MAT187H1F Lec0101 Burbulla

Chapter 7 Lecture Notes

Spring 2017

Introduction

Compared to differentiation, integration is much more difficult. Differentiation rules like the product rule, the quotient rule, the chain rule, or the Fundamental Theorem of Calculus permit almost any combination of functions to be easily differentiated in a mechanical, algorithmic way. Integration is a whole different story. In Chapter 7 we shall cover the basic techniques of integration, but even after we have learnt them – and practiced them – there will still be many functions that will be difficult – or impossible! – to integrate. Putting it another way, MAT187H is much more difficult than MAT186H.

Some Basic Integral Formulas

Every differentiation formula you know, produces an integration formula. For example,

$$\frac{de^x}{dx} = e^x \Rightarrow \int e^x \, dx = e^x + C.$$

And

$$\frac{dx^{n+1}}{dx} = (n+1)x^n \Rightarrow \int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Even this is tricky! Obviously, $n \neq -1$. What is

$$\int \frac{1}{x} \, dx?$$

Integral of $1/x$ with respect to x

We do know a function that has derivative $1/x$, namely $\ln x$. But $\ln x$ is only defined for $x > 0$. So

$$\int \frac{1}{x}\,dx = \ln x + C,$$

if $x > 0$. What about $x < 0$? Then $-x > 0$, and

$$\frac{d\ln(-x)}{dx} = \frac{1}{-x}(-1) = \frac{1}{x}.$$

So in general,

$$\int \frac{1}{x}\,dx = \ln|x| + C,$$

for $x \neq 0$. Memorize it!

Six Trigonometric Integrals

1. $\frac{d\sin x}{dx} = \cos x \Rightarrow \int \cos x\,dx = \sin x + C$
2. $\frac{d\cos x}{dx} = -\sin x \Rightarrow \int \sin x\,dx = -\cos x + C$
3. $\frac{d\tan x}{dx} = \sec^2 x \Rightarrow \int \sec^2 x\,dx = \tan x + C$
4. $\frac{d\sec x}{dx} = \sec x\tan x \Rightarrow \int \sec x\tan x\,dx = \sec x + C$
5. $\frac{d\cot x}{dx} = -\csc^2 x \Rightarrow \int \csc^2 x\,dx = -\cot x + C$
6. $\frac{d\csc x}{dx} = -\csc x\cot x \Rightarrow \int \csc x\cot x\,dx = -\csc x + C$

Integral of $\tan x$

What about the integral of $\tan x$? We can integrate it by using a simple substitution:

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx\\ &= \int \frac{1}{u}(-du) \text{ if } u = \cos x\\ &= -\int \frac{1}{u}\, du\\ &= -\ln|u| + C\\ &= -\ln|\cos x| + C\\ &= \ln|\sec x| + C\end{aligned}$$

Integral of $\sec x$

This is trickier:

$$\begin{aligned}\int \sec x\, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x}\, dx\\ &= \int \frac{1}{u}\, du \text{ if } u = \sec x + \tan x\\ &= \ln|u| + C\\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

Two Other Trig integrals

Similarly, you can prove

$$\int \cot x \, dx = -\ln|\csc x| + C$$

and

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C.$$

Both of these formulas have alternate forms:

$$\int \cot x \, dx = \ln|\sin x| + C$$

and

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C.$$

Three Integral Formulas with Inverse Trig Functions

1. $\dfrac{d \sin^{-1} x}{dx} = \dfrac{1}{\sqrt{1-x^2}} \Rightarrow \displaystyle\int \dfrac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$
2. $\dfrac{d \tan^{-1} x}{dx} = \dfrac{1}{1+x^2} \Rightarrow \displaystyle\int \dfrac{1}{1+x^2} \, dx = \tan^{-1} x + C$
3. $\dfrac{d \sec^{-1} x}{dx} = \dfrac{1}{|x|\sqrt{x^2-1}} \Rightarrow \displaystyle\int \dfrac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} |x| + C$

To have these formulas is the main reason we covered the derivatives of the inverse trig functions in Chapter 3.

Sixteen Integral Formulas

Every one of the formulas developed above should be memorized. The rest of Chapter 7 will be all about reducing more complicated integrals to one of the above basic sixteen formulas. Since you will often have to make a simple substitution before using one of the basic sixteen formulas, they are often written in terms of u rather than x. Here they are:

1. $\int e^u\, du = e^u + C$
2. $\displaystyle\int u^n\, du = \frac{u^{n+1}}{n+1} + C, n \neq -1$
3. $\displaystyle\int \frac{1}{u}\, du = \ln|u| + C$

4. $\displaystyle\int \cos u\, du = \sin u + C$
5. $\displaystyle\int \sin u\, du = -\cos u + C$
6. $\displaystyle\int \sec^2 u\, du = \tan u + C$
7. $\displaystyle\int \sec u \tan u\, du = \sec u + C$
8. $\displaystyle\int \csc^2 u\, du = -\cot u + C$
9. $\displaystyle\int \csc u \cot u\, du = -\csc u + C$

10. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$

11. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

12. $\displaystyle\int \cot u\,du = -\ln|\csc u| + C$

13. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$

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14. $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \sin^{-1}\frac{u}{a} + C$

15. $\displaystyle\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

16. $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

Memorize all of them!

Briggs et al list only 12 formulas on p 503 in Section 7.1. The ones missing are formulas 10-13 from above. These four are covered in Section 7.3

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Example 1

$$
\begin{aligned}
\int \frac{x}{x^4+9}\,dx &= \frac{1}{2}\int \frac{2x}{(x^2)^2+9}\,dx \\
&= \frac{1}{2}\int \frac{1}{u^2+3^2}\,du, \text{ if } u = x^2 \\
&= \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\frac{u}{3} + C, \text{ by Formula 15} \\
&= \frac{1}{6}\tan^{-1}\frac{x^2}{3} + C
\end{aligned}
$$

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Example 2: Getting Rid of Radicals

$$
\begin{aligned}
\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx &= \int \frac{\sqrt{u^6}}{1+\sqrt[3]{u^6}}\,6u^5\,du, \text{if } x = u^6 \Leftrightarrow u = x^{1/6} \\
&= \int \frac{u^3}{1+u^2}\,6u^5\,du = 6\int \frac{u^8}{1+u^2}\,du \\
&= 6\int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}\right) du \\
&= \frac{6u^7}{7} - \frac{6u^5}{5} + 2u^3 - 6u + 6\tan^{-1} u + C \\
&= \frac{6x^{7/6}}{7} - \frac{6x^{5/6}}{5} + 2x^{1/2} - 6x^{1/6} + 6\tan^{-1} x^{1/6} + C
\end{aligned}
$$

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Example 3

$$\begin{aligned}
\int \frac{1}{1+e^x}\,dx &= \int \frac{1+e^x-e^x}{1+e^x}\,dx = \int \left(1-\frac{e^x}{1+e^x}\right)\,dx \\
&= \int dx - \int \frac{e^x}{1+e^x}\,dx \\
&= x - \int \frac{1}{u}\,du, \text{ if } u = 1+e^x \\
&= x - \ln|u| + C, \text{ by Formula 3} \\
&= x - \ln|1+e^x| + C \\
&= x - \ln(1+e^x) + C, \text{ since } 1+e^x > 0, \text{ for all } x
\end{aligned}$$

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Example 3; Alternate Approach

$$\begin{aligned}
\int \frac{1}{1+e^x}\,dx &= \int \frac{e^{-x}}{(1+e^x)e^{-x}}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx \\
&= \int \frac{1}{u}(-du), \text{ if } u = 1+e^{-x} \\
&= -\ln|u| + C, \text{ by Formula 3} \\
&= -\ln|1+e^{-x}| + C \\
&= -\ln(1+e^{-x}) + C, \text{ since } 1+e^{-x} > 0, \text{ for all } x \\
\text{(optional)} \quad &= -\ln\left(\frac{e^x+1}{e^x}\right) + C \\
&= -\ln(1+e^x) + \ln(e^x) + C = x - \ln(1+e^x) + C
\end{aligned}$$

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Example 4

As the previous examples illustrate, integration can be quite tricky. Here's another example that requires some algebra you may not have thought of!

$$\begin{aligned}\int \frac{1}{1-\sin x}\,dx &= \int \frac{1+\sin x}{1-\sin^2 x}\,dx \\ &= \int \frac{1+\sin x}{\cos^2 x}\,dx \\ &= \int \sec^2 x\,dx + \int \tan x \sec x\,dx \\ &= \tan x + \sec x + C\end{aligned}$$

Example 5

In integrals involving quadratic expressions, you may have to complete the square before you make use of one of the basic 16 formulas. For example:

$$\begin{aligned}\int \frac{1}{x^2+6x+34}\,dx &= \int \frac{1}{(x+3)^2+25}\,dx \\ (\text{let } u = x+3) &= \int \frac{1}{u^2+5^2}\,du \\ (\text{use Formula 15 }) &= \frac{1}{5}\tan^{-1}\left(\frac{u}{5}\right) + C \\ &= \frac{1}{5}\tan^{-1}\left(\frac{x+3}{5}\right) + C\end{aligned}$$

Integration by Parts Formula

Recall the product rule:

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x).$$

If you integrate both sides of this equation with respect to x you obtain

$$\int (fg)'(x)\,dx = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx.$$

Simplifying the left side, and letting $u = f(x)$, $v = g(x)$, you obtain

$$uv = \int u\,dv + \int v\,du \Leftrightarrow \int u\,dv = uv - \int v\,du,$$

which is the integration by parts formula.

Using $\int u\,dv = uv - \int v\,du$

The hope is that if you can't do $\int u\,dv$, you will be able to do $\int v\,du$. However, to apply integration by parts, you always have to make a choice – what is u? what is dv? – and you have to be able to calculate both

$$\frac{du}{dx} \text{ and } v = \int dv.$$

You will find that some choices are better than others. You will also find that although integration by parts is a very useful method, not all integrals can be solved by this method. Indeed, there is no single method that can solve all integrals. That is why you have to learn all the different methods in Chapter 7.

Example 1

$$\begin{aligned}\int xe^x\,dx &= \int u\,dv,\ \text{ for } u = x, dv = e^x\,dx \\ &= uv - \int v\,du,\ \text{ by the parts formula} \\ &= xe^x - \int e^x\,dx,\ \text{ since } du = dx, v = \int e^x\,dx = e^x \\ &= xe^x - e^x + C\end{aligned}$$

Note that the constant of integration, C, was not added in until the last step. You could have used $v = e^x + K$, but then

$$uv - \int v\,du = x(e^x + K) - (e^x + Kx) + C = xe^x - e^x + C.$$

Example 2

For $\int x \sin x\,dx$, let $u = x, dv = \sin x\,dx$. Then

$$du = dx \text{ and } v = \int \sin x\,dx = -\cos x.$$

So

$$\begin{aligned}\int x\sin x\,dx = \int u\,dv &= uv - \int v\,du \\ &= x(-\cos x) - \int (-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + C\end{aligned}$$

Example 3

Sometimes you have to use parts more than once. For example, for $\int x^2 e^x\,dx$, let $u = x^2, dv = e^x\,dx$. Then $du = 2x\,dx$, $v = e^x$, and

$$\begin{aligned}\int x^2 e^x\,dx = \int u\,dv &= uv - \int v\,du \\ &= x^2e^x - \int 2xe^x\,dx \\ &= x^2e^x - 2\int xe^x\,dx \\ &= x^2e^x - 2\left[xe^x - e^x\right] + C, \text{ by Example 1} \\ &= x^2e^x - 2xe^x + 2e^x + C\end{aligned}$$

Example 4

What about $\int x^7 \ln x\,dx$? You might think you should let $u = x^7$ and use parts seven times. But then $dv = \ln x\,dx$, and $v = \int \ln x\,dx$, which we don't know. Instead, let $u = \ln x$, and $dv = x^7\,dx$. Then

$$\begin{aligned}\int x^7 \ln x\,dx = \int u\,dv &= uv - \int v\,du \\ &= \frac{1}{8}x^8\ln x - \int \frac{1}{8}x^8 \cdot \frac{1}{x}\,dx \\ &= \frac{1}{8}x^8\ln x - \frac{1}{8}\int x^7\,dx \\ &= \frac{1}{8}x^8\ln x - \frac{1}{64}x^8 + C\end{aligned}$$

Example 5

The method of the previous example can be used to find $\int \ln x \, dx$. Let

$$u = \ln x, \text{ and } dv = dx.$$

Then

$$\begin{aligned} \int \ln x \, dx = \int u \, dv &= uv - \int v \, du \\ &= x \ln x - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + C \end{aligned}$$

Example 6

Use the same approach for $\int x \tan^{-1} x \, dx$. That is, let $u = \tan^{-1} x$, let $dv = x \, dx$. Then

$$\begin{aligned} \int x \tan^{-1} x \, dx = \int u \, dv &= uv - \int v \, du \\ &= \frac{1}{2}x^2 \tan^{-1} x - \int \frac{1}{2}x^2 \cdot \frac{1}{1+x^2} \, dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2} \left(x - \tan^{-1} x\right) + C \end{aligned}$$

Some Generalizations

You can generalize the previous six examples as follows. Let n be a non-negative integer. To integrate integrals of the form

$$\int x^n e^x \, dx, \int x^n \sin x \, dx, \int x^n \cos x \, dx,$$

let $u = x^n$ and use parts n times. To integrate integrals of the form

$$\int x^n \ln x \, dx, \int x^n \tan^{-1} x \, dx, \int x^n \sin^{-1} x \, dx,$$

let $dv = x^n dx$, and use parts once.

Integrating Definite Integrals by Parts

Consider the definite integral, based on Example 4 above:

$$\begin{aligned}
\int_1^e x^7 \ln x \, dx = \int_1^e u \, dv &= [uv]_1^e - \int_1^e v \, du \\
&= \left[\frac{1}{8}x^8 \ln x\right]_1^e - \int_1^e \frac{1}{8}x^8 \cdot \frac{1}{x} \, dx \\
&= \frac{e^8}{8} - \frac{1}{8}\int_1^e x^7 \, dx \\
&= \frac{e^8}{8} - \frac{1}{64}\left[x^8\right]_1^e = \frac{e^8}{8} - \frac{e^8 - 1}{64} \\
&= \frac{1 + 7\,e^8}{64}
\end{aligned}$$

Example 7: A Tricky Example

$$\begin{aligned}
\int e^x \sin x\,dx &= \int u\,dv, \text{ with } u = e^x, dv = \sin x\,dx \\
&= e^x(-\cos x) - \int e^x(-\cos x)\,dx \\
&= -e^x \cos x + \int e^x \cos x\,dx \\
&= -e^x \cos x + \int s\,dt, \text{ with } s = e^x, dt = \cos x\,dx \\
&= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x\,dx\right)
\end{aligned}$$

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Example 7, Continued

So we have:

$$\begin{aligned}
\int e^x \sin x\,dx &= -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx \\
\Leftrightarrow 2\int e^x \sin x\,dx &= -e^x \cos x + e^x \sin x + C \\
\Leftrightarrow \int e^x \sin x\,dx &= -\frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x + C
\end{aligned}$$

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Example 8; Another Tricky Example: $\int \sec^3 x \, dx$

$$
\begin{aligned}
\int \sec x \sec^2 x \, dx &= \int u \, dv, \text{ with } u = \sec x, dv = \sec^2 x \, dx \\
&= \sec x \tan x - \int \tan x \sec x \tan x \, dx \\
&= \sec x \tan x - \int \sec x \tan^2 x \, dx \\
&= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) \, dx \\
&= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx
\end{aligned}
$$

Example 8, Continued

So we have:

$$
\begin{aligned}
\int \sec^3 x \, dx &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \\
\Leftrightarrow 2 \int \sec^3 x \, dx &= \sec x \tan x + \int \sec x \, dx \\
\Leftrightarrow 2 \int \sec^3 x \, dx &= \sec x \tan x + \ln|\sec x + \tan x| + C \\
\Leftrightarrow \int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C
\end{aligned}
$$

Reduction Formulas

A reduction formula is an integral formula that gives an integral with a higher power – of some part of it – in terms of a very similar integral with a lower – or reduced – power of the same part. For example,

1. $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$
2. $\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1}x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$

are both reduction formulas. Almost all reduction formulas are proved by using integration by parts. Note: in reduction formulas, n is almost always a non-negative integer.

Example 9; Reduction Formula for $\int x^n e^x\,dx$

Let $u = x^n, dv = e^x\,dx$:

$$\int x^n e^x\,dx = \int u\,dv = x^n e^x - n\int x^{n-1}e^x\,dx$$

$$\begin{aligned}
\text{So } \int x^3 e^x\,dx &= x^3e^x - 3\int x^2e^x\,dx, \text{ since } n=3 \\
&= x^3e^x - 3\left[x^2e^x - 2\int xe^x\,dx\right], \text{ using } n=2 \\
&= x^3e^x - 3x^2e^x + 6\left[xe^x - \int e^x\,dx\right], \text{ using } n=1 \\
&= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C
\end{aligned}$$

Example 10; Reduction Formula for $\int \cos^n x\, dx$

$$\begin{aligned}
\int \cos x \cos^{n-1} x\, dx &= \int u\, dv, \text{ with } u = \cos^{n-1} x, dv = \cos x\, dx \\
&= \sin x \cos^{n-1} x - (n-1) \int \cos^{n-2} x(-\sin x) \sin x\, dx \\
&= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \sin^2 x\, dx \\
&= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \left(1 - \cos^2 x\right) dx \\
&= \sin x \cos^{n-1} x + (n-1) \int \left(\cos^{n-2} x - \cos^n x\right) dx
\end{aligned}$$

Example 10, Continued

So we have

$$\begin{aligned}
(n-1+1) \int \cos^n x\, dx &= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx \\
\Leftrightarrow n \int \cos^n x\, dx &= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x\, dx \\
\Leftrightarrow \int \cos^n x\, dx &= \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x\, dx
\end{aligned}$$

For Example, if $n = 2$:

$$\int \cos^2 x\, dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} x + C$$

Example 11

$$
\begin{aligned}
&\int \cos^6 x \, dx \\
=\; & \frac{1}{6}\sin x \cos^5 x + \frac{5}{6}\int \cos^4 x \, dx, \text{ with } n = 6 \\
=\; & \frac{1}{6}\sin x \cos^5 x + \frac{5}{6}\left[\frac{1}{4}\sin x \cos^3 x + \frac{3}{4}\int \cos^2 x \, dx\right], \text{ with } n = 4 \\
=\; & \frac{1}{6}\sin x \cos^5 x + \frac{5}{24}\sin x \cos^3 x + \frac{5}{8}\int \cos^2 x \, dx \\
=\; & \frac{1}{6}\sin x \cos^5 x + \frac{5}{24}\sin x \cos^3 x + \frac{5}{16}\sin x \cos x + \frac{5}{16}x + C
\end{aligned}
$$

A Note About Reduction Formulas

Section 7.3 will include many more reduction formulas. But don't memorize any of them; or any from this section.

Connections Between Sine and Cosine

Recall the following results about $\sin x$ and $\cos x$:

1. $\sin^2 x + \cos^2 x = 1$
2. $\sin'(x) = \cos x$
3. $\cos'(x) = -\sin x$

These results can be used to integrate integrals of the form

$$\int \sin^m x \cos^n x \, dx,$$

for m, n both non-negative integers.

How to Integrate Integrals of the form $\int \sin^m x \cos^n x \, dx$

1. If m is odd, let $u = \cos x$.
2. If n is odd, let $u = \sin x$.
3. If m and n are both even, use the double angle formulas

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \text{ and } \cos^2 x = \frac{1 + \cos(2x)}{2},$$

   or some other method.

4. Note: if both m and n are odd, let $u = \sin x$, or let $u = \cos x$.

Example 1

$$\begin{aligned}
\int \sin^4 x \cos^3 x \, dx &= \int \sin^4 x \cos^2 x \cos x \, dx \\
&= \int u^4 (1 - u^2) \, du, \text{ with } u = \sin x \\
&= \int \left(u^4 - u^6\right) \, du \\
&= \frac{1}{5} u^5 - \frac{1}{7} u^7 + C \\
&= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C
\end{aligned}$$

Example 2

$$\begin{aligned}
\int \sin^3 x \, dx &= \int \sin^2 x \sin x \, dx = \int \left(1 - \cos^2 x\right) \sin x \, dx \\
&= \int \left(1 - u^2\right) (-du), \text{ with } u = \cos x \\
&= \int \left(u^2 - 1\right) \, du \\
&= \frac{1}{3} u^3 - u + C \\
&= \frac{1}{3} \cos^3 x - \cos x + C
\end{aligned}$$

Example 3

$$\begin{aligned}
\int \sin^2 x \cos^5 x \, dx &= \int \sin^2 x \cos^4 x \cos x \, dx \\
&= \int \sin^2 x (\cos^2 x)^2 \cos x \, dx \\
&= \int u^2 (1 - u^2)^2 \, du, \text{ with } u = \sin x \\
&= \int u^2 (1 - 2u^2 + u^4) \, du \\
&= \int \left(u^2 - 2u^4 + u^6\right) \, du
\end{aligned}$$

Example 3, Continued

So

$$\begin{aligned}
\int \sin^2 x \cos^5 x \, dx &= \int \left(u^2 - 2u^4 + u^6\right) \, du \\
&= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C \\
&= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C
\end{aligned}$$

Example 4

$$\begin{aligned} \int \cos^2 x \, dx &= \int \frac{1}{2}(1+\cos(2x))\, dx \\ &= \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{2}\sin(2x) + C \\ &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C \end{aligned}$$

If you substitute $\sin(2x) = 2\sin x \cos x$, then this answer becomes

$$\int \cos^2 x \, dx = \frac{1}{2}x + \frac{1}{4}(2\sin x \cos x) + C = \frac{1}{2}x + \frac{1}{2}\sin x \cos x + C,$$

which is the result we got using integration by parts.

Example 5

$$\begin{aligned} \int \sin^2 x \cos^2 x \, dx &= \int \frac{1}{2}(1-\cos(2x))\cdot\frac{1}{2}(1+\cos(2x))\, dx \\ &= \frac{1}{4}\int (1-\cos^2(2x))\, dx \\ &= \frac{1}{4}\int \sin^2(2x)\, dx \\ &= \frac{1}{4}\int \frac{1}{2}(1-\cos(4x))\, dx \\ &= \frac{1}{8}\left(x - \frac{1}{4}\sin(4x)\right) + C \end{aligned}$$

Connections Between Tangent and Secant

Recall the following results about $\tan x$ and $\sec x$:

1. $\tan^2 x + 1 = \sec^2 x$
2. $\tan'(x) = \sec^2 x$
3. $\sec'(x) = \sec x \tan x$

These results can be used to integrate integrals of the form

$$\int \tan^m x \sec^n x \, dx,$$

for m, n both non-negative integers.

How to Integrate Integrals of the form $\int \tan^m x \sec^n x \, dx$

1. If n is even, let $u = \tan x$.
2. If m is odd, let $u = \sec x$.
3. If m is even and n is odd, use integration by parts, or some other method.
4. Note: if m is odd and n is even, let $u = \tan x$, or let $u = \sec x$.

Example 6

$$\begin{aligned}\int \tan^2 x \sec^4 x \, dx &= \int \tan^2 x \sec^2 x \sec^2 x \, dx \\ &= \int u^2(1+u^2)\, du, \text{ with } u = \tan x \\ &= \int (u^2 + u^4)\, du \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C\end{aligned}$$

Example 7

$$\begin{aligned}\int \tan^3 x \sec^3 x \, dx &= \int \tan^2 x \sec^2 x \tan x \sec x \, dx \\ &= \int (u^2-1)u^2\, du, \text{ with } u = \sec x \\ &= \int (u^4 - u^2)\, du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C\end{aligned}$$

Example 8

$$\begin{aligned}\int \tan x \sec^8 x \, dx &= \int \sec^7 x \tan x \sec x \, dx \\ &= \int u^7 \, du, \text{ with } u = \sec x \\ &= \frac{1}{8} u^8 + C \\ &= \frac{1}{8} \sec^8 x + C\end{aligned}$$

You could use the substitution, $u = \tan x$, but that'd be messier:

$$\int \tan x \sec^8 x \, dx = \int \tan x \left(\sec^2 x\right)^3 \sec^2 x \, dx = \int u(1+u^2)^3 \, du.$$

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Example 9

We have already seen how $\int \sec^3 x \, dx$ can be done by parts. Similarly,

$$\int \tan^2 x \sec^3 x \, dx$$

can also be done by parts, or by using a reduction formula. First:

$$\begin{aligned}\int \tan^2 x \sec^3 x \, dx &= \int (\sec^2 x - 1) \sec^3 x \, dx \\ &= \int \sec^5 x \, dx - \int \sec^3 x \, dx\end{aligned}$$

Now look up the reduction formula for

$$\int \sec^n x \, dx.$$

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Example 10

$$\begin{aligned}
\int \tan^4 x\,dx &= \int \tan^2 x \tan^2 x\,dx \\
&= \int \tan^2 x(\sec^2 x - 1)\,dx \\
&= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx \\
&= \frac{1}{3}\tan^3 x - \int (\sec^2 x - 1)\,dx \\
&= \frac{1}{3}\tan^3 x - \tan x + x + C
\end{aligned}$$

Integrals of the Form $\int \cot^m x \csc^n x\,dx$

These are handled similarly to integrals of the form $\int \tan^m x \sec^n x\,dx$, using

1. $1 + \cot^2 x = \csc^2 x$
2. $\cot'(x) = -\csc^2 x$
3. $\csc'(x) = -\cot x \csc x$

In particular,

If n is even, let $u = \cot x$.

If m is odd, let $u = \csc x$.

If n is odd, and m is even, use integration by parts, or some other method.

Example 11: A Very Tricky Example

$$\int \frac{\sin x}{\cos x - \sin x}\,dx = \int \frac{\sin x(\cos x + \sin x)}{\cos^2 x - \sin^2 x}\,dx$$

$$= \int \frac{\sin x \cos x + \sin^2 x}{\cos^2 x - \sin^2 x}\,dx$$

$$= \frac{1}{2}\int \frac{\sin(2x) + 1 - \cos(2x)}{\cos(2x)}\,dx$$

$$= \frac{1}{2}\int \tan(2x)\,dx + \frac{1}{2}\int \sec(2x)\,dx - \frac{1}{2}\int dx$$

$$= \frac{1}{4}\ln|\sec(2x)| + \frac{1}{4}\ln|\sec(2x) + \tan(2x)| - \frac{1}{2}x + C$$

Example 11; An Alternate Approach

$$\int \frac{\sin x}{\cos x - \sin x}\,dx = \int \frac{\frac{1}{2}\sin x + \frac{1}{2}\cos x + \frac{1}{2}\sin x - \frac{1}{2}\cos x}{\cos x - \sin x}\,dx$$

$$= \frac{1}{2}\int \frac{\sin x + \cos x}{\cos x - \sin x}\,dx + \frac{1}{2}\int \frac{\sin x - \cos x}{\cos x - \sin x}\,dx$$

$$= -\frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{2}\int dx, \text{ with } u = \cos x - \sin x$$

$$= -\frac{1}{2}\ln|u| - \frac{1}{2}x + C$$

$$= -\frac{1}{2}\ln|\cos x - \sin x| - \frac{1}{2}x + C$$

This gives a different answer; but both are correct.

Trigonometric Substitutions; aka Inverse Trig Substitutions

Integrand Contains	Trig Substitution	Inverse Trig Substitution
$\sqrt{a^2 - x^2}$	try $x = a\sin\theta$	or $\theta = \sin^{-1}\left(\frac{x}{a}\right)$
$\sqrt{a^2 + x^2}$	try $x = a\tan\theta$	or $\theta = \tan^{-1}\left(\frac{x}{a}\right)$
$\sqrt{x^2 - a^2}$	try $x = a\sec\theta$	or $\theta = \sec^{-1}\left(\frac{x}{a}\right)$

Example 1; $x = \tan\theta$; $dx = \sec^2\theta\, d\theta$

$$\begin{aligned}
\int \frac{1}{(1+x^2)^{3/2}}\,dx &= \int \frac{1}{(1+\tan^2\theta)^{3/2}}\sec^2\theta\,d\theta \\
&= \int \frac{1}{(\sec^2\theta)^{3/2}}\sec^2\theta\,d\theta \\
&= \int \frac{\sec^2\theta}{\sec^3\theta}\,d\theta \\
&= \int \frac{1}{\sec\theta}\,d\theta \\
&= \int \cos\theta\,d\theta \\
&= \sin\theta + C
\end{aligned}$$

Example 1, Continued

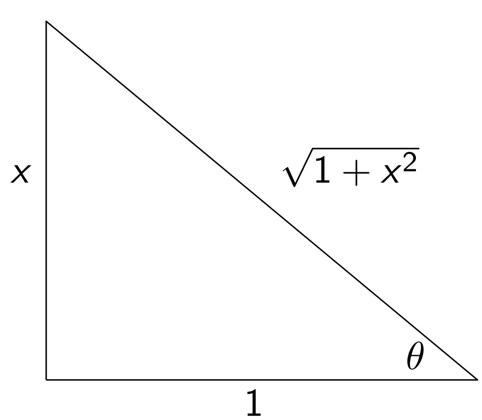
So far we have

$$\int \frac{1}{(1+x^2)^{3/2}}\,dx = \sin\theta + C.$$

At this point two comments are in order:

1. In simplifying $(\sec^2\theta)^{3/2} = \sec^3\theta$, we assumed $\sec\theta > 0$. Unless we have information otherwise, this will be our usual approach in simplifying trig integrals after making a trig substitution.
2. To finish the problem, we must put our answer back in terms of x. To do this, we need to find $\sin\theta$ in terms of x, given that $x = \tan\theta$. This is most commonly done with the help of a triangle.

Example 1, Concluded



- In the triangle to the left, $x = \tan\theta$.
- The length of the hypotenuse is $\sqrt{1+x^2}$.
- So

$$\sin\theta = \frac{x}{\sqrt{1+x^2}}.$$

- Thus

$$\int \frac{1}{(1+x^2)^{3/2}}\,dx = \frac{x}{\sqrt{1+x^2}} + C$$

Example 2: $x = 2\sin\theta$; $dx = 2\cos\theta \, d\theta$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}}\,dx &= \int \frac{8\sin^3\theta}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta \\
&= 8\int \frac{\sin^3\theta}{\cos\theta}\,\cos\theta\,d\theta \\
&= 8\int \sin^3\theta\,d\theta = 8\int \sin^2\theta\sin\theta\,d\theta \\
&= 8\int (1-u^2)\,(-du), \text{ if } u = \cos\theta \\
&= -8u + \frac{8}{3}u^3 + C
\end{aligned}$$

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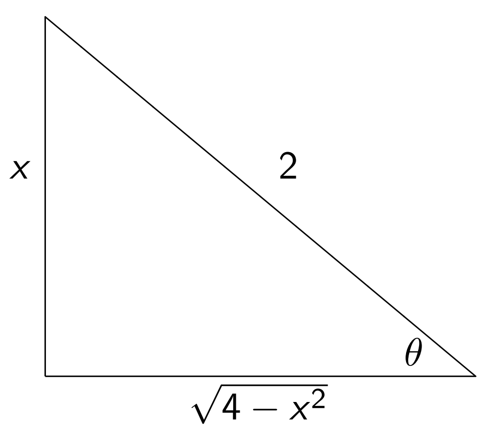
Example 2, Continued

$$\int \frac{x^3}{\sqrt{4-x^2}}\,dx = -8u + \frac{8}{3}u^3 + C = -8\cos\theta + \frac{8}{3}\cos^3\theta + C.$$



- In the triangle to the left,
$$x = 2\sin\theta \Leftrightarrow \sin\theta = \frac{x}{2}.$$
- So
$$\cos\theta = \frac{\sqrt{4-x^2}}{2}.$$

Example 2, Concluded

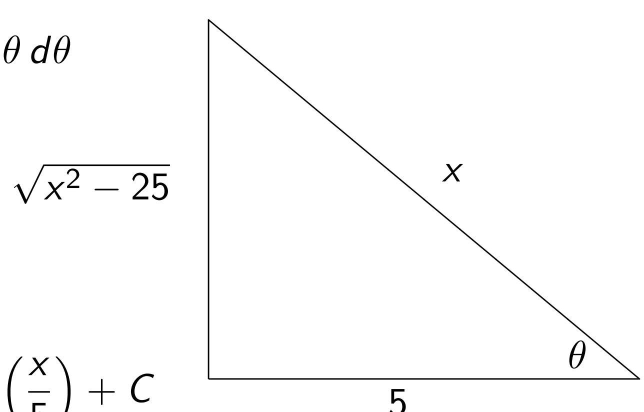
Thus

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}}\,dx &= -8\cos\theta + \frac{8}{3}\cos^3\theta + C \\
&= -8\left(\frac{\sqrt{4-x^2}}{2}\right) + \frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C \\
&= -4\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2} + C \\
\text{(optionally)} &= \sqrt{4-x^2}\left(-4+\frac{1}{3}(4-x^2)\right) + C \\
&= -\frac{1}{3}\sqrt{4-x^2}\,(x^2+8) + C
\end{aligned}$$

Example 3: $x = 5\sec\theta$; $dx = 5\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{x^2-25}}{x}\,dx = \int \frac{\sqrt{25\sec^2\theta - 25}}{5\sec\theta}\,5\sec\theta\tan\theta\,d\theta$$

$$\begin{aligned}
&= 5\int \sqrt{\sec^2\theta - 1}\tan\theta\,d\theta \\
&= 5\int \tan^2\theta\,d\theta \\
&= 5\int (\sec^2\theta - 1)\,d\theta \\
&= 5\tan\theta - 5\theta + C \\
&= \sqrt{x^2-25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C
\end{aligned}$$



Example 4; Example 1 Revisited, $x = \tan\theta$

In a definite integral, you should change the limits as you make the substitution. This will make it unnecessary to change back to x :

$$\begin{aligned}
\int_0^1 \frac{1}{(1+x^2)^{3/2}}\,dx &= \int_0^{\pi/4} \frac{1}{(1+\tan^2\theta)^{3/2}}\sec^2\theta\,d\theta \\
&= \int_0^{\pi/4} \cos\theta\,d\theta \\
&= [\sin\theta]_0^{\pi/4} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

It will not always be as easy to change the limits as in this example.

Example 5: An Integral That Can De Done Four Ways

Consider the integral

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx.$$

In keeping with the methods of this section, we could let $x = \sin\theta$:

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta \\
&= \int \sin^3\theta\,d\theta \\
&= -\cos\theta + \frac{1}{3}\cos^3\theta + C \\
&= -\sqrt{1-x^2} + \frac{1}{3}\left(1-x^2\right)^{3/2} + C
\end{aligned}$$

Example 5, Using Integration by Parts

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int x^2 \cdot \frac{x}{\sqrt{1-x^2}}\,dx \\
&= \int u\,dv, \text{ with } u = x^2, dv = \frac{x}{\sqrt{1-x^2}}\,dx \\
&= x^2\left(-\sqrt{1-x^2}\right) - \int \left(-\sqrt{1-x^2}\right)\cdot 2x\,dx \\
&= -x^2\sqrt{1-x^2} + \int \sqrt{1-x^2}\cdot 2x\,dx \\
&= -x^2\sqrt{1-x^2} - \frac{2}{3}\left(1-x^2\right)^{3/2} + C
\end{aligned}$$

Example 5, Using a Simple Substitution

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= -\frac{1}{2}\int x^2 \cdot \frac{-2x}{\sqrt{1-x^2}}\,dx \\
&= -\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\,du, \text{ with } u = 1-x^2 \\
&= -\frac{1}{2}\int \left(u^{-1/2} - u^{1/2}\right)\,du \\
&= -\sqrt{u} + \frac{1}{3}u^{3/2} + C \\
&= -\sqrt{1-x^2} + \frac{1}{3}\left(1-x^2\right)^{3/2} + C
\end{aligned}$$

Example 5, Using Another Substitution

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= -\int x^2 \cdot \frac{-x}{\sqrt{1-x^2}}\,dx \\
&= -\int \frac{1-u^2}{u} \cdot u\,du, \text{ with } u^2 = 1-x^2 \\
&= \int (u^2-1)\,du \\
&= \frac{1}{3}u^3 - u + C \\
&= \frac{1}{3}(1-x^2)^{3/2} - \sqrt{1-x^2} + C
\end{aligned}$$

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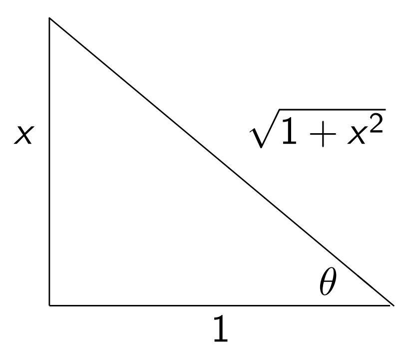
Example 6: $x = \tan\theta;\ dx = \sec^2\theta\, d\theta$



$$\begin{aligned}
&\int \frac{1}{(x^2+1)^2}\,dx \\
=\ &\int \frac{1}{(1+\tan^2\theta)^2} \cdot \sec^2\theta\,d\theta \\
=\ &\int \cos^2\theta\,d\theta = \frac{1}{2}\int (1+\cos(2\theta))\ d\theta \\
=\ &\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C = \frac{1}{2}\theta + \frac{2}{4}\sin\theta\cos\theta + C \\
=\ &\frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{\sqrt{x^2+1}} \cdot \frac{1}{\sqrt{x^2+1}} + C
\end{aligned}$$

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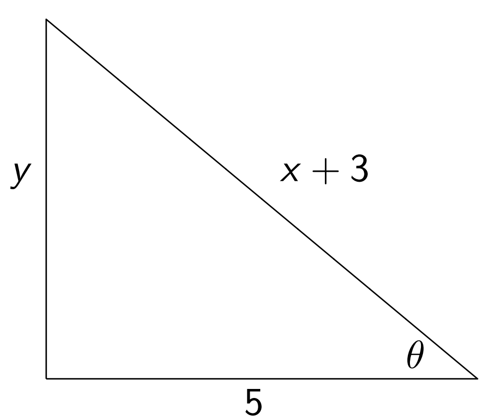
Example 7

In integrals involving quadratic expressions, you may have to complete the square before you make a trig substitution.

$$\begin{aligned}
\int \frac{1}{x^2+6x-16}\,dx &= \int \frac{1}{(x+3)^2-25}\,dx \\
(\text{let } x+3 = 5\sec\theta) &= \int \frac{1}{25\sec^2\theta - 25}\cdot 5\sec\theta\tan\theta\,d\theta \\
&= \frac{1}{5}\int \frac{\sec\theta\tan\theta}{\tan^2\theta}\,d\theta \\
&= \frac{1}{5}\int \csc\theta\,d\theta \\
&= -\frac{1}{5}\ln|\csc\theta+\cot\theta| + C
\end{aligned}$$

Example 7, Continued



- In the triangle to the left, $x+3 = 5\sec\theta$.
- The length of the third side is $y = \sqrt{x^2+6x-16}$.
- So
$$\csc\theta = \frac{x+3}{\sqrt{x^2+6x-16}}.$$
- And
$$\cot\theta = \frac{5}{\sqrt{x^2+6x-16}}.$$

Example 7, Concluded

$$\begin{aligned}
\int \frac{1}{x^2+6x-16}\,dx &= -\frac{1}{5}\ln|\csc\theta+\cot\theta| + C, \text{ so far} \\
&= -\frac{1}{5}\ln\left|\frac{x+3}{\sqrt{x^2+6x-16}} + \frac{5}{\sqrt{x^2+6x-16}}\right| + C \\
&= \frac{1}{5}\ln\left|\frac{\sqrt{x^2+6x-16}}{x+8}\right| + C \\
\text{(optionally)} &= \frac{1}{5}\ln\left|\sqrt{\frac{x-2}{x+8}}\right| + C = \frac{1}{10}\ln\left|\frac{x-2}{x+8}\right| + C
\end{aligned}$$

Example 8

$$\begin{aligned}
\int \frac{1}{x^2+2x+2}\,dx &= \int \frac{1}{(x+1)^2+1}\,dx, \text{ let } x+1=\tan\theta \\
&= \int \frac{1}{\tan^2+1}\cdot\sec^2\theta\,d\theta \\
&= \int d\theta, \text{ since } \tan^2\theta+1=\sec^2\theta \\
&= \theta + C \\
&= \tan^{-1}(x+1) + C
\end{aligned}$$

Example 9

$$\begin{aligned}\int \frac{1}{\sqrt{9+16x-4x^2}}\,dx &= \int \frac{1}{\sqrt{-4\left(x^2-4x-\frac{9}{4}\right)}}\,dx \\ &= \int \frac{1}{\sqrt{-4\left(x^2-4x+4-\frac{25}{4}\right)}}\,dx \\ &= \int \frac{1}{\sqrt{-4(x-2)^2+25}}\,dx \\ &= \int \frac{1}{\sqrt{5^2-(2x-4)^2}}\,dx\end{aligned}$$

So let $2x-4=5\sin\theta \Leftrightarrow x=\frac{5}{2}\sin\theta+2$.

Example 9, Concluded

We have

$$2x-4=5\sin\theta \Leftrightarrow x=\frac{5}{2}\sin\theta+2.$$

Then

$$\begin{aligned}\int \frac{1}{\sqrt{5^2-(2x-4)^2}}\,dx &= \int \frac{1}{\sqrt{25-25\sin^2\theta}}\cdot\frac{5}{2}\cos\theta\,d\theta \\ &= \frac{1}{2}\int d\theta \\ &= \frac{1}{2}\theta + C \\ &= \frac{1}{2}\sin^{-1}\left(\frac{2x-4}{5}\right)+C\end{aligned}$$

Example 10

$$\begin{aligned}
\int \frac{x-3}{(x^2+4x+13)^{3/2}}\,dx &= \int \frac{x-3}{((x+2)^2+3^2)^{3/2}}\,dx \\
(\text{ let } x+2 = 3\tan\theta) &= \int \frac{3\tan\theta - 2 - 3}{(9\tan^2\theta+9)^{3/2}} \cdot 3\sec^2\theta\,d\theta \\
&= \frac{1}{9}\int \frac{3\tan\theta - 5}{\sec\theta}\,d\theta \\
&= \frac{1}{3}\int \sin\theta\,d\theta - \frac{5}{9}\int \cos\theta\,d\theta \\
&= -\frac{1}{3}\cos\theta - \frac{5}{9}\sin\theta + C
\end{aligned}$$

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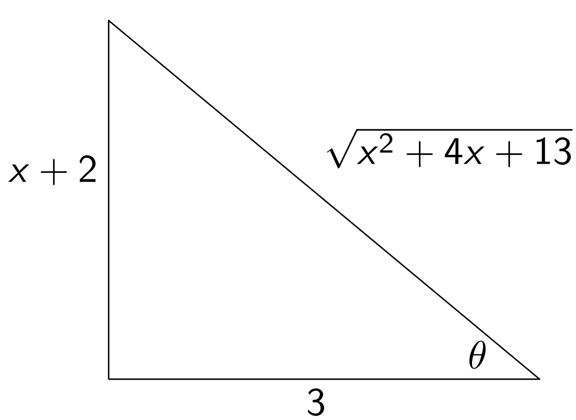
Example 10, Continued



- $$\cos\theta = \frac{3}{\sqrt{x^2+4x+13}}$$
- $$\sin\theta = \frac{x+2}{\sqrt{x^2+4x+13}}$$

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Example 10, Concluded

So $\displaystyle\int \frac{x-3}{(x^2+4x+13)^{3/2}}\,dx$

$$= -\frac{1}{3}\cos\theta - \frac{5}{9}\sin\theta + C$$

$$= -\frac{1}{3}\cdot\frac{3}{\sqrt{x^2+4x+13}} - \frac{5}{9}\cdot\frac{x+2}{\sqrt{x^2+4x+13}} + C$$

$$= -\frac{1}{9}\cdot\frac{5x+19}{\sqrt{x^2+4x+13}} + C$$

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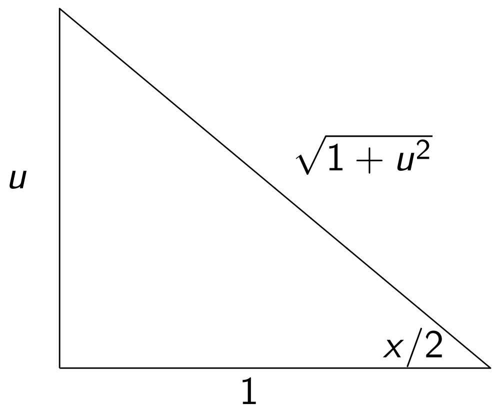
The Substitution $u = \tan(x/2)$

This is sometimes referred to as the world's sneakiest substitution. Check that:



$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2u}{1+u^2}$$

$$\cos x = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \frac{1-u^2}{1+u^2}$$

$$x = 2\tan^{-1} u \Rightarrow dx = \frac{2du}{1+u^2}$$

It can be used to transform integrands which are rational functions of $\sin x$ and $\cos x$.

Example 11

$$\begin{aligned}
&\int \frac{dx}{1-\sin x+\cos x} \\
&= \int \frac{\frac{2du}{1+u^2}}{1-\left(\frac{2u}{1+u^2}\right)+\left(\frac{1-u^2}{1+u^2}\right)}, \text{ with } u=\tan(x/2) \\
&= \int \frac{du}{1-u} = -\ln|1-u|+C \\
&= -\ln|1-\tan(x/2)|+C
\end{aligned}$$

Example 12: Alternate Approach to Example 11, Sec. 7.3

$$\begin{aligned}
\int \frac{\sin x}{\cos x-\sin x}\,dx &= \int \frac{\left(\frac{2u}{1+u^2}\right)}{\left(\frac{1-u^2}{1+u^2}\right)-\left(\frac{2u}{1+u^2}\right)}\frac{2du}{1+u^2} \\
&= -\int \frac{4u}{(u^2+2u-1)(1+u^2)}\,du \\
\text{(partial fractions)} &= \frac{1}{2}\ln(u^2+1)-\frac{1}{2}\ln|u^2+2u-1|-\tan^{-1}u+C \\
\left(u=\tan\frac{x}{2}\right) &= \ln\left|\sec\frac{x}{2}\right|-\frac{1}{2}\ln\left|\tan^2\frac{x}{2}+2\tan\frac{x}{2}-1\right|-\frac{1}{2}x+C
\end{aligned}$$

Rational Functions

This section is about how to integrate a rational function.

- If $p(x)$ and $q(x)$ are polynomials, then
$$\frac{p(x)}{q(x)}$$
is called a rational function.
- If the degree of $p(x)$ is greater than or equal to the degree of $q(x)$, then long division will result in a polynomial plus a rational expression in which the highest degree is in the denominator.
- The method of partial fractions applies to rational functions in which the highest degree is in the denominator.

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Example 1

$$\begin{aligned}\int \frac{x^2+6x-5}{x+7}\,dx &= \int \left(x-1+\frac{2}{x+7}\right)dx, \text{ by long division}\\ &= \frac{1}{2}x^2 - x + 2\ln|x+7| + C\end{aligned}$$

For all remaining examples in this section, the rational function to be integrated will have the highest power in the denominator, so that no long division will be necessary. But the rational functions to be integrated will be much more complicated than this example, so that some new techniques will have to be developed.

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Example 2; The Idea Behind Method of Partial Fractions

Since

$$\frac{2}{x+1} - \frac{3}{(x+1)^2} + \frac{x-1}{x^2+1} = \frac{3x^3+x-2}{(x+1)^2(x^2+1)},$$

as you can check,

$$\int \frac{3x^3+x-2}{(x+1)^2(x^2+1)}\,dx = \int \frac{2}{x+1}\,dx - \int \frac{3}{(x+1)^2}\,dx + \int \frac{x-1}{x^2+1}\,dx.$$

To integrate the left side, we will integrate the right side term by term.

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Example 2: Commentary

$$\frac{2}{x+1} - \frac{3}{(x+1)^2} + \frac{x-1}{x^2+1}$$

are called the partial fractions of

$$\frac{3x^3+x-2}{(x+1)^2(x^2+1)}.$$

To integrate

$$\frac{3x^3+x-2}{(x+1)^2(x^2+1)}$$

we integrate the partial fractions, each of which is easier than the original expression.

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Example 2; Concluded

$$
\begin{aligned}
&\int \frac{3x^3+x-2}{(x+1)^2(x^2+1)}\,dx \\
=\;& \int \frac{2}{x+1}\,dx - \int \frac{3}{(x+1)^2}\,dx + \int \frac{x-1}{x^2+1}\,dx \\
=\;& \int \frac{2}{x+1}\,dx - \int \frac{3}{(x+1)^2}\,dx + \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx \\
=\;& 2\ln|x+1| + \frac{3}{x+1} + \frac{1}{2}\ln(x^2+1) - \tan^{-1}x + C
\end{aligned}
$$

This example also exhibits all possible terms in the answer for the integral of a rational function: rational expressions, logarithms, and inverse tangents.

How to Find Partial Fractions of $p(x)/q(x)$

First you must factor the denominator, $q(x)$.

1. For every linear factor $ax+b$ which is repeated m times, the partial fraction decomposition includes
$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m},$$
for constants $A_1, A_2, \ldots, A_m$, which need to be determined.
2. For every irreducible quadratic factor ax^2+bx+c which is repeated m times, the partial fraction decomposition includes
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m},$$
for constants $A_1, B_1, A_2, B_2, \ldots, A_m, B_m$, to be determined.

Example 2 Revisited: Finding The Partial Fractions

The factors of the denominator were given: $x+1$ is a linear factor, repeated twice; x^2+1 is an irreducible factor. Thus we need to find constants A, B, C and D such that

$$\begin{aligned}
&\frac{3x^3+x-2}{(x+1)^2(x^2+1)} \\
=\ & \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+1} \\
=\ & \frac{A(x+1)(x^2+1)+B(x^2+1)+(Cx+D)(x+1)^2}{(x+1)^2(x^2+1)} \\
=\ & \frac{(A+C)x^3+(A+B+2C+D)x^2+(A+C+2D)x+A+B+D}{(x+1)^2(x^2+1)}
\end{aligned}$$

An Application of Linear Algebra

Since the denominators of two equal fractions are the same, the numerators must be equal. Comparing coefficients, we obtain the system of 4 equations in 4 unknowns:

$$\begin{cases}
A + C = 3 \\
A + B + 2C + D = 0 \\
A + C + 2D = 1 \\
A + B + D = -2
\end{cases}$$

Solve this system any way you like! By subtracting the third from the first equation, you obtain $-2D = 2 \Leftrightarrow D = -1$. By subtracting the fourth from the second equation, you obtain $2C = 2 \Leftrightarrow C = 1$. Then $A = 2$, and $B = -3$ quickly follow.

Example 2; the Last Word

We have $A = 2, B = -3, C = 1, D = -1$, so that

$$\begin{aligned} \frac{3x^3 + x - 2}{(x+1)^2(x^2+1)} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} \\ &= \frac{2}{x+1} - \frac{3}{(x+1)^2} + \frac{x-1}{x^2+1}, \end{aligned}$$

which are precisely the partial fractions we started with, in Example 2. Other examples follow. Note: needless to say, the method of partial fractions involves lots and lots of algebra. That may be why it is not very popular!

Example 3: $\int \frac{2x+1}{x^2+3x+2}\,dx$

$x^2 + 3x + 2 = (x+2)(x+1)$. So let

$$\begin{aligned} \frac{2x+1}{x^2+3x+2} &= \frac{A}{x+2} + \frac{B}{x+1} = \frac{A(x+1)+B(x+2)}{(x+2)(x+1)} \\ \Leftrightarrow 2x+1 &= (A+B)x + A + 2B \end{aligned}$$

$$\left\{ \begin{array}{rcrcl} A & + & B & = & 2 \\ A & + & 2B & = & 1 \end{array} \right. \Leftrightarrow A = 3 \text{ and } B = -1, \text{ and so}$$

$$\int \frac{2x+1}{x^2+3x+2}\,dx = \int \frac{3}{x+2}\,dx - \int \frac{1}{x+1}\,dx = 3\ln|x+2| - \ln|x+1| + C$$

Example 4: $\displaystyle\int \frac{6x^2+5x-3}{x^3-x}\,dx$

$x^3 - x = x(x^2-1) = x(x-1)(x+1)$. So let

$$\begin{aligned} \frac{6x^2+5x-3}{x^3-x} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \\ &= \frac{A(x^2-1)+Bx(x+1)+Cx(x-1)}{x(x-1)(x+1)} \\ \Leftrightarrow 6x^2+5x-3 &= (A+B+C)x^2 + (B-C)x - A \end{aligned}$$

$$\left\{ \begin{array}{rcrcrcr} A & + & B & + & C & = & 6 \\ & & B & - & C & = & 5 \\ -A & & & & & = & -3 \end{array} \right. \Leftrightarrow (A,B,C) = (3,4,-1)$$

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Example 4, Continued

Thus

$$\frac{6x^2+5x-3}{x^3-x} = \frac{3}{x} + \frac{4}{x-1} - \frac{1}{x+1};$$

and so

$$\begin{aligned} \int \frac{6x^2+5x-3}{x^3-x}\,dx &= \int \frac{3}{x}\,dx + \int \frac{4}{x-1}\,dx - \int \frac{1}{x+1}\,dx \\ &= 3\ln|x| + 4\ln|x-1| - \ln|x+1| + C \end{aligned}$$

As this example shows, the integration is usually the easiest part!

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Example 5: $\int \frac{2x^2 - 4x + 3}{x^3 - x^2}\, dx$

$x^3 - x^2 = x^2(x-1)$. So let

$$\begin{aligned} \frac{2x^2 - 4x + 3}{x^3 - x^2} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \\ &= \frac{Ax(x-1) + B(x-1) + Cx^2}{x^2(x-1)} \\ \Leftrightarrow 2x^2 - 4x + 3 &= (A+C)x^2 + (-A+B)x - B \end{aligned}$$

$$\left\{ \begin{array}{rcrcrcr} A & & & + & C & = & 2 \\ -A & + & B & & & = & -4 \\ & - & B & & & = & 3 \end{array} \right. \Leftrightarrow (A, B, C) = (1, -3, 1).$$

Example 5, Continued

Thus

$$\frac{2x^2 - 4x + 3}{x^3 - x^2} = \frac{1}{x} - \frac{3}{x^2} + \frac{1}{x-1};$$

and so

$$\begin{aligned} \int \frac{2x^2 - 4x + 3}{x^3 - x^2}\, dx &= \int \frac{1}{x}\, dx - \int \frac{3}{x^2}\, dx + \int \frac{1}{x-1}\, dx \\ &= \ln|x| + \frac{3}{x} + \ln|x-1| + C \end{aligned}$$

Example 6: $\int \frac{6x^2+29x+36}{x^3+6x^2+9x}\,dx$

$x^3+6x^2+9x = x(x^2+6x+9) = x(x+3)^2$. So let

$$\begin{aligned}\frac{6x^2+29x+36}{x^3+6x^2+9x} &= \frac{A}{x}+\frac{B}{x+3}+\frac{C}{(x+3)^2}\\ &= \frac{A(x+3)^2+Bx(x+3)+Cx}{x(x+3)^2}\\ \Leftrightarrow 6x^2+29x+36 &= (A+B)x^2+(6A+3B+C)x+9A\end{aligned}$$

By inspection, you can see that

$$A=4,\ B=2,\ \text{and}\ C=-1.$$

Example 6, Continued

Thus

$$\frac{6x^2+29x+36}{x^3+6x^2+9x} = \frac{4}{x}+\frac{2}{x+3}-\frac{1}{(x+3)^2};$$

and so

$$\begin{aligned}\int \frac{6x^2+29x+36}{x^3+6x^2+9x}\,dx &= \int \frac{4}{x}\,dx+\int\frac{2}{x+3}\,dx-\int\frac{1}{(x+3)^2}\,dx\\ &= 4\ln|x|+2\ln|x+3|+\frac{1}{x+3}+C\end{aligned}$$

Example 7: $\int \frac{6x^2 - x + 5}{x^3 + x}\,dx$

$x^3 + x = x(x^2+1)$. $x^2 + 1$ is an irreducible quadratic; so let

$$\begin{aligned} \frac{6x^2 - x + 5}{x^3 + x} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \\ &= \frac{A(x^2+1) + (Bx + C)x}{x(x^2+1)} \\ \Leftrightarrow 6x^2 - x + 5 &= (A+B)x^2 + Cx + A \end{aligned}$$

By inspection, you can see that

$$A = 5, B = 1, \text{ and } C = -1.$$

Example 7, Continued

Thus

$$\frac{6x^2 - x + 5}{x^3 + x} = \frac{5}{x} + \frac{x-1}{x^2+1};$$

and so

$$\begin{aligned} \int \frac{6x^2 - x + 5}{x^3 + x}\,dx &= \int \frac{5}{x}\,dx + \int \frac{x-1}{x^2+1}\,dx \\ &= \int \frac{5}{x}\,dx + \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx \\ &= 5\ln|x| + \frac{1}{2}\ln(x^2+1) - \tan^{-1} x + C \end{aligned}$$

Example 8: $\int \frac{x^3+6x^2+x}{x^4-1}\,dx$

$$x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1).$$

Then

$$\frac{x^3+6x^2+x}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$=\quad \frac{A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)}{(x-1)(x+1)(x^2+1)}$$

$$\Rightarrow \quad x^3+6x^2+x$$

$$= \quad (A+B+C)x^3+(A-B+D)x^2+(A+B-C)x+A-B-D$$

Example 8, Continued

We must solve the system of 4 equations in 4 unknowns:

$$\begin{cases} A + B + C = 1 \\ A - B + D = 6 \\ A + B - C = 1 \\ A - B - D = 0 \end{cases} \Leftrightarrow (A,B,C,D)=(2,-1,0,3),$$

as you may check. Thus

$$\frac{x^3+6x^2+x}{x^4-1}=\frac{2}{x-1}-\frac{1}{x+1}+\frac{3}{x^2+1}.$$

Example 8, Conclusion

Finallly,

$$\begin{aligned}\int \frac{x^3+6x^2+x}{x^4-1}\,dx &= \int \frac{2}{x-1}\,dx - \int \frac{1}{x+1}\,dx + \int \frac{3}{x^2+1}\,dx \\ &= 2\ln|x-1| - \ln|x+1| + 3\tan^{-1} x + C\end{aligned}$$

Example 9: $\int \frac{2x^3+7x^2-x+13}{x^4+5x^2+4}\,dx$

$x^4+5x^2+4=(x^2+4)(x^2+1)$, which are both irreducible quadratic factors. So let

$$\begin{aligned}\frac{2x^3+7x^2-x+13}{x^4+5x^2+4} &= \frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+1} \\ &= \frac{(Ax+B)(x^2+1)+(Cx+D)(x^2+4)}{(x^2+4)(x^2+1)}\end{aligned}$$

$$\Leftrightarrow 2x^3+7x^2-x+13=(A+C)x^3+(B+D)x^2+(A+4C)x+B+4D$$

Example 9, Continued

We must solve the system of 4 equations in 4 unknowns:

$$\begin{cases} A + C = 2 \\ B + D = 7 \\ A + 4C = -1 \\ B + 4D = 13 \end{cases} \Leftrightarrow (A, B, C, D) = (3, 5, -1, 2),$$

as you may check. Thus

$$\frac{2x^3 + 7x^2 - x + 13}{x^4 + 5x^2 + 4} = \frac{3x + 5}{x^2 + 4} + \frac{2 - x}{x^2 + 1}.$$

Example 9, Concluded

Finally,

$$\begin{aligned} & \int \frac{2x^3 + 7x^2 - x + 13}{x^4 + 5x^2 + 4}\, dx \\ = & \int \frac{3x + 5}{x^2 + 4}\, dx + \int \frac{2 - x}{x^2 + 1}\, dx \\ = & \int \frac{3x}{x^2 + 4}\, dx + \int \frac{5}{x^2 + 4}\, dx + \int \frac{2}{x^2 + 1}\, dx - \int \frac{x}{x^2 + 1}\, dx \\ = & \frac{3}{2}\ln(x^2 + 4) + \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + 2\tan^{-1} x - \frac{1}{2}\ln(x^2 + 1) + C \end{aligned}$$

Example 10: $\int \frac{3x^4 + x^3 + 4x^2 + 1}{x(x^2+1)^2}\,dx$

Since the denominator has already been factored, let

$$\begin{aligned}\frac{3x^4 + x^3 + 4x^2 + 1}{x(x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\\ &= \frac{A(x^2+1)^2 + (Bx+C)(x^3+x) + Dx^2 + Ex}{x(x^2+1)^2}\end{aligned}$$

$$\Leftrightarrow 3x^4+x^3+4x^2+1 = (A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A,$$

from which we see $A = 1, C = 1, B = 2, D = 0,$ and $E = -1$.

Example 10, Continued (using Example 6, from Sec 7.4)

Thus

$$\begin{aligned}&\int \frac{3x^4 + x^3 + 4x^2 + 1}{x(x^2+1)^2}\,dx\\ =\;& \int \frac{1}{x}\,dx + \int \frac{2x+1}{x^2+1}\,dx - \int \frac{1}{(x^2+1)^2}\,dx\\ =\;& \int \frac{1}{x}\,dx + \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx - \int \frac{1}{(x^2+1)^2}\,dx\\ =\;& \ln|x| + \ln(x^2+1) + \tan^{-1}x - \frac{1}{2}\tan^{-1}x - \frac{1}{2}\frac{x}{x^2+1} + C\\ =\;& \ln|x| + \ln(x^2+1) + \frac{1}{2}\tan^{-1}x - \frac{1}{2}\frac{x}{x^2+1} + C\end{aligned}$$

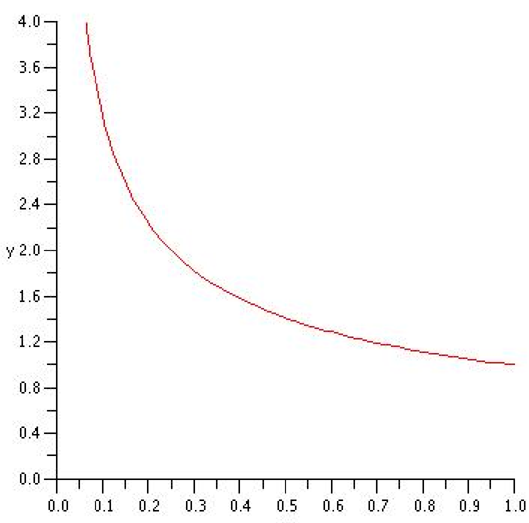
Definitions of Improper Integrals: $\int_a^b f(x)\,dx$

An improper integral is a definite integral in which one of two things occurs:

1. the integrand, $f(x)$, has an infinite discontinuity at some point in the interval $[a, b]$.
2. one, or both, of the limits of integration, a or b, are infinite.

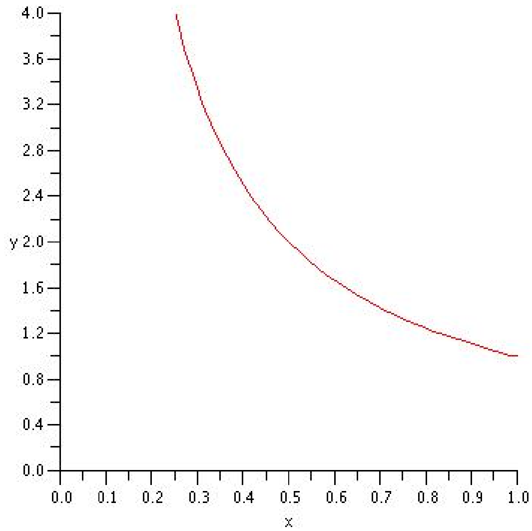
In either case, an improper integral is evaluated by calculating a limit of a proper definite integral. If the limit exists, the improper integral exists, or converges, or is finite. If the limit doesn't exist, then the improper integral doesn't exist, or diverges, or is infinite. Some examples follow.

Example 1: $\int_0^1 \frac{1}{\sqrt{x}}\,dx$; Vertical Asymptote at $x = 0$



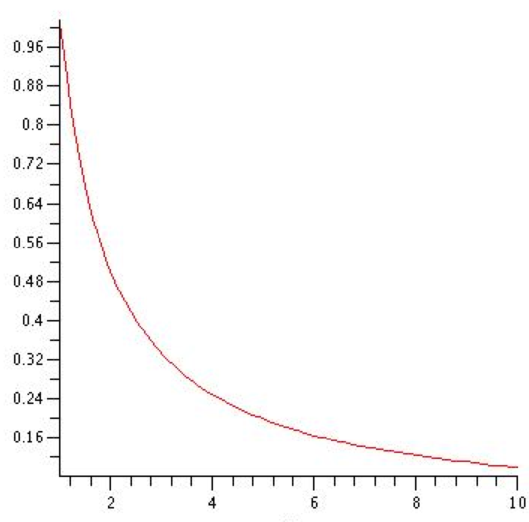
$$\begin{aligned}
&\int_0^1 \frac{1}{\sqrt{x}}\,dx \\
&= \lim_{a\to0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx \\
&= \lim_{a\to0^+} \left[2\sqrt{x}\right]_a^1 \\
&= \lim_{a\to0^+} \left(2 - 2\sqrt{a}\right) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

Example 2: $\int_0^1 \frac{1}{x}\,dx$; Vertical Asymptote at $x = 0$



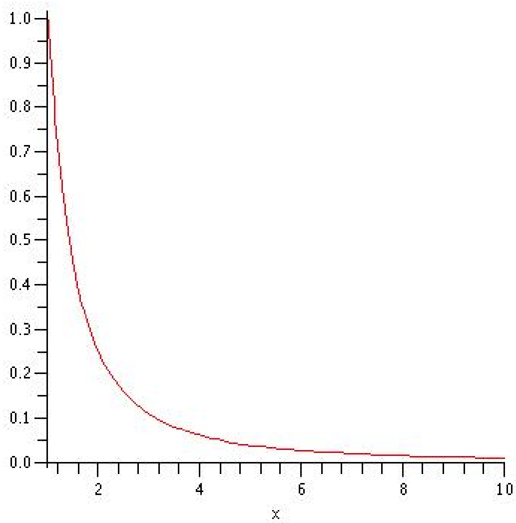
$$\begin{aligned}
&\int_0^1 \frac{1}{x}\,dx \\
= &\lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx \\
= &\lim_{a\to 0^+} [\ln x]_a^1 \\
= &\lim_{a\to 0^+} (\ln 1 - \ln a) \\
= &\ 0 - (-\infty) \\
= &\ \infty
\end{aligned}$$

Example 3: $\int_1^\infty \frac{1}{x}\,dx$; Infinite Limit of Integration



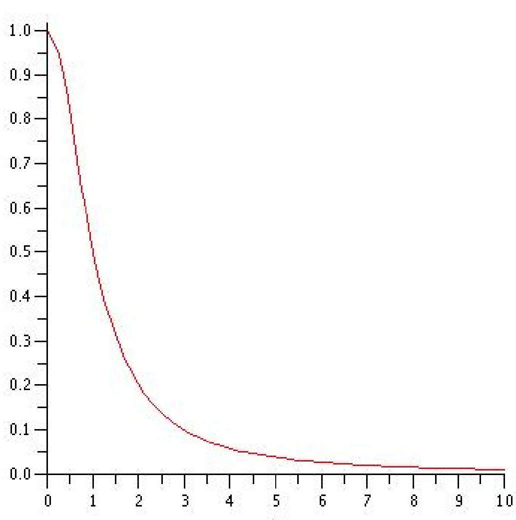
$$\begin{aligned}
&\int_1^\infty \frac{1}{x}\,dx \\
= &\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx \\
= &\lim_{b\to\infty} [\ln x]_1^b \\
= &\lim_{b\to\infty} (\ln b - \ln 1) \\
= &\ \infty - 0 \\
= &\ \infty
\end{aligned}$$

Example 4: $\int_1^\infty \frac{1}{x^2}\,dx$; Infinite Limit of Integration



$$\begin{aligned}
&\int_1^\infty \frac{1}{x^2}\,dx \\
=\;& \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx \\
=\;& \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b \\
=\;& \lim_{b\to\infty} \left(-\frac{1}{b}+\frac{1}{1}\right) \\
=\;& 0+1 \\
=\;& 1
\end{aligned}$$

Example 5: $\int_0^\infty \frac{1}{1+x^2}\,dx$; Infinite Limit of Integration



$$\begin{aligned}
&\int_0^\infty \frac{1}{1+x^2}\,dx \\
=\;& \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx \\
=\;& \lim_{b\to\infty} \left[\tan^{-1} x\right]_0^b \\
=\;& \lim_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) \\
=\;& \frac{\pi}{2} - 0 \\
=\;& \frac{\pi}{2}
\end{aligned}$$

Example 6: $\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$

This is the mirror image of the previous integral, that is, the reflection of the previous one in the y-axis. It is calculated as follows:

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx \\
&= \lim_{a\to-\infty} \left[\tan^{-1} x\right]_{a}^{0} \\
&= \lim_{a\to-\infty} \left(\tan^{-1} 0 - \tan^{-1} a\right) \\
&= 0 - \left(-\frac{\pi}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

Improper Integrals of the Type $\int_{-\infty}^{\infty} f(x)\,dx$

This type is evaluated by calculating two improper integrals:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx \\
&= \lim_{a\to-\infty} \int_{a}^{0} f(x)\,dx + \lim_{b\to\infty} \int_{0}^{b} f(x)\,dx
\end{aligned}$$

Both of the improper integrals

$$\int_{-\infty}^{0} f(x)\,dx \text{ and } \int_{0}^{\infty} f(x)\,dx$$

must exist independently of each other for $\int_{-\infty}^{\infty} f(x)\,dx$ to exist.

Example 7: $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx \\ &= \frac{\pi}{2} + \frac{\pi}{2}, \text{ by previous examples} \\ &= \pi\end{aligned}$$

Note: sometimes a change of variable will transform an improper integral to a proper integral:

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \int_0^{\pi/2} \frac{1}{1+\tan^2\theta}\cdot \sec^2\theta\,d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$$

Escape Velocity

Suppose an object of mass M_2 is fired with speed v off a planet with mass M_1 and radius R. What is the object's escape velocity? the speed needed to become free of the planet's gravity?

$$\begin{aligned}\frac{1}{2}M_2v^2 &= \int_R^{\infty} \frac{GM_1M_2}{r^2}\,dr \\ &= \left[-\frac{GM_1M_2}{r}\right]_R^{\infty} = \frac{GM_1M_2}{R} \\ \Rightarrow \frac{1}{2}v^2 &= \frac{GM_1}{R} \Rightarrow v = \sqrt{\frac{2GM_1}{R}}\end{aligned}$$

For the earth, this works out to be 11.2 km/sec. (This is also the speed with which an asteroid would hit the earth.)

Integral Tables

Inside the back cover of the textbook there is a short table of integrals that lists 106 integral formulas. Using the methods of Chapter 7 we could prove all of these 106 formulas. In practice, if you are a working scientist and you need an integral, you would probably consult a table of integrals, or some mathematical software like Maple or Mathematica. If you consult an integral table to solve an integral, you will often first have to do some manipulation, or make a substitution, before you can use the integral formula. An example follows. By the way, there are tables of integrals that are much longer than the 106 formulas listed in our book!

Example 1

Formula 66 inside the back cover of our textbook says

$$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{a^2}{2}\sin^{-1}\frac{u}{a} - \frac{u}{2}\sqrt{a^2-u^2} + C.$$

To apply this formula to

$$\int \frac{x^2}{\sqrt{25-16x^2}}\,dx,$$

you have to first make a substitution, namely $u = 4x$, so that $du = 4dx$. Then

$$\int \frac{x^2}{\sqrt{25-16x^2}}\,dx = \frac{1}{4}\int \frac{\left(\frac{1}{4}u\right)^2}{\sqrt{25-u^2}}\,du$$

So

$$\begin{aligned}
\int \frac{x^2}{\sqrt{25-16x^2}}\,dx &= \frac{1}{4}\int \frac{\left(\frac{1}{4}u\right)^2}{\sqrt{25-u^2}}\,du \\
&= \frac{1}{64}\int \frac{u^2}{\sqrt{5^2-u^2}}\,du \\
&= \frac{1}{64}\left(\frac{25}{2}\sin^{-1}\frac{u}{5} - \frac{u}{2}\sqrt{25-u^2}\right) + C \\
&= \frac{25}{128}\sin^{-1}\frac{4x}{5} - \frac{x}{32}\sqrt{25-16x^2} + C
\end{aligned}$$

Example 1 with Maple

Maple is a very powerful computer algebra system, which can basically do all calculus:

$$> \text{integrate}(x^2/\text{sqrt}(25-16*x^2), x);$$

$$-\frac{1}{32}x\ \text{sqrt}(25-16x^2) + \frac{25}{128}\ \text{arcsin}\left(\frac{4}{5}x\right)$$

Note: no constant of integration. Other commonly used computer algebra systems are Matlab and Mathematica. These packages use the same methods we have covered, plus many other substitutions that we have not explicitly stated.

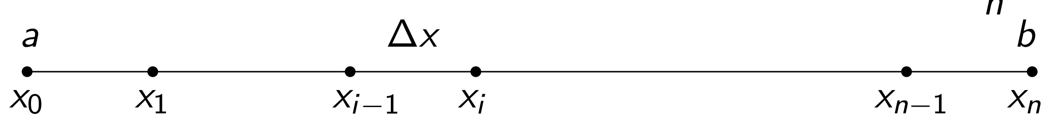
Approximating Definite Integrals

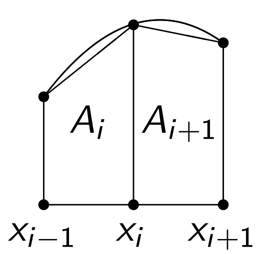
We know that if $F' = f$ on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$. However, there are at least two reasons why this approach may be impractical.

1. For one thing, it may be extremely difficult – it may be impossible – to find the antiderivative of $f(x)$. For example, it is known that none of these functions – e^{-x^2}, $\cos(\cos x)$, or $(1+x^2)^{2/3}$ – has an elementary antiderivative in terms of x.
2. Secondly, even if you know F, it may not be easy to evaluate $F(a)$ or $F(b)$. For example, $\int_1^5 \frac{1}{x}\,dx = \ln 5$; but what is $\ln 5$?

The Trapezoid Rule: To Approximate $\int_a^b f(x)\,dx$

Use a regular partition of $[a, b]$ into n subintervals; $\Delta x = \dfrac{b-a}{n}$:



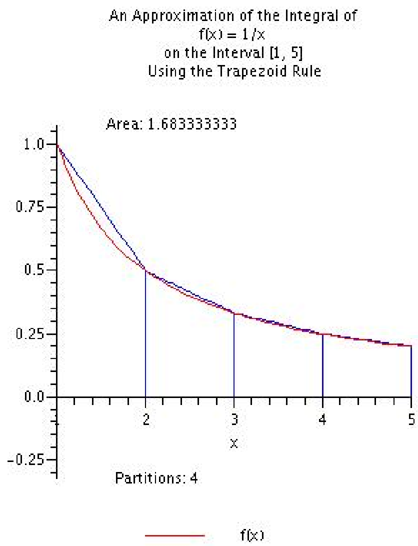


$$A_i = \frac{1}{2}(f(x_{i-1}) + f(x_i))\,\Delta x;\text{ so } A = \int_a^b f(x)\,dx$$

$$\simeq \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{2}(f(x_{i-1}) + f(x_i))\,\Delta x$$

$$= \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)) = T_n$$

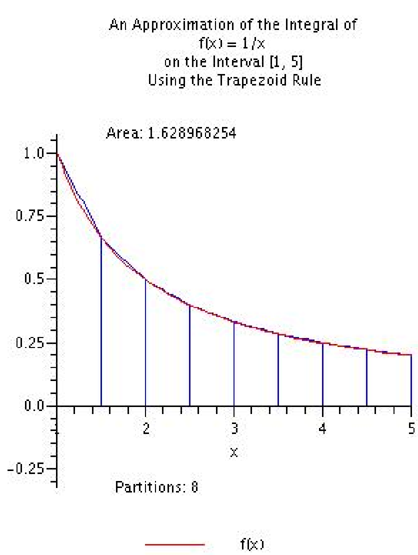
Example 1: Approximate $\int_1^5 \frac{1}{x}\, dx$ with $n = 4$; $y_i = f(x_i)$.



$$f(x) = \frac{1}{x};\ n = 4;\ \Delta x = \frac{5-1}{4} = 1$$

$$\begin{aligned} T_4 &= \frac{1}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4) \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{1}{5}\right) \\ &= \frac{101}{60} \simeq 1.683333\ldots \end{aligned}$$

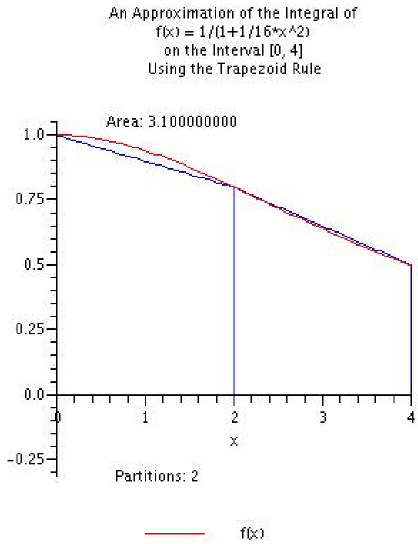
Example 2: Approximate $\int_1^5 \frac{1}{x}\, dx$ with $n = 8$



$$f(x) = \frac{1}{x};\ n = 8;\ \Delta x = \frac{5-1}{8} = 0.5$$

$$\begin{aligned} T_8 &= \frac{1}{4}\left(\frac{1}{1} + \frac{2}{1.5} + \frac{2}{2} + \frac{2}{2.5} + \right. \\ &\qquad \left. \frac{2}{3} + \frac{2}{3.5} + \frac{2}{4} + \frac{2}{4.5} + \frac{1}{5}\right) \\ &\simeq 1.628968254\ldots \end{aligned}$$

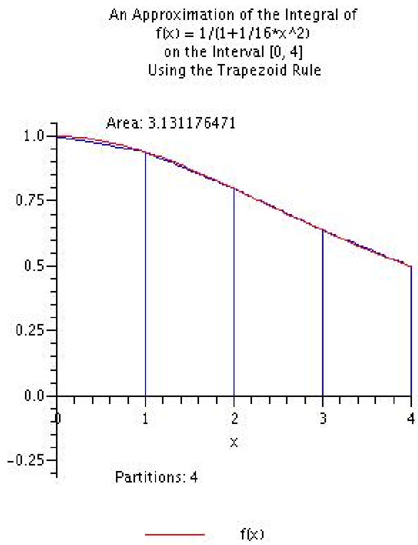
Example 3: Approximate $\int_0^4 \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx$ with $n=2$



$$f(x) = \frac{1}{1+\left(\frac{x}{4}\right)^2};\ n=2;\ \Delta x = \frac{4-0}{2} = 2$$

$$\begin{aligned} T_2 &= \frac{2}{2}\left(f(0)+2f(2)+f(4)\right) \\ &= 1 + \frac{2}{1+(0.5)^2} + \frac{1}{1+1^2} \\ &= 3.1 \end{aligned}$$

Example 4: Approximate $\int_0^4 \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx;\ n=4;\ y_i = f(x_i)$

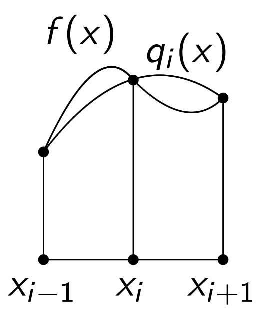


$$f(x) = \frac{1}{1+\left(\frac{x}{4}\right)^2};\ n=4;\ \Delta x = \frac{4-0}{4} = 1$$

$$\begin{aligned} T_4 &= \frac{1}{2}\left(y_0+2y_1+2y_2+2y_3+y_4\right) \\ &= \frac{1}{2}\left(1 + \frac{2}{1+(0.25)^2} + \frac{2}{1+(0.5)^2} \right. \\ &\qquad \left. + \frac{2}{1+(0.75)^2} + \frac{1}{1+1^2}\right) \\ &\simeq 3.131176471\ldots \end{aligned}$$

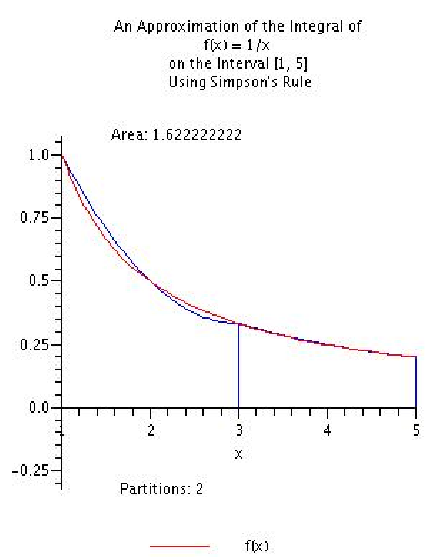
Simpson's Rule. Let $y_i = f(x_i)$. Let n be even.

Let (x_{i-1}, y_{i-1}), (x_i, y_i) and (x_{i+1}, y_{i+1}) be three consecutive data points; and $q_i(x)$ the quadratic function that passes through them. Let $\int_{x_{i-1}}^{x_{i+1}} f(x)dx \simeq \int_{x_{i-1}}^{x_{i+1}} q_i(x)\,dx = A_i = \frac{\Delta x}{3}(y_{i-1} + 4y_i + y_{i+1})$.



$$\begin{aligned}\int_a^b f(x)\,dx &\simeq \sum_{i=1}^{n/2} A_{2i-1} = A_1 + A_3 + \cdots + A_{n-1} \\ &= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots \\ &\qquad + \cdots + 2y_{n-2} + 4y_{n-1} + y_n) \\ &= S_n\end{aligned}$$

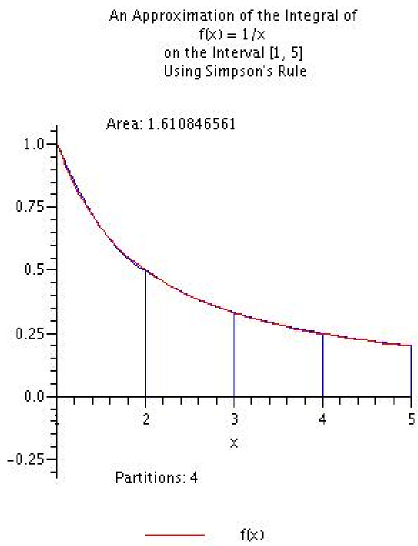
Example 5: Approximate $\int_1^5 \frac{1}{x}\,dx$ with $n = 4$; $y_i = f(x_i)$



$$f(x) = \frac{1}{x};\ n = 4;\ \Delta x = \frac{5-1}{4} = 1$$

$$\begin{aligned}S_4 &= \frac{1}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) \\ &= \frac{1}{3}\left(\frac{1}{1} + \frac{4}{2} + \frac{2}{3} + \frac{4}{4} + \frac{1}{5}\right) \\ &= \frac{73}{45} \simeq 1.6222222\ldots\end{aligned}$$

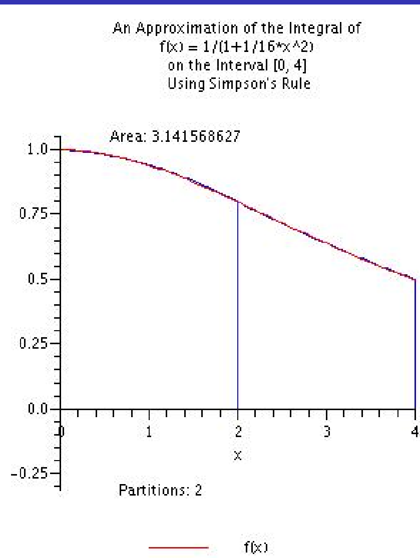
Example 6: Approximate $\int_1^5 \frac{1}{x}\,dx$ with $n = 8$



$$f(x) = \frac{1}{x};\ n = 8;\ \Delta x = \frac{5-1}{8} = 0.5$$

$$\begin{aligned} S_8 &= \frac{1}{6}\left(\frac{1}{1} + \frac{4}{1.5} + \frac{2}{2} + \frac{4}{2.5} + \frac{2}{3} + \frac{4}{3.5} + \frac{2}{4} + \frac{4}{4.5} + \frac{1}{5}\right) \\ &\simeq 1.610846561\ldots \end{aligned}$$

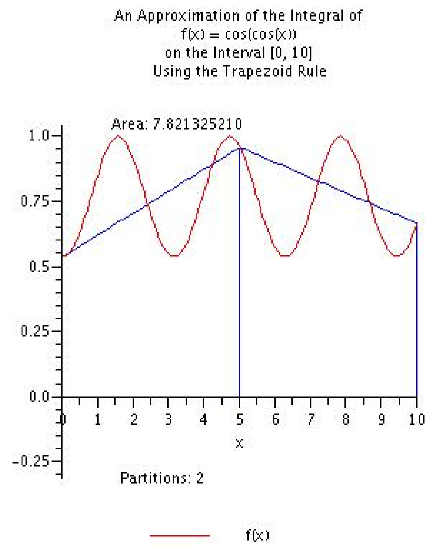
Example 7: Approximate $\int_0^4 \frac{1}{1+\left(\frac{x}{4}\right)^2}\,dx;\ n = 4;\ y_i = f(x_i)$

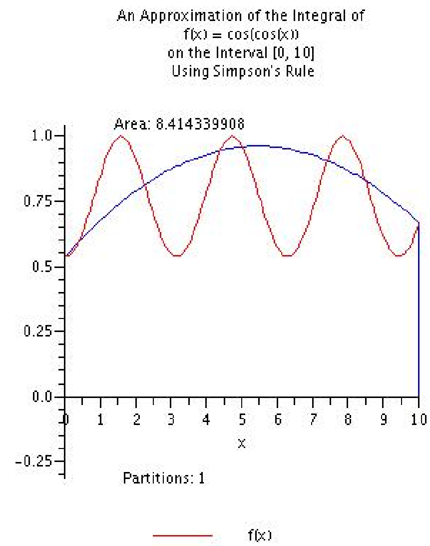


$$f(x) = \frac{1}{1+\left(\frac{x}{4}\right)^2};\ n = 4;\ \Delta x = \frac{4-0}{4} = 1$$

$$\begin{aligned} S_4 &= \frac{1}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \\ &= \frac{1}{3}\left(1 + \frac{4}{1+(0.25)^2} + \frac{2}{1+(0.5)^2} + \frac{4}{1+(0.75)^2} + \frac{1}{1+1^2}\right) \\ &\simeq 3.141592502\ldots \end{aligned}$$

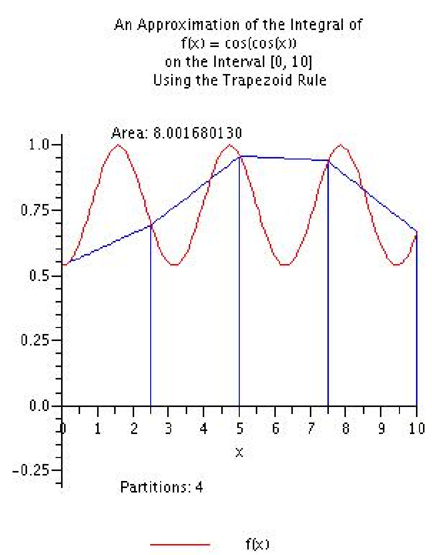
Example 8: Approximate $\int_0^{10} \cos(\cos x)\, dx$; with $n = 2$

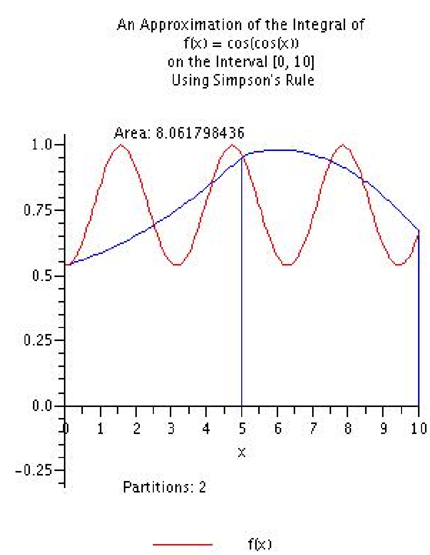




Chapter 7: Integration Techniques
7.1 Basic Approaches
7.2 Integration by Parts
7.3 Trigonometric Integrals
7.4 Trigonometric Substitutions
7.5 Partial Fractions
7.8 Improper Integrals
7.6 Other Integration Strategies
7.7 Numerical Integration

Example 9: Approximate $\int_0^{10} \cos(\cos x)\, dx$; with $n = 4$





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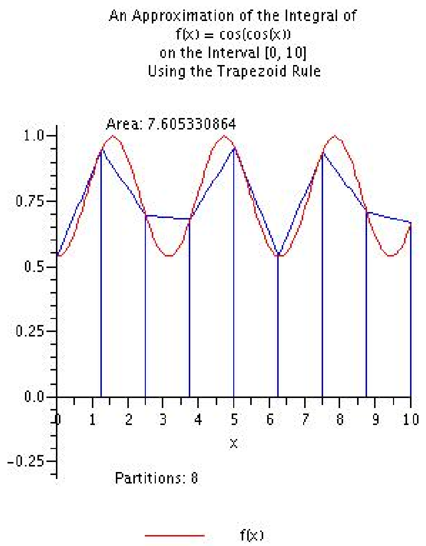
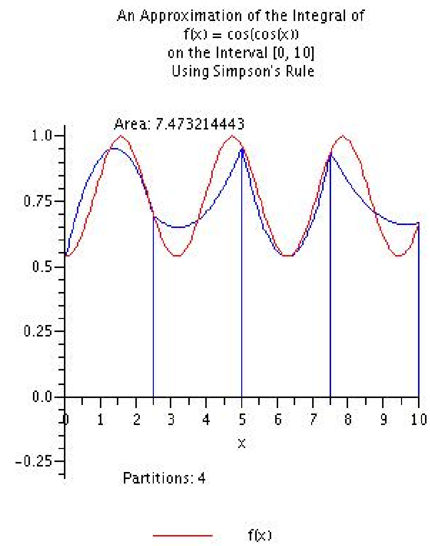
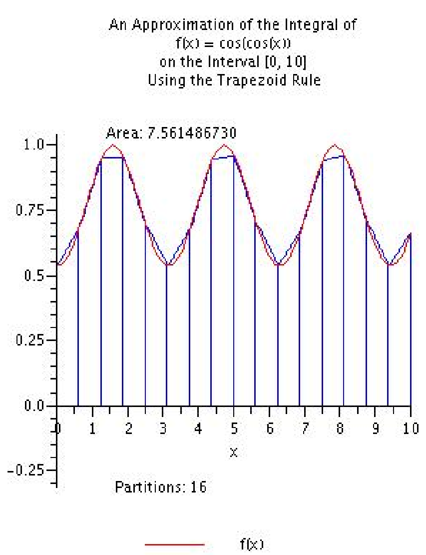
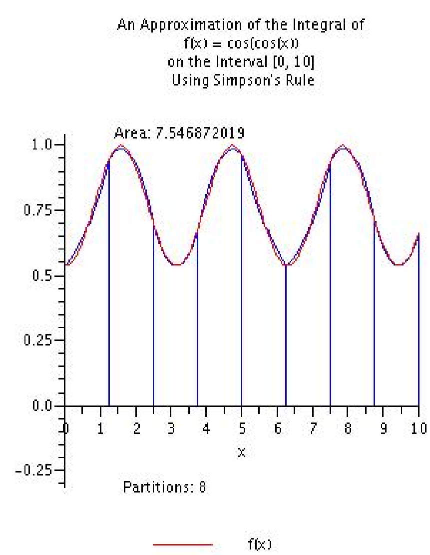
Example 10: Approximate $\int_0^{10} \cos(\cos x)\,dx$; with $n = 8$





Example 11: Approximate $\int_0^{10} \cos(\cos x)\,dx$; with $n = 16$





Error Estimates

The accuracy of Trapezoid or Simpson's Rule approximations can be calculated with the help of the following error estimates:

1. $$\left| \int_a^b f(x)\,dx - T_n \right| \le \frac{M_2(b-a)^3}{12n^2}$$
2. $$\left| \int_a^b f(x)\,dx - S_n \right| \le \frac{M_4(b-a)^5}{180n^4}$$

where M_k is the maximum of the absolute value of the kth derivative of f on the interval $[a, b]$.

Example 12: Approximating $\int_1^5 f(x)\,dx;\ a = 1;\ b = 5$

$$f(x) = \frac{1}{x};\ f'(x) = -\frac{1}{x^2};\ f''(x) = \frac{2}{x^3};\ f^{(3)}(x) = -\frac{6}{x^4};\ f^{(4)} = \frac{24}{x^5}.$$

1. $M_2 = 2 \Rightarrow \left| \int_1^5 f(x)\,dx - T_n \right| \le \frac{2(5-1)^3}{12n^2} = \frac{32}{3n^2}$
2. $M_4 = 24 \Rightarrow \left| \int_1^5 f(x)\,dx - S_n \right| \le \frac{24(5-1)^5}{180n^4} = \frac{2048}{15n^4}$

If $n = 8$ then $\left| \int_1^5 f(x)\,dx - T_n \right| \le \frac{32}{3 \cdot 8^2} = 0.16666666\ldots$, and $\left| \int_1^5 f(x)\,dx - S_n \right| \le \frac{2048}{15 \cdot 8^4} = 0.03333333\ldots$