a formula to describe something – be it geometrical or physical – we use a Riemann sum to first set things up in a simple, approximate way.
- These approximations usually start with a regular partition.
- Then as the limit of the norm of the partition goes to zero, the Riemann sum approaches an integral.
- In Chapter 6, formulas for area, volumes, length, surface area, work – to name a few – will be derived this way.

Chapter 6: Applications of Integration

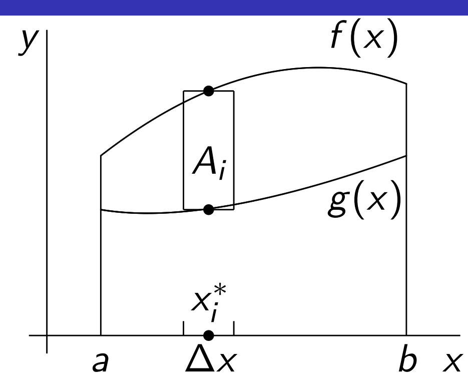
6.1 Velocity and Net Change
6.2 Regions Between Curves
6.3 Volume by Slicing
6.4 Volume by Shells
6.5 Length of Curves
6.6 Surface Area
6.7 Physical Applications
6.9 Exponential Models
6.10 Hyperbolic Functions: An Overview

Area Between Two Curves: The Basic Formula

If $f(x) \geq g(x)$ for $x \in [a, b]$, then

$$\int_a^b (f(x) - g(x))\,dx$$

is the area of the region below f, above g, for $a \leq x \leq b$. **Proof:**

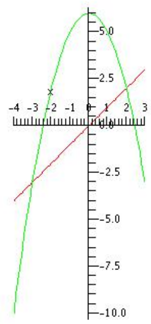


$$A_i = (f(x_i^*) - g(x_i^*))\Delta x \Rightarrow A \simeq \sum A_i = \sum (f(x_i^*) - g(x_i^*))\Delta x;$$

$$A = \lim_{\Delta x \to 0} \sum A_i = \lim_{\Delta x \to 0} \sum (f(x_i^*) - g(x_i^*))\Delta x = \int_a^b (f(x) - g(x))\,dx.$$

Example 1

Find the area of the region bounded by $y = x$ and $y = 6 - x^2$.
Solution: find intersection points:
$6 - x^2 = x \Leftrightarrow 0 = x^2 + x - 6 \Leftrightarrow x = -3$ or $x = 2$.

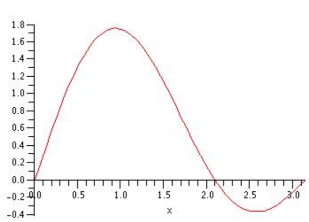


$$\begin{aligned} A &= \int_{-3}^{2} \left(6 - x^2 - x\right)\, dx \\ &= \left[6x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]_{-3}^{2} \\ &= \frac{125}{6} \end{aligned}$$

Example 2

Find the area between the x-axis and the curve $y = \sin(2x) + \sin x$ on $[0, \pi]$. **Solution:** $2 \sin x \cos x + \sin x = 0 \Rightarrow x = 0, \pi, 2\pi/3$.
Then $A = A_1 + A_2$ where

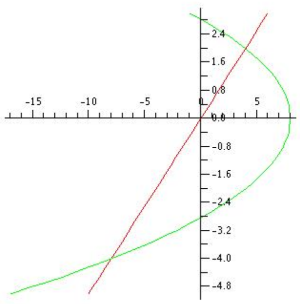
$$A_1 = \int_0^{2\pi/3} \left(\sin(2x) + \sin x\right)\, dx = \left[-\frac{1}{2}\cos(2x) - \cos x\right]_0^{2\pi/3} = \frac{9}{4},$$



$$\begin{aligned} A_2 &= \int_{2\pi/3}^{\pi} \left(-\sin(2x) - \sin x\right)\, dx \\ &= \left[\frac{1}{2}\cos(2x) + \cos x\right]_{2\pi/3}^{\pi} = \frac{1}{4} \end{aligned}$$

Example 3: Integrating with Respect to y

Find area of region bounded by the curves with equation $2y = x$ and $y^2 = 8 - x$. **Solution:** find intersection points: $y^2 = 8 - 2y \Leftrightarrow y^2 + 2y - 8 = 0 \Leftrightarrow y = -4$ or $y = 2$.



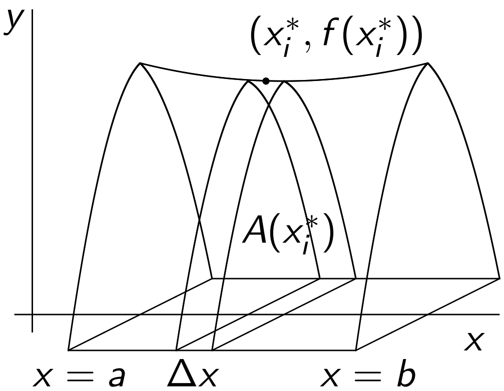
$$\begin{aligned} A &= \int_{-4}^{2} \left(8 - y^2 - 2y\right)\, dy \\ &= \left[8y - \frac{1}{3}y^3 - y^2\right]_{-4}^{2} \\ &= 16 - \frac{8}{3} - 4 + 32 - \frac{64}{3} + 16 \\ &= 36 \end{aligned}$$

Example 3, Integrating with Respect to x; the Hard Way

$$\begin{aligned} A &= \int_{-8}^{4} \left(\frac{x}{2} - (-\sqrt{8-x})\right)\, dx + \int_{4}^{8} \left(\sqrt{8-x} - (-\sqrt{8-x})\right)\, dx \\ &= \int_{-8}^{4} \left(\frac{x}{2} + \sqrt{8-x}\right)\, dx + \int_{4}^{8} 2\sqrt{8-x}\, dx \\ &= \left[\frac{x^2}{4} - \frac{2(8-x)^{3/2}}{3}\right]_{-8}^{4} + \left[-\frac{4(8-x)^{3/2}}{3}\right]_{4}^{8} \\ &= 4 - \frac{16}{3} - 16 + \frac{128}{3} - 0 + \frac{32}{3} \\ &= 36, \text{ as before.} \end{aligned}$$

The Method of Slicing

Let x_i^* be any point in a subinterval of length Δx; let the area of a cross-sectional slice perpendicular to the x-axis with base Δx be $A(x_i^*)$. Add up the volumes of all the slices:



$$\begin{aligned} V &\simeq \sum \Delta V = \sum A(x_i^*)\,\Delta x \\ \Rightarrow V &= \lim_{\Delta x \to 0} \sum A(x_i^*)\,\Delta x \\ &= \int_a^b A(x)\,dx \end{aligned}$$

Chapter 6: Applications of Integration

6.1 Velocity and Net Change
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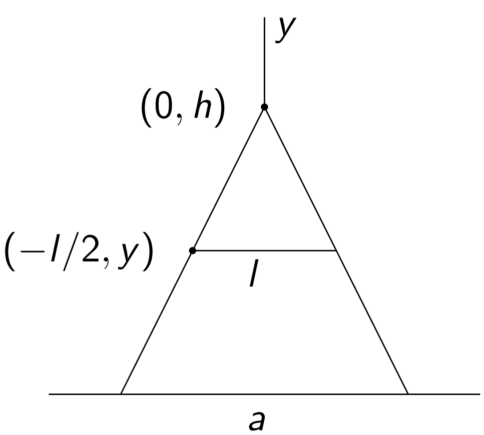
Example 1: Volume of a Pyramid, $V = \frac{1}{3}a^2h$

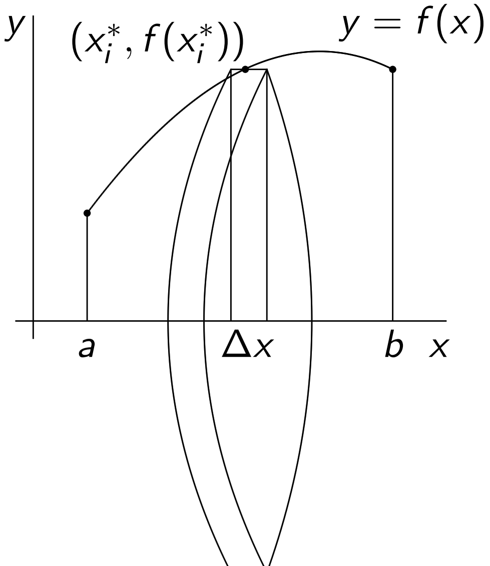
Let l be the length of the side of a cross-section of the pyramid at height y above the base. By similar triangles, $\dfrac{l}{h-y} = \dfrac{a}{h}$.



The cross section is a square: $A(y) = l^2$.

$$\begin{aligned} V &= \int_0^h A(y)\,dy = \int_0^h a^2\left(1 - \frac{y}{h}\right)^2 dy \\ &= a^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2}\right) dy \\ &= a^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2} \right]_0^h = \frac{1}{3}a^2 h \end{aligned}$$

Solids of Revolution: Method of Disks



Let x_i^* be any point in a subinterval of length Δx. The volume of the disc obtained by revolving about the x-axis the rectangle with base Δx and radius $f(x_i^*)$ is $\Delta V = \pi f(x_i^*)^2 \, \Delta x$.

$$\begin{aligned} V &\simeq \sum \Delta V = \sum \pi f(x_i^*)^2 \, \Delta x \\ \Rightarrow V &= \lim_{\Delta x \to 0} \sum \pi f(x_i^*)^2 \, \Delta x \\ &= \int_a^b \pi f(x)^2 \, dx \end{aligned}$$

Example 2: Volume of a Sphere

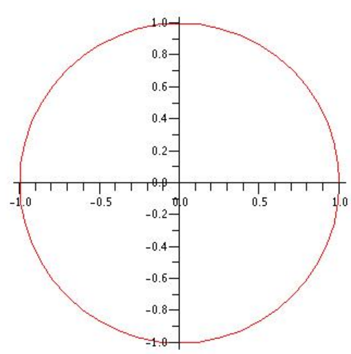
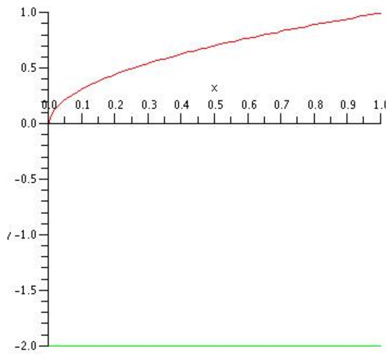


Figure: $x^2 + y^2 = r^2$

$$\begin{aligned} V &= 2\int_0^r \pi y^2 \, dx \\ &= 2\pi \int_0^r (r^2 - x^2) \, dx \\ &= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \\ &= 2\pi \left(r^3 - \frac{r^3}{3} \right) \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

Example 3

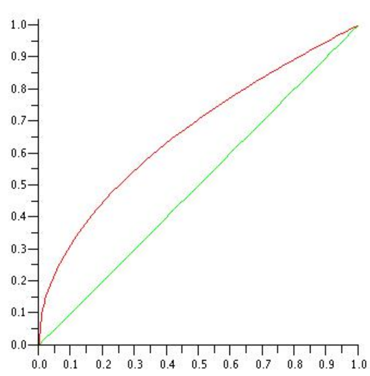
Find the volume of the solid obtained by revolving around the line $y = -2$ the curve $y = \sqrt{x}$ for $x \in [0, 1]$.



$$
\begin{aligned}
V &= \int_0^1 \pi(\sqrt{x}+2)^2\,dx \\
&= \pi \int_0^1 (x + 4\sqrt{x} + 4)\,dx \\
&= \pi \left[\frac{x^2}{2} + \frac{8}{3}x^{3/2} + 4x \right]_0^1 = \frac{43}{6}\pi
\end{aligned}
$$

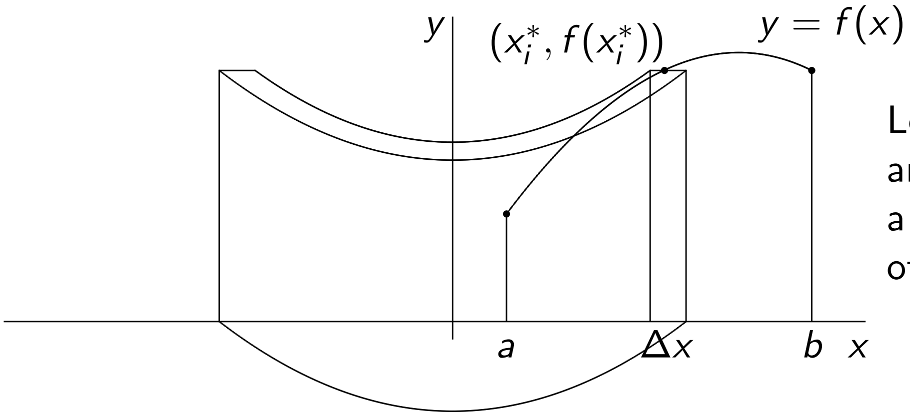
Example 4

Find the volume of the solid obtained by revolving about the x-axis the region bounded by the curves $y = \sqrt{x}$ and $y = x$ for $x \in [0, 1]$.



$$
\begin{aligned}
V &= \int_0^1 \pi(r_o^2 - r_i^2)\,dx \\
&= \pi \int_0^1 (x - x^2)\,dx \\
&= \pi \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}\pi
\end{aligned}
$$

Suppose $y = f(x)$ on $[a, b]$ is Revolved Around the y-axis



Let x_i^* be any point in a subinterval of length Δx.

The approximate volume of the above cylindrical shell is $\Delta V \simeq C \cdot h \cdot w = 2\pi x_i^* \cdot f(x_i^*) \cdot \Delta x$.

Volumes By Cylindrical Shells

Then

$$\begin{aligned} V &\simeq \sum \Delta V \\ &= \sum 2\pi x_i^* \cdot f(x_i^*) \cdot \Delta x \\ \Rightarrow V &= \lim_{\Delta x \to 0} \sum 2\pi x_i^* \cdot f(x_i^*) \cdot \Delta x \\ &= \int_a^b 2\pi x \cdot f(x)\, dx \end{aligned}$$

Example 1: Volume of a Cone

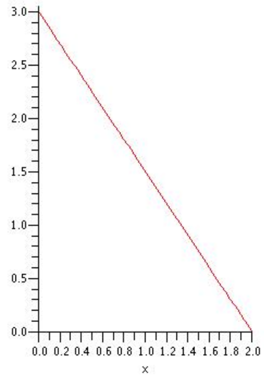


Figure: $\frac{x}{r} + \frac{y}{h} = 1$

$$\begin{aligned} V &= \int_0^r 2\pi x\, h\left(1 - \frac{x}{r}\right)\, dx \\ &= 2\pi h \int_0^r \left(x - \frac{x^2}{r}\right)\, dx \\ &= 2\pi h \left[\frac{x^2}{2} - \frac{x^3}{3r}\right]_0^r \\ &= 2\pi h \left(\frac{r^2}{2} - \frac{r^2}{3}\right) = \frac{1}{3}\pi h r^2 \end{aligned}$$

Example 2: Volume of a Sphere

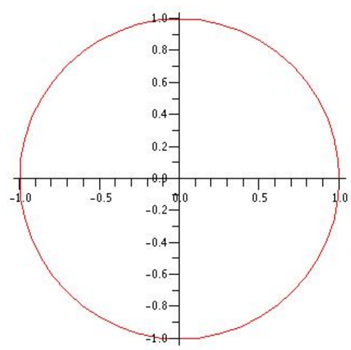
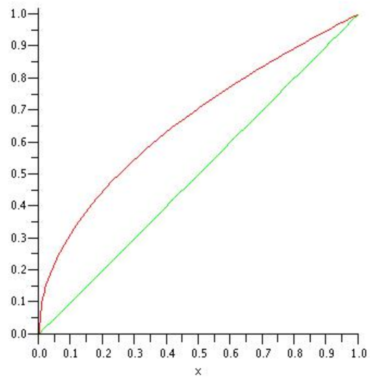


Figure: $x^2 + y^2 = r^2$

$$\begin{aligned} V &= 2\int_0^r 2\pi x \cdot y\, dx \\ &= 2\pi \int_0^r 2x\sqrt{r^2 - x^2}\, dx \\ &= 2\pi \int_{r^2}^0 \sqrt{u}\,(-du) = 2\pi \int_0^{r^2} \sqrt{u}\, du \\ &= 2\pi \left[\frac{2}{3}u^{3/2}\right]_0^{r^2} = \frac{4}{3}\pi r^3 \end{aligned}$$

Example 3

Find the volume of the solid obtained by revolving around the y-axis the region bounded by $y = \sqrt{x}$ and $y = x$, for $x \in [0, 1]$.

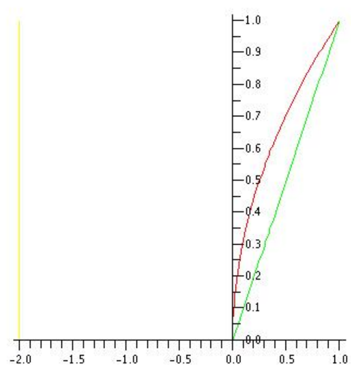


$$\begin{aligned} V &= \int_0^1 2\pi x\,(\sqrt{x} - x)\,dx \\ &= 2\pi \int_0^1 (x^{3/2} - x^2)\,dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{x^3}{3}\right]_0^1 = \frac{2}{15}\pi \end{aligned}$$

Chapter 6: Applications of Integration
6.1 Velocity and Net Change
6.2 Regions Between Curves
6.3 Volume by Slicing
6.4 Volume by Shells
6.5 Length of Curves
6.6 Surface Area
6.7 Physical Applications
6.9 Exponential Models
6.10 Hyperbolic Functions: An Overview

Example 4

Find the volume of the solid obtained by revolving around the line $x = -2$ the region bounded by $y = \sqrt{x}$ and $y = x$, for $x \in [0, 1]$.



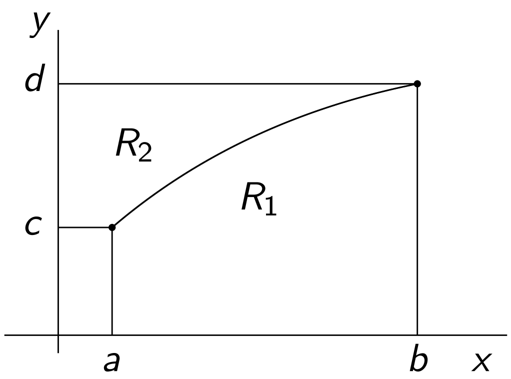
$$\begin{aligned} V &= \int_0^1 2\pi(x+2)\,(\sqrt{x} - x)\,dx \\ &= 2\pi \int_0^1 (x^{3/2} + 2\sqrt{x} - x^2 - 2x)\,dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} - \frac{x^3}{3} - x^2\right]_0^1 = \frac{4}{5}\pi \end{aligned}$$

Chapter 6 Lecture Notes Review and Two New Sections MAT187H1F Lec0101 Burbulla

Summary

Consider $y = f(x)$ on $[a, b]$, assume f is invertible, with $f^{-1} = g$, and that $c = f(a)$ and $d = f(b)$. What do these four integrals represent?

1. $\displaystyle\int_a^b \pi\,(f(x))^2\,dx$
2. $\displaystyle\int_a^b 2\pi\,x\,f(x)\,dx$
3. $\displaystyle\int_c^d \pi\,(g(y))^2\,dy$
4. $\displaystyle\int_c^d 2\pi\,y\,g(y)\,dy$



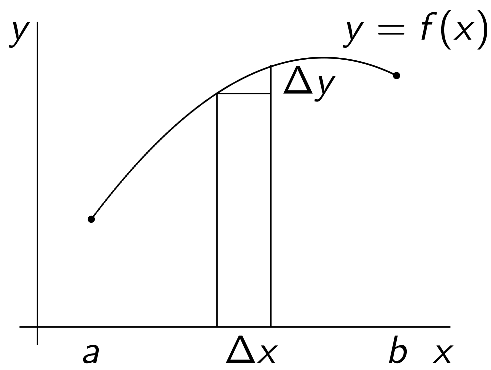
$\displaystyle\int_a^b \pi\,(f(x))^2\,dx$ is the volume of the region R_1 revolved about the x-axis.

$\displaystyle\int_a^b 2\pi\,x\,f(x)\,dx$ is the volume of the region R_1 revolved about the y-axis.

$\displaystyle\int_c^d \pi\,(g(y))^2\,dy$ is the volume of the region R_2 revolved about the y-axis.

$\displaystyle\int_c^d 2\pi\,y\,g(y)\,dy$ is the volume of the region R_2 revolved about the x-axis.

The Length of a Curve



If Δs is the length of a small part of the curve, then

$$\Delta s^2 \simeq \Delta x^2 + \Delta y^2$$
$$\Rightarrow \Delta s \simeq \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

Now let $\Delta x \to 0$: $S = \lim_{\Delta x \to 0} \sum \Delta s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$. S is called the arc length, or simply length, of the curve.

Example 1

The expression $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ is called the arc length differential. Then $S = \int_a^b ds$. **Example:** let $f(x) = x^{2/3}$ on $[0, 1]$.

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{2}{3}x^{-1/3}\right)^2}\,dx = \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx$$

$$\Rightarrow S = \int_0^1 ds = \int_0^1 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx = \frac{1}{18}\int_4^{13} \sqrt{u}\,du$$

$$(u = 9x^{2/3} + 4) = \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_4^{13} = \frac{1}{27}(13^{3/2} - 8)$$

Example 2: Circumference of a Circle

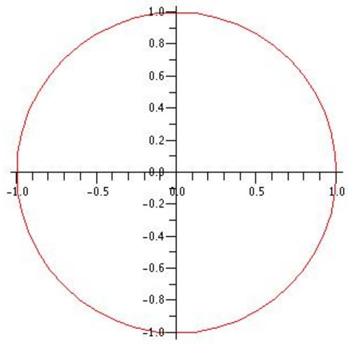


Figure: $x^2 + y^2 = r^2$

$$\begin{aligned} C &= 4\int_0^r \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= 4\int_0^r \sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx \\ &= 4\int_0^r \sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx \\ &= 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx, \text{ since } r>0 \end{aligned}$$

Chapter 6: Applications of Integration
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6.9 Exponential Models
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Example 2, Continued

To evaluate this integral, we shall use inverse trigonometric functions. In particular:

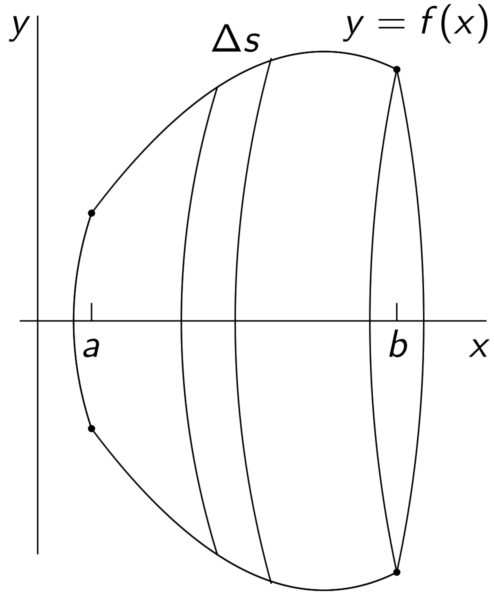
$$\int \frac{dx}{\sqrt{r^2-x^2}} = \sin^{-1}\left(\frac{x}{r}\right) + C,$$

as you may check. Then

$$\begin{aligned} C &= 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx = 4r\left[\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r \\ &= 4r(\sin^{-1}(1)-\sin^{-1}(0)) = 2\pi r, \text{ as you may check.} \end{aligned}$$

Chapter 6 Lecture Notes Review and Two New Sections MAT187H1F Lec0101 Burbulla

Surface Area



Partition the interval $[a, b]$ into n subintervals of equal length Δx. In each subinterval pick a value x_i^* and consider the strip of surface obtained by revolving the curve $y = f(x)$ on the interval $[x_{i-}, x_1]$ around the x-axis. It's approximate radius is $r_i = f(x_i^*)$, and its approximate width is Δs. So the approximate area of the strip is

$$\Delta A = 2\pi f(x_i^*)\Delta s.$$

Surface Area Formulas

Then the surface area of the solid of revolution obtained by revolving the curve $y = f(x)$ on the interval $[a, b]$ around the x-axis is given by

$$SA = \lim_{\Delta x \to 0} \sum \Delta A = \int_a^b 2\pi f(x)\, ds = \int_a^b 2\pi f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

If you revolve the curve around the y-axis, then the approximate radius of the strip becomes x_i^* and the formula for the surface area is

$$SA = \int_a^b 2\pi x\, ds = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Example 1: Surface Area of a Sphere is $4\pi r^2$

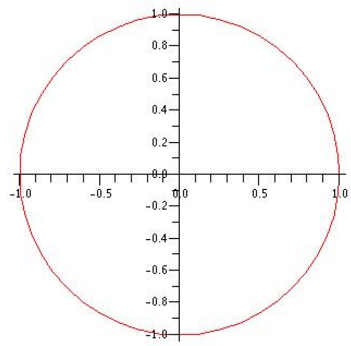


Figure: $x^2 + y^2 = r^2$

$$
\begin{aligned}
SA &= 2\int_0^r 2\pi y\, ds \\
&= 4\pi \int_0^r \sqrt{r^2 - x^2}\sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\, dx \\
&= 4\pi \int_0^r \sqrt{r^2 - x^2 + x^2}\, dx \\
&= 4\pi \int_0^r r\, dx, \text{ since } r > 0 \\
&= 4\pi r [x]_0^r = 4\pi r^2
\end{aligned}
$$

Chapter 6: Applications of Integration
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6.4 Volume by Shells
6.5 Length of Curves
6.6 Surface Area
6.7 Physical Applications
6.9 Exponential Models
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Example 2: Surface Area of a Cone is $\pi r\sqrt{r^2 + h^2}$

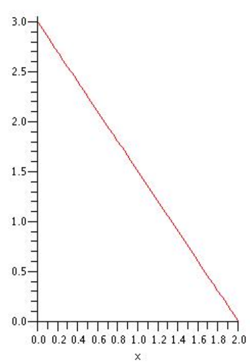


Figure: $\frac{x}{r} + \frac{y}{h} = 1$

$$
\begin{aligned}
SA &= \int_0^r 2\pi x\, ds \\
&= 2\pi \int_0^r x\sqrt{1 + \left(\frac{-h}{r}\right)^2}\, dx \\
&= 2\pi \int_0^r x\,\frac{\sqrt{r^2 + h^2}}{r}\, dx \\
&= 2\pi \frac{\sqrt{r^2 + h^2}}{r}\left[\frac{x^2}{2}\right]_0^r \\
&= \pi r\sqrt{r^2 + h^2}
\end{aligned}
$$

Chapter 6 Lecture Notes Review and Two New Sections MAT187H1F Lec0101 Burbulla

Work in Physics

The work required to move an object through a distance d by applying a constant force F is given by $W = F\,d$. Suppose an object is moved from $x = a$ to $x = b$ by applying a non-constant force $F(x)$. You can approximate the work on a small subinterval of length Δx, by picking a point x_i^* in the subinterval and taking the force to be constant, $F(x_i^*)$, over that subinterval. Then on each subinterval the work done is $\Delta W \simeq F(x_i^*)\Delta x$. The total work done is

$$W = \lim_{\Delta x \to 0} \sum \Delta W = \int_a^b F(x)\,dx.$$

Two Formulas from Physics

To use the formula $W = \int_a^b F(x)\,dx$ you have to know the force in terms of x. Here are two examples in which such a force is known:

1. Hooke's Law. If you stretch a mass on a spring the force required is proportional to the displacement: $F(x) = kx$, where $x = 0$ is the equilibrium position of the spring.
2. Newton's Law of Gravity. If m_1 and m_2 are separated by a distance x then the gravitational force of attraction between the two masses is given by

$$F = \frac{Gm_1 m_2}{x^2},$$

where G is the gravitational constant.

Example 1

1. The work done to stretch a spring from $x = a$ to $x = b$ is

$$W = \int_a^b kx\,dx = k\left[\frac{x^2}{2}\right]_a^b = \frac{k}{2}(b^2 - a^2).$$

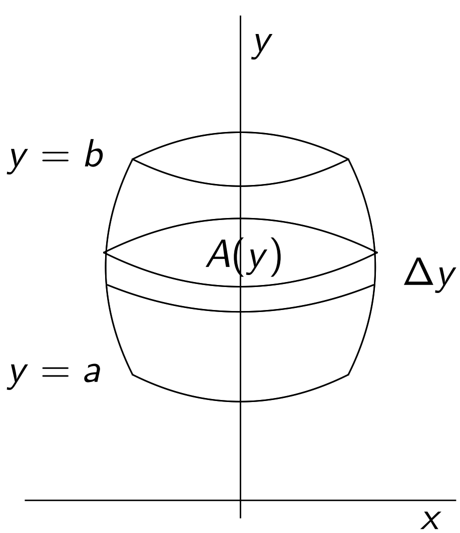
2. If R is the radius of the earth, and m_2 its mass, then the work required to put a satellite of mass m_1 into an orbit of height h above the earth's surface is

$$W = \int_R^{R+h} \frac{Gm_1m_2}{x^2}\,dx = \left[-\frac{Gm_1m_2}{x}\right]_R^{R+h} = \frac{Gm_1m_2h}{R(R+h)}.$$

Chapter 6: Applications of Integration

6.1 Velocity and Net Change
6.2 Regions Between Curves
6.3 Volume by Slicing
6.4 Volume by Shells
6.5 Length of Curves
6.6 Surface Area
6.7 Physical Applications
6.9 Exponential Models
6.10 Hyperbolic Functions: An Overview

Work Done in Filling a Tank



Suppose a fluid of density ρ is pumped from ground level $y = 0$ up into a tank, with base at $y = a$ and top at $y = b$. Suppose the cross-sectional area of the tank at height y is $A(y)$. Consider a thin shell of the fluid of thickness Δy. The volume of this shell is approximately $\Delta V = A(y)\,\Delta y$. Its mass is approximately $\rho\Delta V$ and the work required to pump this thin shell of liquid up to height y is approximately $\Delta W = \rho\Delta V\,g\,y$.

Chapter 6 Lecture Notes Review and Two New Sections MAT187H1F Lec0101 Burbulla

Formulas for Work Done in Filling or Emptying a Tank

1. Thus the work required to pump the tank full of fluid is

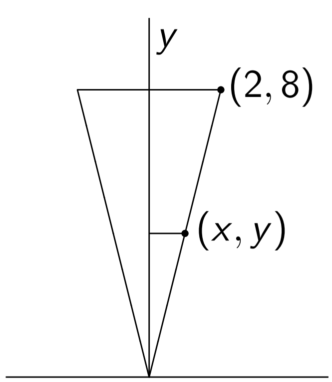
$$\begin{aligned} W = \lim_{\Delta y \to 0} \sum \Delta W &= \lim_{\Delta y \to 0} \sum \rho A(y) \Delta y \, g \, y \\ &= \lim_{\Delta y \to 0} \sum \rho g \, A(y) \, y \, \Delta y \\ &= \int_a^b \rho g \, A(y) \, y \, dy \end{aligned}$$

2. If the tank is emptied by pumping all the liquid up to a pipe or conduit above the tank at height $y = h$ then the work done in emptying the tank is $W = \displaystyle\int_a^b \rho g \, A(y) \, (h - y) \, dy$.

Example 2

Find the work done in pumping fluid of density ρ from ground level into a conical tank with radius at the top 2 m and height 8 m.

By similar triangles, $\dfrac{x}{y} = \dfrac{2}{8} \Leftrightarrow x = \dfrac{1}{4} y.$

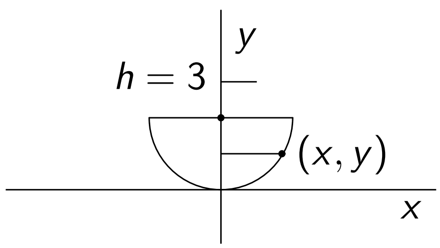


$$A(y) = \pi x^2 = \frac{\pi}{16} y^2$$

$$\begin{aligned} \Rightarrow W &= \int_0^8 \rho g \, \frac{\pi}{16} y^2 \, y \, dy = \frac{\pi \rho g}{16} \int_0^8 y^3 \, dy \\ &= \frac{\pi \rho g}{16} \left[\frac{y^4}{4} \right]_0^8 = 64 \pi \rho g \text{ (Joules)} \end{aligned}$$

Example 3

A hemispherical tank of radius 2 m is full of liquid with density ρ. How much work is required to empty the tank by pumping all the liquid up to a pipe 1 m above the top of the tank?



$x^2 + (y-2)^2 = 4$

$$
\begin{aligned}
A(y) &= \pi x^2 \\
&= \pi\left(4 - (y-2)^2\right) \\
&= \pi(4 - y^2 + 4y - 4) \\
&= \pi(4y - y^2)
\end{aligned}
$$

Chapter 6: Applications of Integration
6.1 Velocity and Net Change
6.2 Regions Between Curves
6.3 Volume by Slicing
6.4 Volume by Shells
6.5 Length of Curves
6.6 Surface Area
6.7 Physical Applications
6.9 Exponential Models
6.10 Hyperbolic Functions: An Overview

Example 3, Concluded

$$
\begin{aligned}
W &= \int_a^b \rho g\, A(y)\,(h-y)\,dy \\
&= \int_0^2 \rho g\, \pi(4y - y^2)\,(3-y)\,dy \\
&= \int_0^2 \rho g \pi(y^3 - 7y^2 + 12y)\,dy \\
&= \rho g \pi \left[\frac{y^4}{4} - \frac{7}{3}y^3 + 6y^2\right]_0^2 = \frac{28}{3}\rho g \pi
\end{aligned}
$$

Chapter 6 Lecture Notes Review and Two New Sections MAT187H1F Lec0101 Burbulla

The Work-Energy Relationship

Recall: $F = ma = m\frac{dv}{dt}$. Suppose an object of mass m is moved by a force F from $x = a$ at time $t = t_i$ to $x = b$ at time $t = t_f$. Then

$$\begin{aligned}
W &= \int_a^b F\,dx = \int_a^b m\frac{dv}{dt}\,dx \\
&= \int_{t_i}^{t_f} m\frac{dv}{dt}\frac{dx}{dt}\,dt = \int_{t_i}^{t_f} m\,v\,\frac{dv}{dt}\,dt \\
\text{(by substitution)} &= \int_{v_i}^{v_f} m\,v\,dv = \left[\frac{mv^2}{2}\right]_{v_i}^{v_f} \\
&= \frac{mv_f^2}{2} - \frac{mv_i^2}{2}, \text{ the change in kinetic energy}
\end{aligned}$$

Chapter 6: Applications of Integration
6.1 Velocity and Net Change
6.2 Regions Between Curves
6.3 Volume by Slicing
6.4 Volume by Shells
6.5 Length of Curves
6.6 Surface Area
6.7 Physical Applications
6.9 Exponential Models
6.10 Hyperbolic Functions: An Overview

Example 4

A mass of 10 kg is moving along the x-axis with speed 5 m/sec. At position $x = 0$ a force $F(x) = 3x^2$ N begins to push the object. What is the speed of the object when it reaches $x = 10$? Assume position along the x-axis is measured in meters.

Solution:

$$W = \int_0^{10} F(x)\,dx = \int_0^{10} 3x^2\,dx = \left[x^3\right]_0^{10} = 1000.$$

$$\frac{10v_f^2}{2} - \frac{10v_i^2}{2} = 1000 \Leftrightarrow 5v_f^2 = 1000 + 125 \Leftrightarrow v_f^2 = 225 \Leftrightarrow v_f = 15$$

Chapter 6 Lecture Notes Review and Two New Sections MAT187H1F Lec0101 Burbulla

Natural Growth Equation

Let x be the amount of some substance present at time t. The following differential equation

$$\frac{dx}{dt} = kx, k \neq 0$$

has many important applications. It can be interpreted as

$$\underbrace{\frac{dx}{dt}}_{the\ \ rate\ \ of\ \ change} \underbrace{= k}_{is\ \ proportional\ \ to\ the} \ \underbrace{x}_{amount\ \ present}.$$

In this case, the substance is said to be growing naturally.

Solution to the Natural Growth Equation

It is easy to verify that $x = x_0 e^{kt}$ is a solution to the natural growth equation, where the initial value of x is x_0 at $t = 0$:

$$x = x_0 e^{kt} \Rightarrow \frac{dx}{dt} = k\, x_0 e^{kt} = kx,$$

and

$$t = 0 \Rightarrow x = x_0 e^0 = x_0.$$

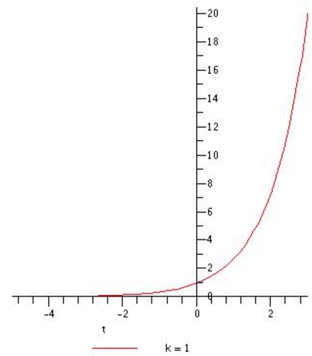
The solutions to the natural growth equation are called exponential models.

Exponential Growth: $k > 0, x_0 > 0$

In this case the value of x is always increasing. Two key features:

1. Doubling time:

$$\begin{aligned} x = 2x_0 \quad &\Leftrightarrow \quad x_0 e^{kt} = 2x_0 \\ &\Leftrightarrow \quad e^{kt} = 2 \\ &\Leftrightarrow \quad kt = \ln 2 \\ &\Leftrightarrow \quad t = \frac{\ln 2}{k} \end{aligned}$$



2. $\lim_{t \to \infty} x = \infty$

Examples: exponential population growth; compound interest.

Example 1

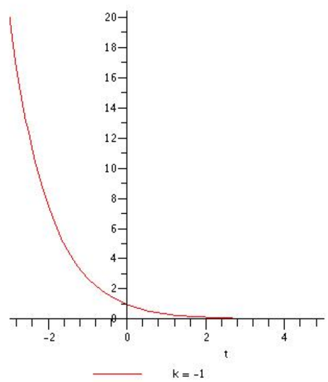
The population of a town is growing exponentially so that its population doubles every 10 years. The population of the town was 10,000 in 1995; what will its population be in the year 2020?
Solution: Let x be the population of the town at time t, where time is measured in years since 1995. So $t = 0$ corresponds to 1995, and $x_0 = 10\,000$. Use the doubling time to find k :

$$10 = \frac{\ln 2}{k} \Leftrightarrow k = \frac{\ln 2}{10} \simeq 0.0693.$$

So $x = x_0 e^{kt} = 10\,000 e^{\frac{\ln 2}{10} t} = 10\,000 \cdot 2^{\frac{t}{10}}$, or $x \simeq 10\,000 e^{0.0693t}$.
Now let $t = 25$: $x = 10\,000 \cdot 2^{\frac{25}{10}} = 10\,000 \cdot 2^{2.5} \simeq 56\,569$.

Exponential Decay: $k < 0, x_0 > 0$

In this case the value of x is always decreasing, and $\lim_{t\to\infty} x = 0$.



Half Life:

$$x = \frac{1}{2}x_0 \Leftrightarrow x_0 e^{kt} = \frac{1}{2}x_0$$
$$\Leftrightarrow e^{kt} = 2^{-1}$$
$$\Leftrightarrow kt = -\ln 2$$
$$\Leftrightarrow t = -\frac{\ln 2}{k}$$

Example: radioactive decay.

Example 2: Carbon-14 Dating

The half life of carbon-14 is 5,700 years. If a specimen of charcoal found in Stonehenge contains only 63% of its original carbon-14, how old is Stonehenge? **Solution:** Let x be the amount of carbon-14 present in the charcoal at time t, with t in years since the charcoal was created. Use the half life to find k :

$$5\,700 = -\frac{\ln 2}{k} \Leftrightarrow k = -\frac{\ln 2}{5\,700} \simeq -0.0001216.$$

Then $x = x_0 e^{kt} = x_0 e^{-0.0001216t}$. Let $x = 0.63x_0$, and solve for t :

$$0.63x_0 = x_0 e^{-0.0001216t} \Leftrightarrow 0.63 = e^{-0.0001216t} \Leftrightarrow \ln 0.63 = -0.0001216t$$

$\Leftrightarrow t \simeq 3\,800$. So the age of Stonehenge is approximately 3,800 yrs.

Two Different Trigonometries

As opposed to the six regular trigonometric functions,

$$\sin\theta, \cos\theta, \tan\theta, \csc\theta, \sec\theta \text{ and } \cot\theta,$$

which can be defined in terms of the circle $x^2 + y^2 = 1$, the six hyperbolic trigonometric functions are defined in terms of the hyperbola $x^2 - y^2 = 1$. However, this is not apparent from their definitions:

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}, \cosh\theta = \frac{e^\theta + e^{-\theta}}{2},$$

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta}, \operatorname{csch}\theta = \frac{1}{\sinh\theta}, \operatorname{sech}\theta = \frac{1}{\cosh\theta}, \coth\theta = \frac{\cosh\theta}{\sinh\theta}.$$

Chapter 6: Applications of Integration
6.1 Velocity and Net Change
6.2 Regions Between Curves
6.3 Volume by Slicing
6.4 Volume by Shells
6.5 Length of Curves
6.6 Surface Area
6.7 Physical Applications
6.9 Exponential Models
6.10 Hyperbolic Functions: An Overview

Basic Hyperbolic Trigonometric Identity

$$\begin{aligned}
\cosh^2\theta - \sinh^2\theta &= \left(\frac{e^\theta + e^{-\theta}}{2}\right)^2 - \left(\frac{e^\theta - e^{-\theta}}{2}\right)^2 \\
&= \frac{e^{2\theta} + 2 + e^{-2\theta}}{4} - \frac{e^{2\theta} - 2 + e^{-2\theta}}{4} \\
&= \frac{4}{4} = 1.
\end{aligned}$$

If $x = \cosh\theta$ and $y = \sinh\theta$, then the basic hyperbolic trig identity states that $x^2 - y^2 = 1$, which is the equation of an hyperbola.

Chapter 6 Lecture Notes Review and Two New Sections MAT187H1F Lec0101 Burbulla

Graphs of $\sinh\theta$ and $\cosh\theta$

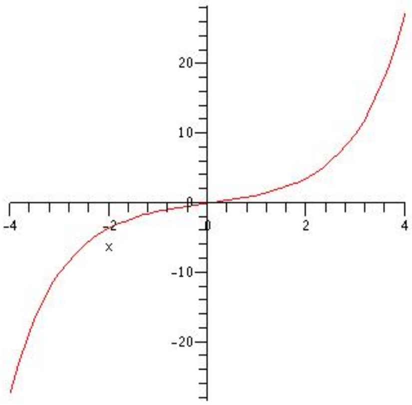


Figure: hyperbolic sine

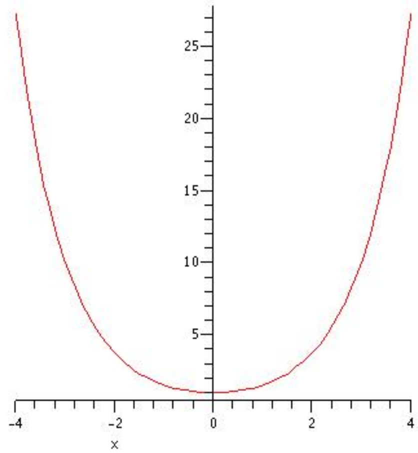


Figure: hyperbolic cosine

Hyperbolic Trigonometric Indentities

The following identities can all be proved by putting all hyperbolic trig functions in terms of e^θ and $e^{-\theta}$ and simplifying:

1. $1 - \tanh^2\theta = \operatorname{sech}^2\theta$
2. $\coth^2\theta - 1 = \operatorname{csch}^2\theta$
3. $\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$
4. $\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta$
5. $\sinh 2\theta = 2\sinh\theta\cosh\theta$
6. $\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$
7. $\cosh^2\theta = \dfrac{1}{2}(\cosh(2\theta)+1)$
8. $\sinh^2\theta = \dfrac{1}{2}(\cosh(2\theta)-1)$

Derivatives of $\sinh\theta$ and $\cosh\theta$

$$\begin{aligned}\sinh'\theta &= \frac{d}{d\theta}\left(\frac{e^\theta - e^{-\theta}}{2}\right)\\ &= \frac{e^\theta + e^{-\theta}}{2}\\ &= \cosh\theta\end{aligned}$$

Similarly,

$$\cosh'\theta = \frac{d}{d\theta}\left(\frac{e^\theta + e^{-\theta}}{2}\right) = \frac{e^\theta - e^{-\theta}}{2} = \sinh\theta$$

Derivatives of the Other Hyperbolic Trig Functions

You can also prove that

1. $\tanh'\theta = \operatorname{sech}^2\theta$
2. $\coth'\theta = -\operatorname{csch}^2\theta$
3. $\operatorname{sech}'\theta = -\operatorname{sech}\theta\tanh\theta$
4. $\operatorname{csch}'\theta = -\operatorname{csch}\theta\coth\theta$

Observe that all the formulas from the last few slides are very similar to the corresponding regular trig identities you are already familiar with, except for the odd plus or minus sign. In fact, you shouldn't memorize any of these formulas; you should simply be aware of what the hyperbolic trig functions are, just in case they show up in the homework, or in your other courses.

Inverse Hyperbolic Trig Functions

Since the hyperbolic trig functions are expressed in terms of exponentials, it should come as no surprise that the inverse hyperbolic trig functions can be expressed in terms of logarithms. For example,

$$\begin{aligned} y = \sinh^{-1} x \Leftrightarrow x = \sinh y \quad &\Leftrightarrow \quad x = \frac{e^y - e^{-y}}{2} \\ &\Leftrightarrow \quad 2x = e^y - \frac{1}{e^y} \\ &\Leftrightarrow \quad 2xe^y = e^{2y} - 1 \\ (\text{ using the quadratic formula }) \quad &\Leftrightarrow \quad e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} \end{aligned}$$

Formula for $y = \sinh^{-1} x$

So far, we have:

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}.$$

Since $e^y > 0$ for all y, we must take

$$e^y = x + \sqrt{x^2+1},$$

from which we get

$$\sinh^{-1} x = y = \ln(x + \sqrt{x^2+1}).$$

There are similar formulas for the other five inverse hyperbolic trig functions.

Derivatives of Inverse Hyperbolic Trig Functions

These can be calculated implicitly, or directly.

$$\begin{aligned} y = \sinh^{-1} x \Rightarrow \sinh y = x \quad &\Rightarrow \quad \cosh y \frac{dy}{dx} = 1 \\ &\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cosh y} \\ &\Rightarrow \quad \frac{d \sinh^{-1} x}{dx} = \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

Alternately,

$$\frac{d \sinh^{-1} x}{dx} = \frac{d}{dx} \ln(x + \sqrt{x^2+1}) = \frac{1 + x/\sqrt{x^2+1}}{x + \sqrt{x^2+1}} = \frac{1}{\sqrt{1+x^2}}.$$

Six New Integration Formulas

1. $\displaystyle\int \frac{du}{\sqrt{u^2+1}} = \sinh^{-1} u + C$
2. $\displaystyle\int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + C, \text{ if } |u| > 1$
3. $\displaystyle\int \frac{du}{1-u^2} = \tanh^{-1} u + C, \text{ if } |u| < 1$
4. $\displaystyle\int \frac{du}{1-u^2} = \coth^{-1} u + C, \text{ if } |u| > 1$
5. $\displaystyle\int \frac{du}{u\sqrt{1-u^2}} = -\text{sech}^{-1} |u| + C, \text{ if } |u| < 1$
6. $\displaystyle\int \frac{du}{u\sqrt{1+u^2}} = -\text{csch}^{-1} |u| + C$