MAT187H1F Lec0101 Burbulla

Chapter 8 Lecture Notes

Spring 2017

What is a Differential Equation?

Any equation that involves variables and derivatives is called a differential equation. Both

1. $\dfrac{dy}{dx} = x^2$
2. $\dfrac{dy}{dx} = y$

are examples of differential equations – very simple ones. If in addition some initial condition – extra information – is given, then the differential equation becomes an initial value problem. For example, the following are both initial value problems:

1. $\dfrac{dy}{dx} = x^2; y = 1$ if $x = -1$
2. $\dfrac{dy}{dx} = y; y = -6$ if $x = 4$

What Is a Solution to a Differential Equation?

Any function – y in terms of x, for the previous examples – that satisfies the differential equation is called a solution to the differential equation. We distinguish between general solutions and particular solutions.

1. a general solution does not need to satisfy any initial condition, and includes one or more arbitrary constants;
2. a particular solution is a solution to the differential equation which also satisfies the initial conditions.

The graph of a solution to a differential equation is called an integral curve.

Example 1

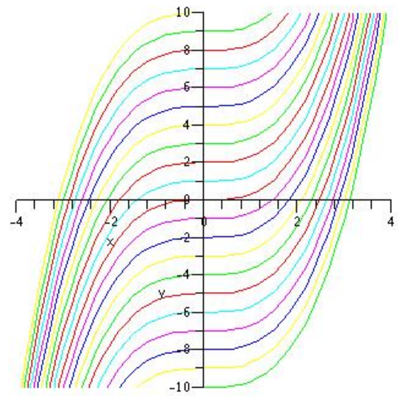
$y = \frac{1}{3}x^3 + C$ is the general solution to the differential equation

$$\frac{dy}{dx} = x^2;$$

$$y = \frac{1}{3}x^3 + \frac{4}{3}$$

is the particular solution to the initial value problem

$$\text{DE: } \frac{dy}{dx} = x^2; \quad \text{IC: } y = 1 \text{ if } x = -1.$$



To find the particular solution from the general solution substitute the initial conditions and solve for $C : 1 = \frac{1}{3}(-1)^3 + C \Leftrightarrow C = \frac{4}{3}$.

Chapter 8: Differential Equations
8.1 Basic Ideas
8.2 Direction Fields
8.3 Separable Differential Equations
8.4 Special First-Order Differential Equations
8.5 Modeling with Differential Equations

Example 2

Verify that

$$y = -6\,e^{x-4}$$

is the solution to the initial value problem:

$$\text{DE: } \frac{dy}{dx} = y; \text{ IC: } y = -6 \text{ if } x = 4.$$

Solution: if $x = 4$ then $y = -6\,e^0 = -6$; so the given function satisfies the initial condition. Also

$$y = -6\,e^{x-4} \quad \Rightarrow \quad \frac{dy}{dx} = -6\,e^{x-4}$$
$$\Rightarrow \quad \frac{dy}{dx} = y$$

Chapter 8 Lecture Notes MAT187H1F Lec0101 Burbulla

Motion In A Gravitational Field

As Galileo established, the acceleration due to gravity, neglecting air resistance, is constant. Its value is

$$g = -9.8 \text{ m/sec}^2 \text{ or } g = -32 \text{ ft/sec}^2.$$

If s is the position of a free-falling body at time t, then

$$\frac{d^2 s}{dt^2} = g.$$

This is known as a second order differential equation, since it involves the second derivative of a function. To solve it, we integrate twice:

$$\frac{ds}{dt} = \int g\, dt = gt + v_0, \;\; s = \int (gt + v_0)\, dt = \frac{1}{2} gt^2 + v_0 t + s_0,$$

where v_0 and s_0 are initial velocity and position at time $t = 0$.

Chapter 8: Differential Equations
8.1 Basic Ideas
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Example 3

If a penny is dropped from the top of a 40-m tall building, how long will it take to hit the ground? With what speed will it hit the ground? Neglect air resistance.
Solution: Take $v_0 = 0, s_0 = 40, g = -9.8$. Then $s = -4.9t^2 + 40$. The penny hits the ground when

$$s = 0 \Leftrightarrow t \approx \pm 2.86.$$

Take $t \approx 2.86$ sec. Then

$$v \approx -9.8(2.86) \approx -28.03$$

So the penny hits the ground with a speed of 28.03m/sec, or about 101 km per hr.

Chapter 8 Lecture Notes MAT187H1F Lec0101 Burbulla

A Harvesting Model

A simple model to describe the harvesting of a natural resource, such as timber or fish, is given by

$$p'(t) = r\,p(t) - H,$$

where

- $p(t)$ is the population of the resource at time $t \geq 0$,
- $r > 0$ is the natural growth rate of the resource,
- $H > 0$ is the harvesting rate,
- and $p_0 = p(0)$ is the initial amount of the resource.

Verify that

$$p(t) = \left(p_0 - \frac{H}{r}\right) e^{rt} + \frac{H}{r}$$

is a solution to this initial value problem.

Chapter 8: Differential Equations
8.1 Basic Ideas
8.2 Direction Fields
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8.4 Special First-Order Differential Equations
8.5 Modeling with Differential Equations

$$\frac{d\left(\left(p_0 - \frac{H}{r}\right) e^{rt} + \frac{H}{r}\right)}{dt} = r\left(p_0 - \frac{H}{r}\right) e^{rt} = (r\,p_0 - H)\,e^{rt};$$

and

$$r\,p(t) - H = r\left(\left(p_0 - \frac{H}{r}\right) e^{rt} + \frac{H}{r}\right) - H = (r\,p_0 - H)\,e^{rt} + H - H;$$

so, yes,

$$p'(t) = r\,p(t) - H.$$

As for the initial condition:

$$p(0) = \left(p_0 - \frac{H}{r}\right) + \frac{H}{r} = p_0.$$

Chapter 8 Lecture Notes MAT187H1F Lec0101 Burbulla

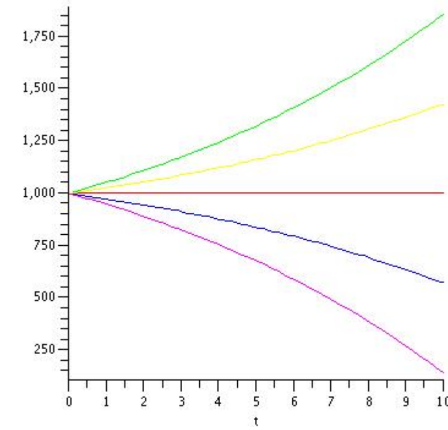
Example 4

Plot graphs of

$$p(t) = \left(p_0 - \frac{H}{r}\right) e^{rt} + \frac{H}{r}$$

for $p_0 = 1000$, $r = 0.1$ and the values of

$$H = 50, 75, 100, 125, 150.$$



Note: if the harvest rate is too great, $H = 125, 150$, (blue and violet) the population of the resource will decline and eventually become extinct; if the harvest rate is not too great, $H = 50, 75$, (green and yellow) the population will increase.

Equilibrium Solutions

If $H = 100$, red graph, then the population remains constant, at $p = 1000$; it is called an equilibrium solution.

Equilibrium solutions to a differential equation can be found by setting $p'(t)$ identically to zero.That is,

$$p'(t) = 0$$

for all t. In this example, we have

$$p'(t) = r\,p(t) - H = 0 \Leftrightarrow H = r\,p(t).$$

In particular, taking $t = 0$ we obtain $H = r\,p_0$. This is the relation between the parameters r, p_0 and H required to ensure an equilibrium solution.

What Is A Direction Field?

Consider the general differential equation

$$\text{DE: } \frac{dy}{dx} = F(x,y),$$

where $F(x,y)$ is some expression in terms of x and y. This equation tells you a lot about the graph of any solution to DE; namely, any solution $y = f(x)$ to DE that passes through the point (x,y), does so with slope $m = F(x,y)$. By plotting many of these slopes – represented by short lines with slope m – for many different points (x,y), you can produce what is called a direction field, or a slope field. The slope field can be used to picture the graphs of solutions to DE.

Isoclines

The relation

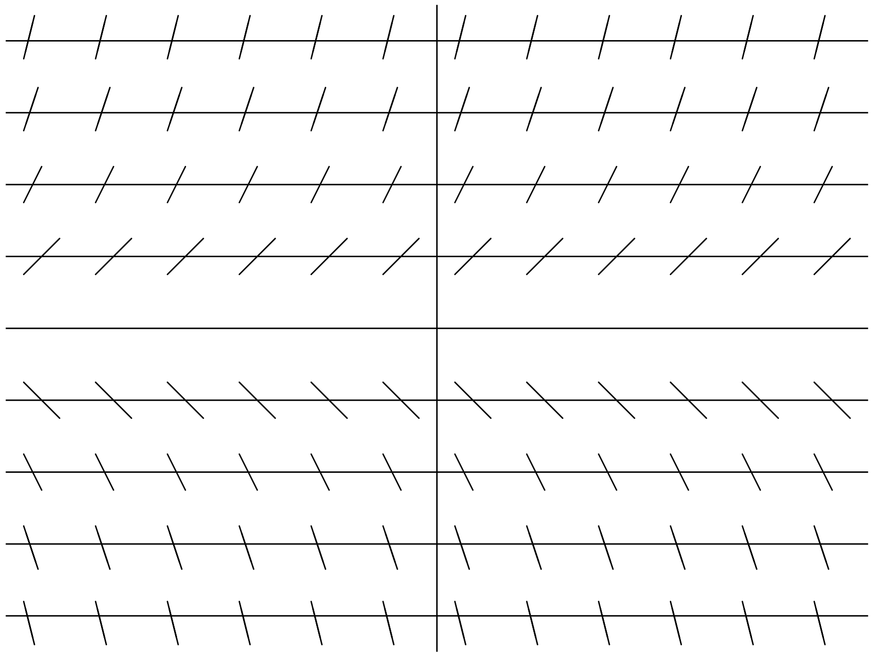
$$m = F(x,y)$$

determines a graph in the Cartesian plane that joins together all points (x,y) for which any solution to the

$$\text{DE: } \frac{dy}{dx} = F(x,y)$$

will pass with slope m. The graph of the relation $m = F(x,y)$ is called an isocline. In the two examples that follow the isoclines are lines; but in general plotting the isoclines can be much more difficult.

Example 1: $\frac{dy}{dx} = 2y$



$y = 2;\ m = 4$
$y = 1.5;\ m = 3$
$y = 1;\ m = 2$
$y = .5;\ m = 1$
$y = 0;\ m = 0$
$y = -.5;\ m = -1$
$y = -1;\ m = -2$
$y = -1.5;\ m = -3$
$y = -2;\ m = -4$

$y = 0$ Is A Special Solution of $\frac{dy}{dx} = 2y$

$y = 0$ is an obvious solution to

$$\frac{dy}{dx} = 2y;$$

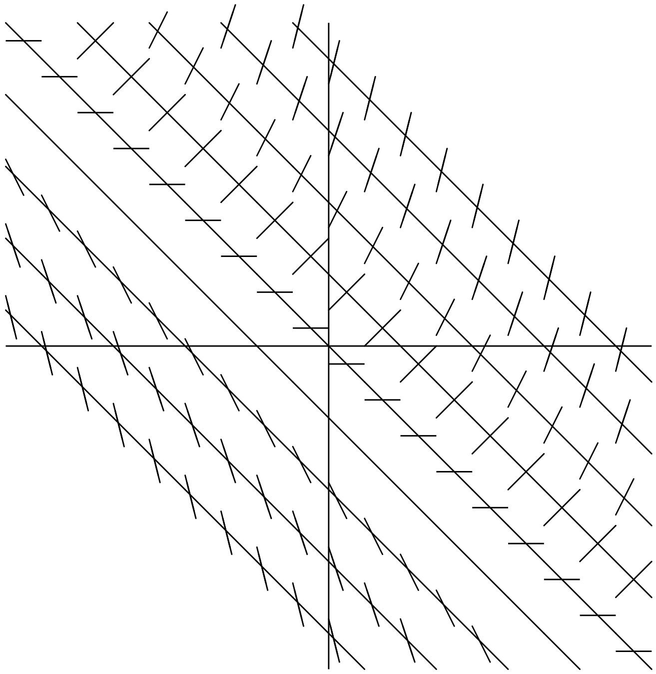
it is the equilibrium solution. On the direction field, the line $y = 0$ is itself a line of slope 0 while the value of m on that line is also 0. This coincidence indicates that the function

$$y = 0$$

is a special solution to

$$\frac{dy}{dx} = 2y.$$

Example 2: $\frac{dy}{dx} = x + y$



$y = 4 - x; m = 4$

$y = 3 - x; m = 3$

$y = 2 - x; m = 2$

$y = 1 - x; m = 1$

$y = -x; m = 0$

$y = -x - 1$ Is A Special Solution to $\frac{dy}{dx} = x + y$

$y = -x - 1$ is an obvious solution to

$$\frac{dy}{dx} = x + y.$$

On the direction field, the line $y = -x - 1$ is itself a line of slope -1 while the value of m on that line is also -1. This coincidence indicates that the function

$$y = -x - 1$$

is a special solution to

$$\frac{dy}{dx} = x + y.$$

Separable Differential Equations

A differential equation is called separable if you can separate the variables, so that only one variable is present on each side of the equation. Here are two examples:

1. $\dfrac{dx}{dt} = kx \Leftrightarrow \dfrac{1}{x}\, dx = k\, dt$
2. $A(y)\,\dfrac{dy}{dt} = -a\sqrt{2gy} \Leftrightarrow \dfrac{A(y)}{\sqrt{y}}\, dy = -a\sqrt{2g}\, dt$

Then you can try to solve the differential equation by integrating both sides of the equation. Note: not all DE's are separable! Here is an example of a differential equation that is not separable:

$$\frac{dy}{dx} = x + y.$$

Example 1: DE: $\dfrac{dy}{dx} = y^2$; IC: $y = 2$ when $x = 0$.

Solution: separate variables.

$$\begin{aligned}
\frac{dy}{dx} = y^2 \quad &\Leftrightarrow \quad \frac{1}{y^2}\, dy = dx, \text{ if } y \neq 0 \\
&\Leftrightarrow \quad \int \frac{1}{y^2}\, dy = \int dx \\
&\Leftrightarrow \quad -\frac{1}{y} = x + C \\
&\Leftrightarrow \quad y = -\frac{1}{x + C}, \text{ the general solution.}
\end{aligned}$$

To find C, use IC: $2 = -\dfrac{1}{0 + C} \Leftrightarrow C = -\dfrac{1}{2}$.

Example 1, Continued

$$\text{DE: } \frac{dy}{dx} = y^2; \quad \text{IC: } y = 0 \text{ when } x = 0.$$

Solution: $y = 0$, obviously! This is the function $y = 0$, for all x. It is called an equilibrium solution. That is, it is a constant function that satisfies the DE and the IC. You can find it by setting

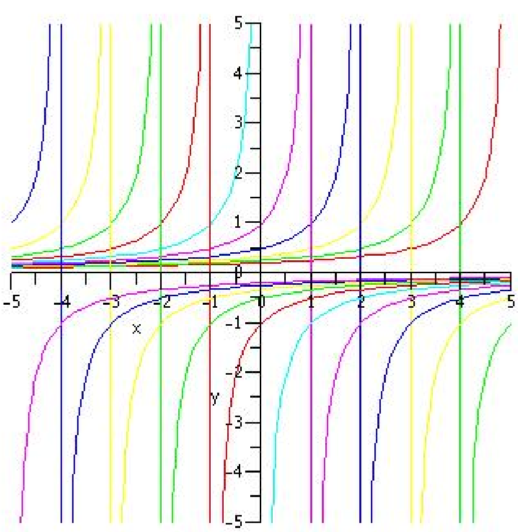
$$\frac{dy}{dx} = 0,$$

and solving for y. The method of the previous slide won't work, since the previous slide assumed $y \neq 0$. Note that this particular solution is not included in the general solution to DE:

$$y = -\frac{1}{x+C} = -\frac{1}{C}, \text{ if } x = 0.$$

But y will never be 0, no matter what C is.

Integral Curves for DE: $\frac{dy}{dx} = y^2$



Note that all curves are asymptotic to the equilibrium solution $y = 0$.

Example 2

$$\text{DE: } \frac{dy}{dx} = (y^2+1)\left(xe^{-x}\right); \quad \text{IC: } y = 1 \text{ when } x = 0.$$

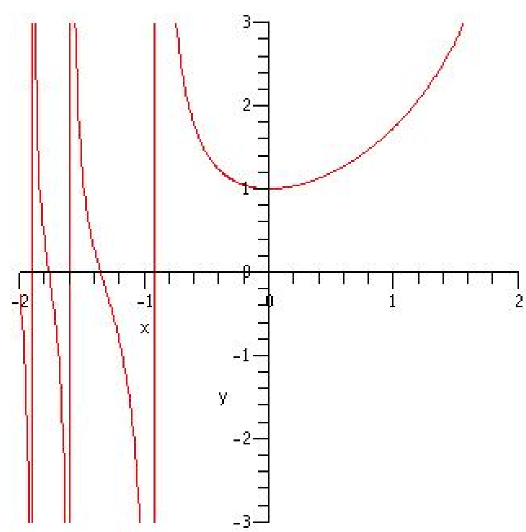
Solution: separate variables.

$$\begin{aligned} \frac{dy}{dx} = (y^2+1)\left(xe^{-x}\right) \quad &\Leftrightarrow \quad \frac{1}{1+y^2}\,dy = xe^{-x}\,dx \\ &\Leftrightarrow \quad \int \frac{1}{1+y^2}\,dy = \int xe^{-x}\,dx \\ &\Leftrightarrow \quad \tan^{-1} y = -xe^{-x} - e^{-x} + C \end{aligned}$$

To find C, use IC: $\tan^{-1} 1 = -1 + C \Leftrightarrow C = \frac{\pi}{4} + 1$. So

$$y = \tan\left(1 - xe^{-x} - e^{-x} + \frac{\pi}{4}\right).$$

The Graph of $y = \tan\left(1 - xe^{-x} - e^{-x} + \frac{\pi}{4}\right)$



Natural Growth Equation

Let x be the amount of some substance present at time t. The following differential equation

$$\frac{dx}{dt} = kx, k \neq 0$$

has many important applications. It can be interpreted as

$$\underbrace{\frac{dx}{dt}}_{the\ \ rate\ \ of\ \ change} \qquad \underbrace{= k}_{is\ \ proportional} \qquad \underbrace{x}_{to\ the\ \ amount\ \ present} \quad .$$

In this case, the substance is said to be growing naturally.

Solution to the Natural Growth Equation

$$\begin{aligned}
\frac{dx}{dt} = kx \quad &\Leftrightarrow \quad \int \frac{1}{x}\, dx = \int k\, dt \\
&\Leftrightarrow \quad \ln|x| = kt + C \\
&\Leftrightarrow \quad |x| = e^{kt+C} \\
&\Leftrightarrow \quad |x| = e^{C}\, e^{kt} \\
&\Leftrightarrow \quad x = \pm e^{C}\, e^{kt} \\
&\Leftrightarrow \quad x = A\, e^{kt}, \text{ for arbitrary constant } A \neq 0
\end{aligned}$$

If in addition $x = x_0$ when $t = 0$, then $A = x_0$, and

$$x = x_0 e^{kt}.$$

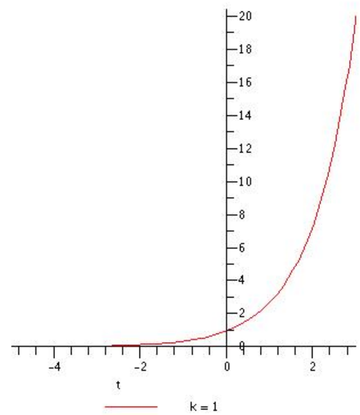
Exponential Growth: $k > 0, x_0 > 0$

In this case the value of x is always increasing. Two key features:

1. Doubling time:

$$\begin{aligned} x = 2x_0 \quad &\Leftrightarrow \quad x_0 e^{kt} = 2x_0 \\ &\Leftrightarrow \quad e^{kt} = 2 \\ &\Leftrightarrow \quad kt = \ln 2 \\ &\Leftrightarrow \quad t = \frac{\ln 2}{k} \end{aligned}$$

2. $\lim_{t\to\infty} x = \infty$



Examples: exponential population growth; compound interest.

Example 3; This Is A Repeat of Example 1 in Sec 6.9

The population of a town is growing exponentially so that its population doubles every 10 years. The population of the town was 10,000 in 1995; what will its population be in the year 2020?
Solution: Let x be the population of the town at time t, where time is measured in years since 1995. So $t = 0$ corresponds to 1995, and $x_0 = 10\,000$. Use the doubling time to find k :

$$10 = \frac{\ln 2}{k} \Leftrightarrow k = \frac{\ln 2}{10} \approx 0.0693.$$

So $x = x_0 e^{kt} = 10\,000 e^{\frac{\ln 2}{10} t} = 10\,000 \cdot 2^{\frac{t}{10}}$, or $x \approx 10\,000 e^{0.0693t}$.
Now let $t = 25$:

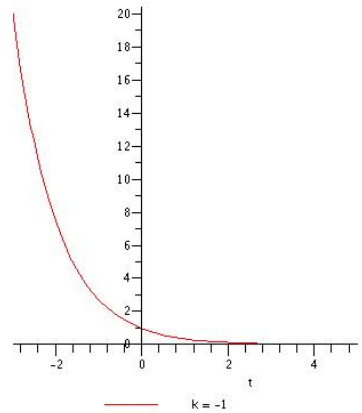
$$x = 10\,000 \cdot 2^{\frac{25}{10}} = 10\,000 \cdot 2^{2.5} \approx 56\,569.$$

Exponential Decay: $k < 0, x_0 > 0$

In this case the value of x is always decreasing.

- Half Life:

$$x = \frac{1}{2}x_0 \quad\Leftrightarrow\quad x_0 e^{kt} = \frac{1}{2}x_0$$
$$\Leftrightarrow\quad e^{kt} = 2^{-1}$$
$$\Leftrightarrow\quad kt = -\ln 2$$
$$\Leftrightarrow\quad t = -\frac{\ln 2}{k}$$



- Long term value:
$$\lim_{t\to\infty} x = 0$$

Example: radioactive decay.

Example 4: Carbon-14 Dating (Repeat of Ex 2 in Sec 6.9)

The half life of carbon-14 is 5,700 years. If a specimen of charcoal found in Stonehenge contains only 63% of its original carbon-14, how old is Stonehenge? **Solution:** Let x be the amount of carbon-14 present in the charcoal at time t, with t in years since the charcoal was created. Use the half life to find k :

$$5\,700 = -\frac{\ln 2}{k} \Leftrightarrow k = -\frac{\ln 2}{5\,700} \approx -0.0001216.$$

Then $x = x_0 e^{kt} = x_0 e^{-0.0001216t}$. Let $x = 0.63x_0$, and solve for t :

$$0.63x_0 = x_0 e^{-0.0001216t} \quad\Leftrightarrow\quad 0.63 = e^{-0.0001216t}$$
$$\Leftrightarrow\quad \ln 0.63 = -0.0001216t$$
$$\Leftrightarrow\quad t = -\frac{\ln 0.63}{0.0001216} \approx 3\,800$$

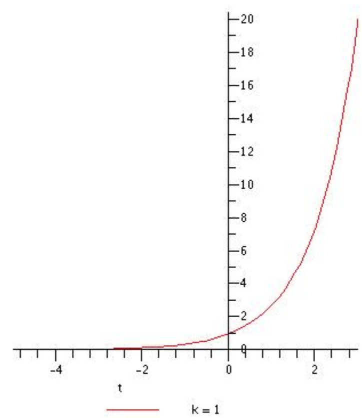
Exponential Growth is Not Realistic

Suppose $k > 0$.

$$x = x_0 e^{kt}$$

$$\Rightarrow \quad \lim_{t\to\infty} x = \infty$$

It is not realistic for a population to grow exponentially for ever; eventually some limits to growth will take effect.



Logistic Growth is More Realistic

Logistic growth is more realistic; it includes built-in limits to growth. The differential equation for logistic growth is

$$\frac{dx}{dt} = kx - \frac{k}{L}x^2 = kx\left(1 - \frac{x}{L}\right),$$

for some positive constants k and L. It is the DE for exponential growth with an additional second degree term. Observe that $x = L$ is an equilibrium solution. It is a stable equilibrium solution since

$$\frac{dx}{dt} > 0 \text{ if } x < L \quad \text{and} \quad \frac{dx}{dt} < 0 \text{ if } x > L$$

and it can be shown that

$$\lim_{t\to\infty} x = L.$$

Solving the DE for Logistic Growth

The differential equation for logistic growth,

$$\frac{dx}{dt} = kx - \frac{k}{L}x^2,$$

for some positive constants k and L, can be solved by separating variables. For simplicity, we shall assume $0 < x < L$.

$$\begin{aligned}
\frac{dx}{dt} = kx\frac{(L-x)}{L} \quad &\Rightarrow \quad \int \frac{L}{x(L-x)}\,dx = \int k\,dt \\
\text{(partial fractions)} \quad &\Rightarrow \quad \int \left(\frac{1}{x} + \frac{1}{L-x}\right) dx = kt + c \\
&\Rightarrow \quad \ln x - \ln(L-x) = kt + c \\
&\Rightarrow \quad \ln\left(\frac{x}{L-x}\right) = kt + c
\end{aligned}$$

The Logistic Equation

$$\begin{aligned}
&\Rightarrow \quad \frac{x}{L-x} = e^{kt+c} = Ae^{kt}, \text{ for } A = e^c \\
&\Rightarrow \quad \frac{L}{x} - 1 = Be^{-kt}, \text{ for } B = 1/A \\
&\Rightarrow \quad \frac{L}{x} = 1 + Be^{-kt} \Rightarrow \frac{x}{L} = \frac{1}{1+Be^{-kt}} \\
&\Rightarrow \quad x = \frac{L}{1+Be^{-kt}}
\end{aligned}$$

which is called the logistic equation. Note:

1. $\lim_{t\to\infty} x = \lim_{t\to\infty} \frac{L}{1+Be^{-kt}} = \frac{L}{1+0} = L.$
2. $x = x_0$ if $t = 0 \Rightarrow x_0 = \frac{L}{1+B} \Leftrightarrow B = \frac{L}{x_0} - 1.$

The Inflection Point of the Logistic Curve

It's not necessary to calculate the second derivative directly. We started with

$$\frac{dx}{dt} = kx - \frac{k}{L}x^2;$$

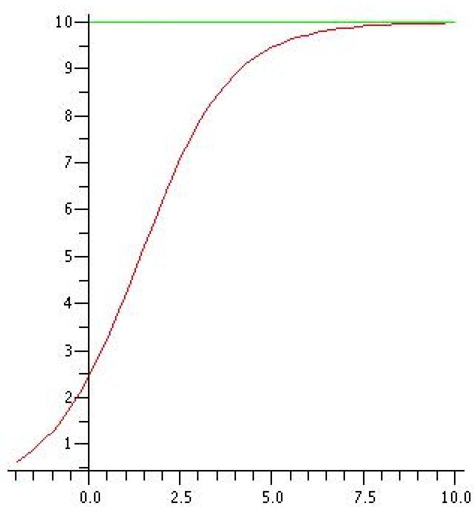
whence

$$\frac{d^2x}{dt^2} = k\frac{dx}{dt} - \frac{k}{L} \cdot 2x\frac{dx}{dt} = k\frac{dx}{dt}\left(1 - \frac{2x}{L}\right).$$

So

$$\frac{d^2x}{dt^2} = 0 \Leftrightarrow x = \frac{L}{2}.$$

Example 5: $x = \dfrac{10}{1 + 3e^{-0.8t}}$; $k = 0.8$; $L = 10$



Example 6

The rate at which a rumour spreads through a group of stockbrokers is proportional to the product of the number who have heard the rumour and the number who have not yet heard of the rumour. 25% of the stockbrokers have heard of the rumour at 1 PM. 30 min later, 40 % of all stockbrokers have heard of it. How long will it take until 75% of all stockbrokers have heard of the rumour? 95% ?

Solution: Let $L = 1 = 100\%$; let x be the percentage of all stockbrokers who have heard of the rumor at time t.

$$\frac{dx}{dt} = kx(1-x) \Rightarrow x = \frac{1}{1+Be^{-kt}}.$$

Since $x_0 = 25\%$ at $t = 0$, we have $\dfrac{1}{4} = \dfrac{1}{1+B} \Leftrightarrow B = 3.$

To find k, use the fact that $x = 40\%$ at $t = 30$:

$$\begin{aligned}
\frac{2}{5} = \frac{1}{1+3e^{-30k}} \quad &\Leftrightarrow \quad \frac{5}{2} = 1 + 3e^{-30k} \\
&\Leftrightarrow \quad \frac{3}{2} = 3e^{-30k} \\
&\Leftrightarrow \quad 2 = e^{30k} \\
&\Leftrightarrow \quad 30k = \ln 2 \\
&\Leftrightarrow \quad k = \frac{\ln 2}{30} \approx 0.0231049
\end{aligned}$$

Thus

$$x = \frac{1}{1+3\cdot e^{-t\ln 2/30}} = \frac{1}{1+3\cdot 2^{-t/30}}.$$

So far,

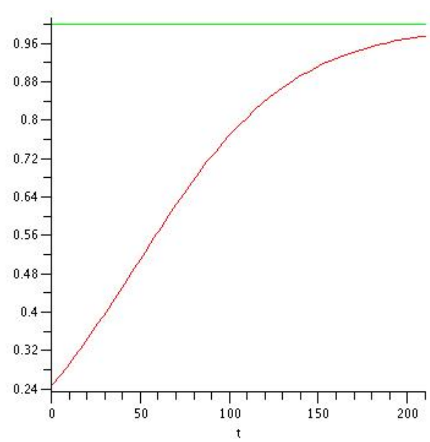
$$x = \frac{1}{1 + 3 \cdot 2^{-t/30}}.$$

Let $x = 75\%$ and solve for t :

$$\begin{aligned}
\frac{3}{4} = \frac{1}{1 + 3 \cdot 2^{-t/30}} \quad &\Leftrightarrow \quad \frac{4}{3} = 1 + 3 \cdot 2^{-t/30} \\
&\Leftrightarrow \quad \frac{1}{3} = 3 \cdot 2^{-t/30} \\
&\Leftrightarrow \quad 9 = 2^{t/30} \\
&\Leftrightarrow \quad \frac{t}{30} = \frac{\ln 9}{\ln 2} \\
&\Leftrightarrow \quad t = 30 \left(\frac{\ln 9}{\ln 2} \right) \approx 95.1
\end{aligned}$$

So it will take 95.1 min, or until approximately 2:35 PM, until 75% of all stockbrokers have heard of the rumour.

Graph for Example 6: $x = \dfrac{1}{1 + 3 \cdot 2^{-t/30}}$



It will take considerably longer until 95% of all stockbrokers have heard of the rumour:

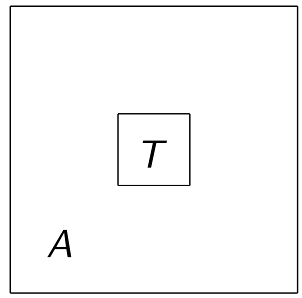
$$\frac{19}{20} = x \Leftrightarrow t = 30 \left(\frac{\ln 57}{\ln 2} \right) \approx 175.$$

Newton's Law of Cooling

Suppose an object with temperature T, at time t, is in a room with constant ambient temperature A. Let the temperature of the object at time $t = 0$ be T_0. Newton's Law of Cooling, or heating, states that

$$\frac{dT}{dt} = -k(T - A),$$

for some positive constant k. That is, the rate of change of T is proportional to the difference of T and the ambient temperature A.



This differential equation can be solved by separating variables:

$$\begin{aligned} \frac{dT}{dt} = -k(T - A) \quad &\Rightarrow \quad \frac{1}{T - A}\, dT = -k\, dt \\ &\Rightarrow \quad \int \frac{1}{T - A}\, dT = -\int k\, dt \\ &\Rightarrow \quad \ln|T - A| = -kt + C \end{aligned}$$

To find C use the initial condition:

$$\ln|T_0 - A| = -k \cdot 0 + C \Leftrightarrow C = \ln|T_0 - A|.$$

To solve for T in terms of t, we consider two cases:

1. cooling: $T_0 > A$
2. heating: $T_0 < A$

Case 1: $T_0 > A$

In this case the object will cool until it is eventually the same temperature as the surrounding medium.

$$\begin{aligned}
\ln|T-A| = -kt + \ln|T_0 - A| \quad &\Rightarrow \quad \ln(T-A) = -kt + \ln(T_0 - A) \\
&\Rightarrow \quad T - A = e^{-kt+\ln(T_0-A)} \\
&\Rightarrow \quad T - A = (T_0 - A)e^{-kt} \\
&\Rightarrow \quad T = A + (T_0 - A)e^{-kt}
\end{aligned}$$

Note that

$$\lim_{t\to\infty} T = \lim_{t\to\infty} \left(A + (T_0 - A)e^{-kt}\right) = A + 0 = A$$

Case 2: $T_0 < A$

In this case the object will heat up until it is eventually the same temperature as the surrounding medium.

$$\begin{aligned}
\ln|T-A| = -kt + \ln|T_0 - A| \quad &\Rightarrow \quad \ln(A-T) = -kt + \ln(A - T_0) \\
&\Rightarrow \quad A - T = e^{-kt+\ln(A-T_0)} \\
&\Rightarrow \quad A - T = (A - T_0)e^{-kt} \\
&\Rightarrow \quad -T = -A + (A - T_0)e^{-kt} \\
&\Rightarrow \quad T = A + (T_0 - A)e^{-kt}
\end{aligned}$$

This solution looks exactly the same as the solution in Case 1. The difference is that now $T_0 - A < 0$, so that as time passes T will increase, instead of decrease.

Example 1

A freshly baked cake is taken out of the oven at 11 AM with temperature 200 C, and put on a table to cool, in a room with constant temperature 20 C. Fifteen minutes later the temperature of the cake is 125 C. How long will it take until the temperature of the cake is 25 C ?

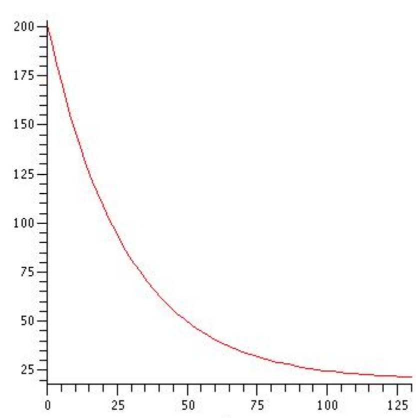
Solution: We have $T_0 = 200, A = 20$. Thus

$$T = 20 + 180e^{-kt}.$$

To find k, let $t = 15$, $T = 125$:

$$125 = 20+180e^{-15k} \Leftrightarrow e^{15k} = \frac{180}{105} \Leftrightarrow k = \frac{1}{15}\ln\left(\frac{12}{7}\right) \approx 0.03593$$

Graph for Example 1



Now let $T = 25$ and solve for t :

$$\begin{aligned} & 25 = 20 + 180e^{-kt} \\ \Rightarrow \quad & e^{kt} = \frac{180}{5} = 36 \\ \Rightarrow \quad & t = \frac{1}{k}\ln 36 \\ \Rightarrow \quad & t \approx 99.7 \text{ min} \end{aligned}$$

So it will take about 1 hr and 40 min for the cake to cool to 25 C.

Example 2

A metal shovel taken from outside, at temperature -10 C, is put in a porch with constant air temperature 15 C. After 5 min the temperature of the shovel is -5 C. When will the temperature of the shovel be 10 C?

Solution: (I will solve this without finding the formula for T.) We have $T_0 = -10, A = 15$.

$$\frac{dT}{dt} = -k(T-15) \quad \Rightarrow \quad \int \frac{1}{T-15}\,dT = -\int k\,dt$$
$$\Rightarrow \quad \ln|T-15| = -kt + C$$

To find C, let $t = 0, T = -10$: $C = \ln|-10-15| = \ln 25.$

To find k, let $t = 5, T = -5$:

$$\ln|-5-15| = -5k + \ln 25 \quad \Rightarrow \quad 5k = \ln 25 - \ln 20$$
$$\Rightarrow \quad k = \frac{1}{5}\ln\left(\frac{5}{4}\right) \approx 0.04463$$

Finally, let $T = 10$ and solve for t :

$$\ln|10-15| = -\frac{t}{5}\ln\left(\frac{5}{4}\right) + \ln 25 \quad \Rightarrow \quad \frac{t}{5}\ln\left(\frac{5}{4}\right) = \ln 25 - \ln 5$$
$$\Rightarrow \quad \frac{t}{5}\ln 1.25 = \ln 5$$
$$\Rightarrow \quad t = \frac{5\ln 5}{\ln 1.25} \approx 36.1$$

First Order Linear Differential Equation

Newton's Law of Cooling is a special case of the general first order linear differential equation

$$\text{DE: } \frac{dy}{dt} = k\,y + b,$$

where y is a function of t, and $k \neq 0$ and b are parameters. It can be solved by separating variables, as above. Or, since

$$\frac{d(y+\frac{b}{k})}{dt} = \frac{dy}{dt} = k\left(y + \frac{b}{k}\right),$$

it can be solved by using the solution for exponential growth:

$$y + \frac{b}{k} = C\,e^{kt} \Rightarrow y = C\,e^{kt} - \frac{b}{k},$$

for some arbitrary constant C, depending on initial conditions. Suppose $y = y_0$ when $t = 0$; then $y_0 = C - \frac{b}{k} \Leftrightarrow C = y_0 + \frac{b}{k}$.

Alternate Solution

$$\text{DE: } y' = k\,y + b \Rightarrow y'' = k\,y'$$

Using the solution for exponential growth, applied to y', we have

$$y' = y_0' e^{kt}.$$

From DE , $y_0' = k\,y_0 + b$, so $y' = (k\,y_0 + b)\,e^{kt}$. Integrating with respect to t gives

$$y = \left(y_0 + \frac{b}{k}\right) e^{kt} + D.$$

Finally, $y = y_0$ when $t = 0$, so

$$D = -\frac{b}{k}.$$

Example 3

A drug administered intravenously to a patient at a rate of 6 mg/hr decays exponentially in the blood at a rate of 3% per hr. If initially none of the drug was present in the patient's bloodstream, how much of the drug will there be in the patient's bloodstream in the long run? How long will it take until the amount of drug in the patient's bloodstream reaches 95% of this stable equilibrium value?

Solution: let $y(t)$ be the amount of the drug in the patient's bloodstream at time t, measured in hours, with initial value $y(0) = 0$. The differential equation is

$$\frac{dy}{dt} = -0.03\,y + 6.$$

This is a first order linear differential equation; its general solution is

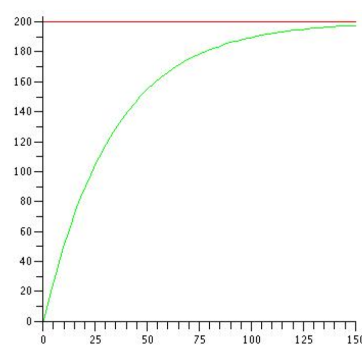
Chapter 8: Differential Equations
8.1 Basic Ideas
8.2 Direction Fields
8.3 Separable Differential Equations
8.4 Special First-Order Differential Equations
8.5 Modeling with Differential Equations

$$y = C\,e^{-0.03t} + \frac{6}{0.03} = C\,e^{-0.03t} + 200.$$

Since $y_0 = 0$, we have $C = -200$. So

$$y = 200\left(1 - e^{-0.03t}\right).$$

The graph of y is to the right.



The stable equilibrium solution is $y = 200$, so in the long run there will be 200 mg of the drug in the patient's bloodstream. To reach 95% of this steady-state level it will take about 100 hours:

$$190 = 200(1 - e^{-0.03t}) \Leftrightarrow 0.95 = 1 - e^{-0.03t} \Leftrightarrow e^{-0.03t} = 0.05$$

$$\Leftrightarrow e^{0.03t} = 20 \Leftrightarrow 0.03t = \ln 20 \Leftrightarrow t \approx 99.858$$

Chapter 8 Lecture Notes MAT187H1F Lec0101 Burbulla

Stable and Unstable Equilibrium Solutions

The equilibrium solution to the general first order linear differential equation

$$\frac{dy}{dt} = k\,y + b \text{ is the constant function } y = -\frac{b}{k}.$$

1. If $k > 0$, then it is an unstable equilibrium solution to DE since
$$y > -\frac{b}{k} \Rightarrow \frac{dy}{dt} > 0; \;\; y < -\frac{b}{k} \Rightarrow \frac{dy}{dt} < 0.$$

2. If $k < 0$, then it is a stable equilibrium solution to DE since
$$y > -\frac{b}{k} \Rightarrow \frac{dy}{dt} < 0; \;\; y < -\frac{b}{k} \Rightarrow \frac{dy}{dt} > 0; \;\text{ and } \lim_{t\to\infty} y = -\frac{b}{k}.$$

Method of the Integrating Factor

$$\text{DE: } \frac{dy}{dx} + P(x)\,y = Q(x).$$

Multiply both sides of DE by the integrating factor, $\mu = e^{\int P(x)\,dx}$:

$$\begin{aligned}
\frac{dy}{dx}\,e^{\int P(x)\,dx} + \left(P(x)e^{\int P(x)\,dx}\right) y &= Q(x)\,e^{\int P(x)\,dx} \\
\Rightarrow \frac{d}{dx}\left(y\,e^{\int P(x)\,dx}\right) &= Q(x)\,e^{\int P(x)\,dx} \\
\Rightarrow y\,e^{\int P(x)\,dx} &= \int \left(Q(x)\,e^{\int P(x)\,dx}\right) dx \\
\Rightarrow y &= \frac{\int \left(Q(x)\,e^{\int P(x)\,dx}\right) dx}{e^{\int P(x)\,dx}}
\end{aligned}$$

Example 4

Consider the non-separable differential equation

$$\frac{dy}{dx} = x + y \Leftrightarrow \frac{dy}{dx} - y = x.$$

The integrating factor is

$$\mu = e^{\int -dx} = e^{-x}$$

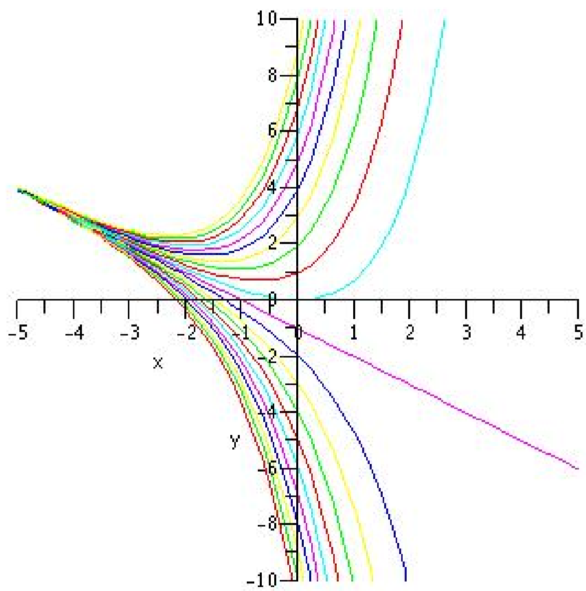
and

$$ye^{-x} = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C, \text{ by parts.}$$

So the general solution is

$$y = -x - 1 + C\,e^{x}.$$

Graphs for Example 4, $C = -10, -9, \ldots, 9, 10$



Example 5

For

$$\frac{dy}{dx} + \frac{2}{x}y = x^3$$

the integrating factor is

$$\mu = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = |x|^2 = x^2.$$

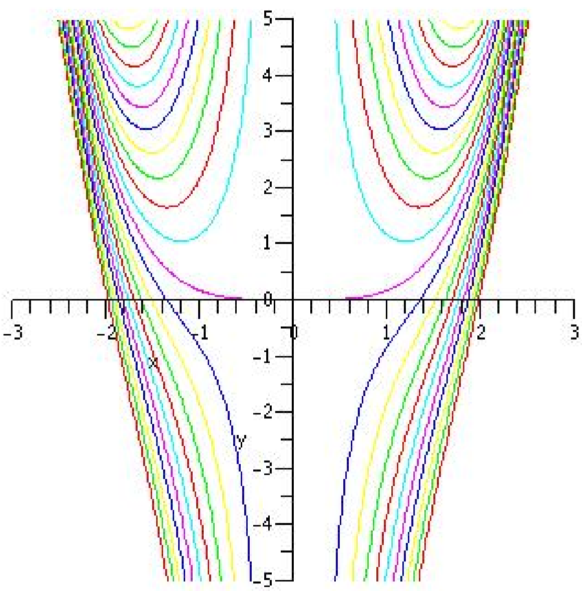
So

$$x^2 y = \int x^5\,dx = \frac{1}{6}x^6 + C$$

and

$$y = \frac{1}{6}x^4 + \frac{C}{x^2}.$$

Graphs for Example 5, $C = -10, -9, \ldots, 9, 10$



Example 6: $\cos x \dfrac{dy}{dx} + y \sin x = \cos^2 x$

First divide the whole equation through by $\cos x$:

$$\frac{dy}{dx} + y \tan x = \cos x.$$

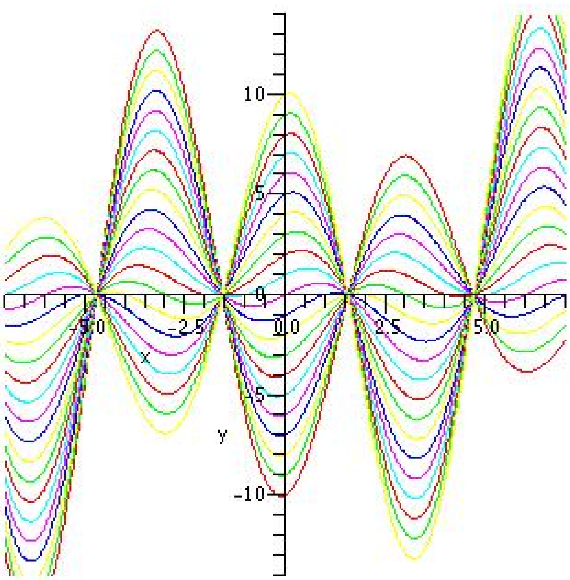
The integrating factor is

$$\mu = e^{\int \tan x \, dx} = e^{\ln|\sec x|} = |\sec x|.$$

Pick either of $\pm \sec x$; it makes no difference.

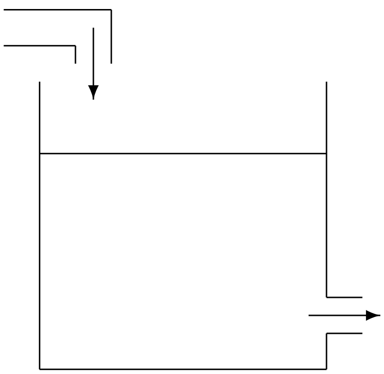
$$y = \frac{\int \cos x \sec x \, dx}{\sec x} = \frac{\int dx}{\sec x} = \frac{x + C}{\sec x} = x \cos x + C \cos x.$$

Graphs for Example 6, $C = -10, -9, \ldots, 9, 10$



Mixing Problems

Let the amount of solute in a mixing tank at time t be x. Let the volume of the solution in the tank at time t be V. More solution is poured into the tank from above, mixed with the solution in the tank, and then poured out from the bottom of the tank.



mixing tank

Let the rate of inflow be r_i with concentration c_i; let the rate of outflow be r_o with concentration c_o. Then

$$\frac{dx}{dt} = r_i c_i - r_o c_o$$
$$= r_i c_i - r_o \frac{x}{V}$$

We then have

$$\text{DE: } \frac{dx}{dt} + \frac{r_o}{V}x = r_i c_i; \text{ IC: } x = x_0, V = V_0 \text{ if } t = 0$$

There are two cases:

1. if $r_i = r_o$, then V is constant, and
$$\frac{dx}{dt} + \frac{r_o}{V}x = r_i c_i$$
can be solved by separating variables.
2. if $r_i \neq r_o$, then $V = V_0 + (r_i - r_o)t$ and
$$\frac{dx}{dt} + \frac{r_o}{V}x = r_i c_i$$
must be solved by other methods.

Example 1

Consider a reservoir with a volume of 1000 m^3 and an initial pollutant concentration of 1%. There is a daily inflow of 50 m^3 of water with a pollutant concentration of 0.05%. The daily outflow rate of the well-mixed water from the reservoir is also 50 m^3. How long will it take to reduce the pollutant concentration in the reservoir to 0.5% ?

Solution: We have $r_i = r_o = 50$, $V = 1000$, $c_i = 0.05\% = 0.0005$.

$$x_0 = 1\% \text{ of } 1000 = 10.$$

The pollutant concentration in the tank is 0.5% if

$$x = 0.5\% \text{ of } 1000 = 5.$$

Chapter 8: Differential Equations
8.1 Basic Ideas
8.2 Direction Fields
8.3 Separable Differential Equations
8.4 Special First-Order Differential Equations
8.5 Modeling with Differential Equations

$$\begin{aligned}
\frac{dx}{dt} + \frac{r_o}{V}x = r_i c_i \quad &\Rightarrow \quad \frac{dx}{dt} + \frac{50}{1000}x = 50(0.0005) \\
&\Rightarrow \quad \frac{dx}{dt} + \frac{1}{20}x = \frac{1}{40} \\
&\Rightarrow \quad \frac{dx}{dt} = -\frac{1}{40}(2x-1) \\
&\Rightarrow \quad \int \frac{1}{2x-1}\,dx = -\int \frac{1}{40}\,dt \\
&\Rightarrow \quad \frac{1}{2}\ln|2x-1| = -\frac{1}{40}t + C
\end{aligned}$$

$x_0 = 10 \Rightarrow C = \frac{1}{2}\ln 19$. Let $x = 5$ and solve for t :

$$\frac{1}{40}t = \frac{1}{2}\ln 19 - \frac{1}{2}\ln 9 \Rightarrow t = 20\ln\frac{19}{9} \approx 14.9$$

Alternate Solution to Example 1

$$\text{DE: } \frac{dx}{dt} + \frac{1}{20}x = \frac{1}{40};\, x_0 = 10 \text{ if } t = 0.$$

The integrating factor is

$$\mu = e^{\int \frac{1}{20}\, dt} = e^{t/20},$$

so

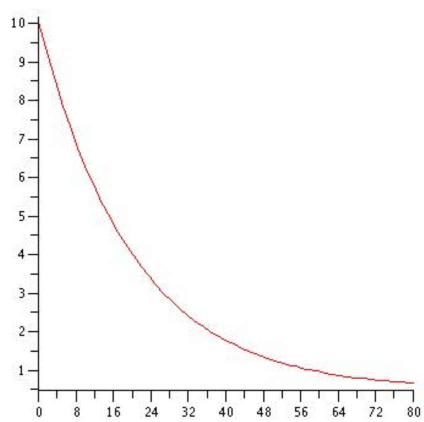
$$x = \frac{\int \frac{1}{40} e^{t/20}\, dt}{e^{t/20}} = e^{-t/20}\left(\frac{20}{40}e^{t/20} + C\right) = \frac{1}{2} + Ce^{-t/20}$$

$$x_0 = 10 \Rightarrow 10 = \frac{1}{2} + C \Rightarrow C = \frac{19}{2} \Rightarrow x = \frac{1}{2} + \frac{19}{2}e^{-t/20}.$$

Graph for Example 1

Letting $x = 5$ and solving for t gives the same answer as before:

$$5 = \frac{1}{2} + \frac{19}{2}e^{-t/20} \Rightarrow \frac{9}{2} = \frac{19}{2}e^{-t/20} \Rightarrow e^{t/20} = \frac{19}{9} \Rightarrow t = 20 \ln \frac{19}{9}.$$



Example 2

A 120 gal tank initially contains 90 lb of salt in 90 gal of water. Brine – salt water – containing 2 lb/gal of salt flows into the tank at a rate of 4 gal/min. The well-stirred mixture flows out at a rate of 3 gal/min. How much salt does the tank contain when it is full?

Solution: We have: $V_0 = 90, x_0 = 90, r_i = 4, r_o = 3, c_i = 2$. Thus $V = 90 + t$, and

$$\frac{dx}{dt} + \frac{3}{90+t}x = 8.$$

The integrating factor is

$$\mu = e^{\int \frac{3}{90+t}\,dt} = e^{3\ln(90+t)} = (90+t)^3.$$

Thus $x =$

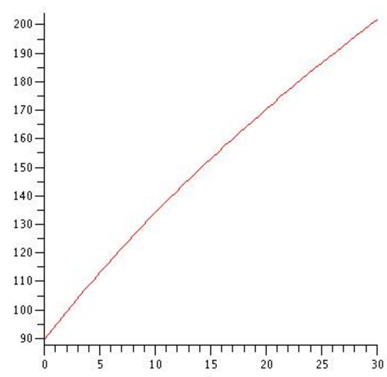
$$\frac{\int 8(90+t)^3\,dt}{(90+t)^3} = \frac{8}{(90+t)^3}\left(\frac{1}{4}(90+t)^4 + C\right) = 2(90+t) + \frac{8C}{(90+t)^3}$$

$$x_0 = 90 \Rightarrow 90 = 180 + \frac{8C}{90^3}$$

$$\Rightarrow 8C = -90^4$$

Therefore

$$x = 2(90+t) - \frac{90^4}{(90+t)^3}.$$



The tank is full when $V = 120 \Leftrightarrow 90 + t = 120 \Leftrightarrow t = 30$. Then

$$x = 2(90+30) - \frac{90^4}{(90+30)^3} \approx 202.$$

Motion With Resistance

Air resistance is typically proportional to the speed, in the direction opposed to motion. Including air resistance is much more realistic, but much messier! To find the equation of motion with resistance, we start with

$$ma = F = -mg - cv,$$

where g is the acceleration due to gravity, m is the mass of the falling object, and c is a positive constant.

$$\begin{aligned} ma = F = -mg - cv \quad &\Rightarrow \quad \frac{dv}{dt} = -g - kv, \text{ for } k = c/m \\ &\Rightarrow \quad \frac{dv}{dt} + kv = -g, \end{aligned}$$

which has integrating factor $\mu = e^{\int k\,dt} = e^{kt}$.

Terminal Velocity

Thus

$$v = \frac{-\int g e^{kt}\,dt}{e^{kt}} = \frac{-\frac{g}{k}e^{kt} + C}{e^{kt}} = -\frac{g}{k} + Ce^{-kt}.$$

If $v = v_0$ when $t = 0$, then

$$v_0 = -\frac{g}{k} + C \Leftrightarrow C = v_0 + \frac{g}{k}.$$

Thus

$$v = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}.$$

The terminal velocity is

$$\lim_{t\to\infty} v = \lim_{t\to\infty}\left(-\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}\right) = -\frac{g}{k}.$$

Since $v = \frac{dx}{dt}$, we can solve for x :

$$x = \int v\,dt = \int \left(-\frac{g}{k} + \left(v_0 + \frac{g}{k} \right) e^{-kt} \right) dt$$
$$= -\frac{g}{k} t - \frac{1}{k} \left(v_0 + \frac{g}{k} \right) e^{-kt} + A$$

If $x = x_0$ when $t = 0$, then

$$A = x_0 + \frac{1}{k} \left(v_0 + \frac{g}{k} \right),$$

as you can check. So the equation of motion with air resistance is

$$x = x_0 - \frac{g}{k} t + \frac{1}{k} \left(v_0 + \frac{g}{k} \right) \left(1 - e^{-kt} \right).$$

Example 3

Let

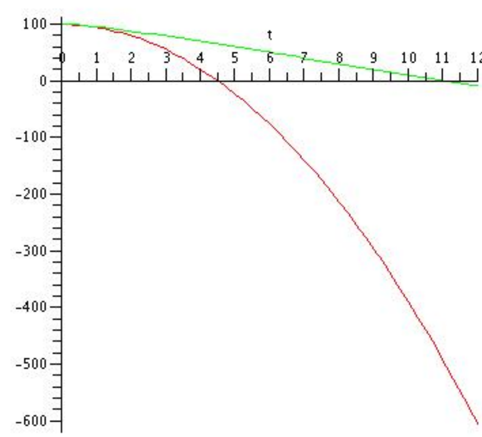
$$x_0 = 100, v_0 = 0, k = 1.$$

Let's compare the trajectories with and without air resistance. That is, compare graphically

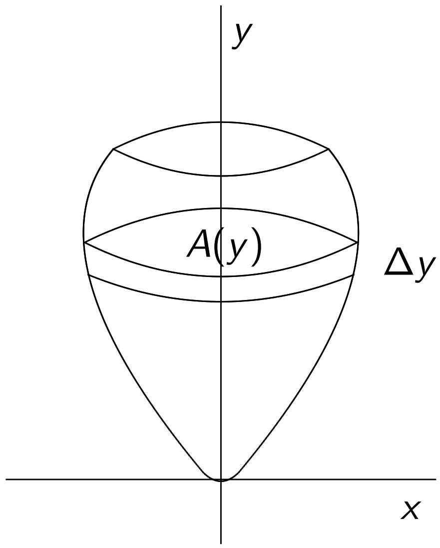
$$x = 100 - \frac{1}{2} g t^2$$

with

$$x = 100 - g \left(t + e^{-t} - 1 \right).$$



Derivation of Torricelli's Law



Suppose a tank is full of liquid. Torricelli's Law is a differential equation that describes how the liquid drains out of the tank. Let $A(y)$ be the cross-sectional area of the tank at height y above the bottom of the tank. Let Δy be the decrease in liquid level over a short time interval Δt. We have $\Delta V \approx A(y)\Delta y$. $\Delta V < 0$, since the liquid level is decreasing.

Torricelli's Law: $A(y)\frac{dy}{dt} = -a\sqrt{2gy}$

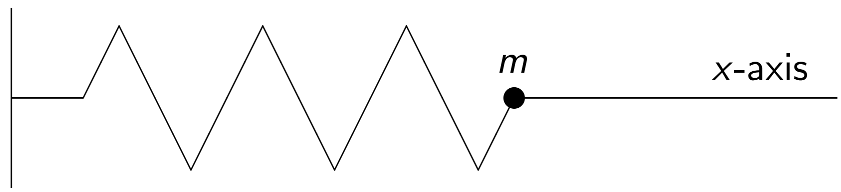
Let a be the area of a small exit hole at the bottom of the tank; let g be the acceleration due to gravity. In the absence of friction, a drop of liquid starting from rest at height y will fall with acceleration g and reach the exit hole with speed $v = \sqrt{2gy}$. So in the short time interval Δt, the volume of liquid leaving through the exit hole is $a\sqrt{2gy}\Delta t$. Thus

$$A(y)\Delta y \approx \Delta V \approx -a\sqrt{2gy}\Delta t \Leftrightarrow A(y)\frac{\Delta y}{\Delta t} \approx \frac{\Delta V}{\Delta t} \approx -a\sqrt{2gy}.$$

As $\Delta t \to 0$, we obtain

$$A(y)\frac{dy}{dt} = \frac{dV}{dt} = -a\sqrt{2gy} \Rightarrow A(y)\frac{dy}{dt} = -a\sqrt{2gy}.$$

Mass On A Spring; Mechanical Vibrations



1. x is the position of a mass on a spring at time t.
2. x is measured as displacement from the equilibrium position, $x = 0$.
3. m is the mass of the object on the spring.
4. c depends on the friction of the surrounding medium.
5. k is the spring constant, from Hooke's Law.

Differential Equation for a Mass on a Spring

We do a force analysis, considering the resisting force of the surrounding medium, with c representing the coefficient of friction; and the restoring force of the spring, with k representing the spring constant.

$$\begin{aligned} F = ma &= F_{\text{resisting}} + F_{\text{restoring}} \\ &= -cv - kx \\ \Leftrightarrow m\frac{d^2x}{dt^2} &= -c\frac{dx}{dt} - kx \\ \Leftrightarrow m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx &= 0, \text{ or } mx''(t) + cx'(t) + kx(t) = 0 \end{aligned}$$

This differential equation is second order, linear, homogeneous, with constant coefficients. We'll solve this later, in Chapter 16.