MAT187H1F Lec0101 Burbulla

Chapter 9 and Chapter 10 Lecture Notes

Spring 2017

What Is An Infinite Sequence?

An infinite sequence is a sequence of numbers, a_n, for which n is a non-negative integer:

$$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$$

Such a sequence can be represented by

$$\{a_n\}_{n=0}^{\infty}, \text{ or simply } \{a_n\}.$$

Warning: sometimes the sequence begins with a_1 instead of a_0. In either case, we call a_n the nth term of the sequence, even if the sequence begins with a_0. Then we call a_0 the 0th term.

Examples

Here are some sequences that are defined in terms of explicit formulas for the nth term:

1. $a_n = (-1)^n, n \geq 0$. The terms of this sequence are
$$1, -1, 1, -1, 1, \ldots$$
2. $b_n = 1/n, n \geq 1$. The terms of this sequence are
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$
3. $c_n = \sin(n\pi), n \geq 0$. The terms of this sequence are
$$0, 0, 0, 0, \ldots$$

The Limit of a Sequence

We say L is the limit of the sequence with n*th* term a_n, or that the sequence converges to L, if $\lim\limits_{n\to\infty} a_n = L$. For the three examples above:

1. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (-1)^n$ doesn't exist; $\{a_n\}_{n=0}^{\infty}$ doesn't converge.
2. $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$
3. $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} \sin(n\pi) = \lim\limits_{n\to\infty} 0 = 0$

NOTE: Suppose $f(x)$ is a function such that $f(n) = a_n$. If $\lim\limits_{x\to\infty} f(x) = L$, then the limit of the sequence is also L. But the converse is not in general true. See Example 3 above.

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Calculating the Limit of a Sequence

Depending on what the formula for the n*th* term of a sequence is, it could be very difficult to calculate the limit of the sequence. You may have to use L'Hopital's rule, the Squeeze Law, or some other techniques for limits that you saw in Calculus I. For example,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\pi/2 - \tan^{-1} n}{\cos\left(\frac{n+1}{n}\frac{\pi}{2}\right)} &= \lim_{n\to\infty} \frac{-\frac{1}{1+n^2}}{-\sin\left(\frac{n+1}{n}\frac{\pi}{2}\right)\left(\frac{-1}{n^2}\right)\left(\frac{\pi}{2}\right)} \\
&= -\frac{2}{\pi} \lim_{n\to\infty} \left(\frac{n^2}{n^2+1}\right) \csc\left(\frac{\pi}{2} \lim_{n\to\infty} \frac{n+1}{n}\right) \\
&= -\frac{2}{\pi} \cdot \csc\left(\frac{\pi}{2}\right) = -\frac{2}{\pi}
\end{aligned}$$

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A Sequence You Should be Familiar With

Somewhere in your mathematical past you should have seen the formula for e in terms of a sequence:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Starting with $n = 1$, the first few terms of this sequence are

$$2, 2.25, 2.370370370\ldots, 2.44140625, 2.48832$$

The $100th$ term in the sequence is $1.01^{100} \simeq 2.704813829\ldots$; the $1000th$ term in the sequence is $1.001^{1000} \simeq 2.716923932$. Correct to 20 digits, $e = 2.71828182845904523 54$

Recursive Sequences

Recall the formula from Newton's method for approximating solutions to the equation $f(x) = 0$:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ for } n = 0, 1, 2, 3, \ldots$$

$x_0, x_1, x_2, x_3, \ldots$ is an infinite sequence, and its limit is equal to an actual solution of the equation. But it is different from all the sequences we have looked at so far: we don't have an explicit formula for x_n; only a rule for calculating x_{n+1} in terms of x_n. Such a sequence is called a recursive sequence. In general, it can be very difficult to find the limit of a recursive sequence, or even to tell if it converges.

The Fibonacci Sequence

The Fibonacci sequence is an example of a recursive sequence:

$$F_0 = 1, F_1 = 1, F_{n+1} = F_n + F_{n-1}, \text{ for } n \geq 1.$$

The first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

The Fibonacci sequence diverges, since $F_n \to \infty$ as $n \to \infty$. Its not obvious, but it can be proved that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

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The Fibonacci Sequence and The Golden Ratio

The Fibonacci sequence is interesting, and shows up in many fascinating contexts. In addition it is related to the golden ratio because

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} = \Phi \simeq 1.618033989\ldots, \text{ the golden ratio.}$$

To see this, let L be the limit.

$$\begin{aligned}
& F_{n+1} = F_n + F_{n-1} \\
\Rightarrow\ & \frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} \Rightarrow \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n} \\
\Rightarrow\ & L = 1 + \frac{1}{L} \Rightarrow L^2 = L + 1 \Rightarrow L = \frac{1 \pm \sqrt{5}}{2}
\end{aligned}$$

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What Is An Infinite Series?

In six words: the sum of an infinite sequence. To be more precise: let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence; let

$$S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + a_2 + \cdots + a_N.$$

S_N is called the N*th* partial sum of the sequence $\{a_n\}_{n=0}^{\infty}$. If $\lim_{N\to\infty} S_N = S$, then we write

$$S = \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

and call it the sum of the infinite series. S may or may not exist.

Convergence and Divergence

If $\lim_{N\to\infty} S_N$ exists we say the infinite series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

converges; if $\lim_{N\to\infty} S_N$ doesn't exist we say the infinite series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

diverges. Whether or not an infinite series converges is one of the central themes of Chapters 9 and 10.

Example 1: $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots+\frac{1}{n(n+1)}+\cdots=1$

Suppose $a_n = \dfrac{1}{n(n+1)}$. What is $\displaystyle\sum_{n=1}^{\infty} a_n$?

Solution: calculate a few partial sums and look for a pattern.

$$S_1 = a_1 = \frac{1}{2};\;\; S_2 = a_1 + a_2 = \frac{1}{2}+\frac{1}{6} = \frac{2}{3};$$

$$S_3 = a_1 + a_2 + a_3 = \frac{1}{2}+\frac{1}{6}+\frac{1}{12} = \frac{3}{4}.$$

It looks like $S_N = \dfrac{N}{N+1}$. (It can be proved by induction.) Thus:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N}{N+1} = 1.$$

Example 2: a series that diverges.

Suppose $a_n = \ln\left(\dfrac{n+1}{n}\right)$. What is $\displaystyle\sum_{n=1}^{\infty} a_n$?

Solution: calculate a few partial sums and look for a pattern.

$$S_1 = a_1 = \ln 2;\;\; S_2 = a_1 + a_2 = \ln 2 + \ln\frac{3}{2} = \ln 3;$$

$$S_3 = a_1 + a_2 + a_3 = \ln 2 + \ln\frac{3}{2} + \ln\frac{4}{3} = \ln 4.$$

It looks like $S_N = \ln(N+1)$. (It can be proved by induction.) Thus:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \ln(N+1) = \infty.$$

Terminology

Definitions:

1. A sequence $\{a_n\}_{n=1}^{\infty}$ is called nondecreasing if
$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$
2. A sequence $\{a_n\}_{n=1}^{\infty}$ is called nonincreasing if
$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$$
3. A sequence that is either nonincreasing or nondecreasing is said to be monotonic.
4. A sequence $\{a_n\}$ is bounded if there is a number M such that $|a_n| < M$, for every n.

Testing for Monotonicity

The infinite sequence $\{a_n\}$ is

1. nondecreasing if $a_{n+1} - a_n \geq 0$,
   or $a_{n+1}/a_n \geq 1$ if $a_n > 0$ for all n.
2. nonincreasing if $a_{n+1} - a_n \leq 0$,
   or $a_{n+1}/a_n \leq 1$ if $a_n > 0$ for all n.

Additionally, if $f(n) = a_n$ then the infinite sequence $\{a_n\}$ is

1. nondecreasing if $f'(x) \geq 0$ for $x \geq 0$.
2. nonincreasing if $f'(x) \leq 0$ for $x \geq 0$.

Example 1

Consider the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$ with nth term $a_n = \frac{n}{n+1}$. Then

$$\begin{aligned} a_{n+1} - a_n &= \frac{n+1}{n+2} - \frac{n}{n+1} \\ &= \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+1)(n+2)} \\ &= \frac{1}{(n+1)(n+2)} > 0, \end{aligned}$$

so the sequence $\{a_n\}$ is nondecreasing.

Example 2

Consider the sequence with nth term $a_n = \sin\left(\frac{n\pi}{n+1}\right)$. From Example 1 we know that the terms of the sequence $\left\{\frac{n}{n+1}\right\}$ are increasing. In particular,

$$\frac{\pi}{2} \leq \frac{n\pi}{n+1} < \pi.$$

Since $\sin x$ is decreasing on the second quadrant, the sequence $\{a_n\}$ is nonincreasing. Or: if $f(x) = \sin\left(\frac{x\pi}{x+1}\right)$, then

$$f'(x) = \frac{\pi}{(x+1)^2} \cos\left(\frac{x\pi}{x+1}\right) < 0,$$

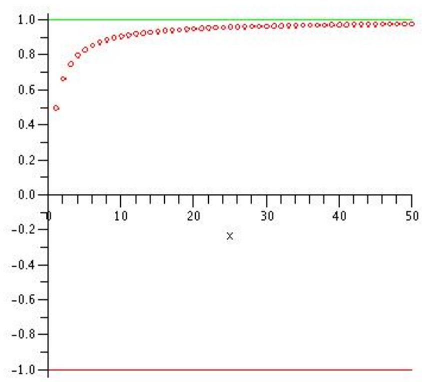
since $\pi/2 \leq x\pi/(x+1) < \pi$.

Bounded Monotonic Sequences

Theorem: Every bounded, monotonic sequence converges. That is

$$\lim_{n\to\infty} a_n$$

exists. (The proof of this result is beyond the scope of this course.)



But the picture to the left illustrates why the result is true, for a bounded, nondecreasing sequence. We are plotting (n, a_n) and assuming

$$|a_n| \leq 1,$$

for all n.

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Example 3: $\lim_{n\to\infty} \sqrt{2+\sqrt{2+\sqrt{2+\cdots}}} = 2$

Consider the recursively defined sequence

$$a_1 = \sqrt{2}, a_{n+1} = \sqrt{2+a_n}, \text{ for } n \geq 1.$$

This sequence is bounded (by 2) and strictly increasing:

1. $a_1 = \sqrt{2} < 2$; $a_n < 2 \Rightarrow a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$.
2. $a_n < a_{n+1} \Leftrightarrow a_n < \sqrt{2+a_n} \Leftrightarrow a_n^2 < 2 + a_n \Leftrightarrow a_n^2 - a_n - 2 < 0$ $\Leftrightarrow (a_n-2)(a_n+1) < 0$, which is true since $0 < a_n < 2$.

So the sequence has a limit, say L. We can find L as we did with the Fibonacci sequence: $\lim_{n\to\infty} a_{n+1} = \sqrt{2 + \lim_{n\to\infty} a_n}$
$\Rightarrow L = \sqrt{2+L} \Rightarrow L^2 - L - 2 = 0 \Rightarrow L = 2$, since $a_n > 0$.

Example 1: $1 + 1/2 + 1/4 + 1/8 + 1/16 + \cdots = 2$

Let $a_n = \left(\frac{1}{2}\right)^n$.

$$S_1 = a_0 + a_1 = 1 + \frac{1}{2} = \frac{3}{2}; S_2 = S_1 + a_2 = \frac{3}{2} + \frac{1}{4} = \frac{7}{4};$$

$$S_3 = S_2 + a_3 = \frac{7}{4} + \frac{1}{8} = \frac{15}{8}; S_4 = S_3 + a_4 = \frac{15}{8} + \frac{1}{16} = \frac{31}{16}.$$

You can prove (by induction) that

$$S_N = \frac{2^{N+1} - 1}{2^N}; \text{ and so } \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(2 - \frac{1}{2^N}\right) = 2.$$

Sum of an Infinite Geometric Series

Example 1 is an example of the general infinite geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots$$

What is the sum of this series? Basic factoring implies

$$r^{N+1} - 1 = (r-1)(r^N + r^{N-1} + \cdots + r + 1) \Rightarrow S_N = \frac{r^{N+1} - 1}{r - 1}.$$

Now $\lim_{N\to\infty} r^{N+1} = 0 \Leftrightarrow |r| < 1$; Thus

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots = \lim_{N\to\infty} S_N = \frac{1}{1-r}, \text{ if } |r| < 1.$$

Example 2

1. $$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} = \frac{2}{2-1} = 2$$
2. $$\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n = \frac{1}{1+2/3} = \frac{3}{3+2} = \frac{3}{5}$$
3. $\sum_{n=0}^{\infty} (-1)^n$ diverges, since $|-1| = 1$; even though you might be tempted to say $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = 0$.

More About the Series $1 - 1 + 1 - 1 + 1 - 1 + \ldots$

1. Writing the series as $(1-1) + (1-1) + (1-1) + \cdots$ suggests its sum should be 0.
2. Writing the series as $1 - (1-1) - (1-1) - (1-1) - \cdots$ suggests its sum should be 1.
3. From our definition, the partial sums S_n are 1 or 0. In 1713 Leibniz suggested the sum of the series should be the average of 0 and 1, namely $1/2$.
4. If you use the formula for the sum of an infinite series with common ratio $r = -1$ you get
$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \sum_{n=0}^{\infty} (-1)^n = \frac{1}{1-(-1)} = \frac{1}{2}$$

Playing Around Some More

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \quad \Rightarrow \quad \frac{d}{dr}\left(\frac{1}{1-r}\right) = \sum_{n=0}^{\infty} \frac{d}{dr} r^n \Rightarrow \frac{1}{(1-r)^2} = \sum_{n=0}^{\infty} n r^{n-1}$$

$$\Rightarrow \quad \frac{1}{(1-r)^2} = 1 + 2r + 3r^2 + 4r^3 + 5r^4 + 6r^5 + \cdots$$

$$r = -1 \quad \Rightarrow \quad \frac{1}{4} = 1 - 2 + 3 - 4 + 5 - 6 + \cdots$$

$$\Rightarrow \quad \frac{1}{4} = (1-2) + (3-4) + (5-6) + \cdots$$

$$\Rightarrow \quad \frac{1}{4} = (-1) + (-1) + (-1) + \cdots = -\infty\ ??$$

The moral is: results about series only make sense if they converge!

Example 3: A Telescoping Series

This is another kind of series in which you can calculate S_N :

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, for which $a_n = \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$.

$$\begin{aligned} S_N &= a_1 + a_2 + \cdots + a_N \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{N} - \frac{1}{N+1} \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1 - \frac{1}{N+1}\right) = 1$$

nth Term Test For Divergence

If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. **Proof:** $a_n = S_n - S_{n-1}$; so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

This result can be restated as follows: If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges. Hence its name! But *not* as: If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges.

Example 1

1. $\sum_{n=1}^{\infty} \frac{n}{3n+1} = \frac{1}{4} + \frac{2}{7} + \frac{3}{10} + \frac{4}{13} + \cdots$ diverges, since

$$\lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \neq 0.$$

2. $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges, since

$$\lim_{n\to\infty} (-1)^n$$

does not exist.

Example 2: The Harmonic Series Diverges

The harmonic series is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

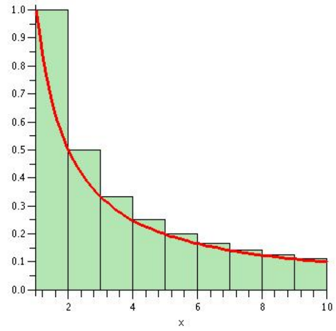
It diverges, even though

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0.$$

This shows that the converse of the n*th* term test for divergence is **not** true. In other words, the condition that the n*th* term of a series goes to zero, is only a necessary condition for its convergence, not a sufficient condition.

Proof that The Harmonic Series Diverges

Let $f(x) = 1/x$; the Riemann sum of f on the interval $[1, N+1]$ determined by N subintervals, with x_i^* chosen to be the left endpoint of each subinterval, is $1 + 1/2 + 1/3 + \cdots + 1/N = S_N$.



From the diagram,

$$\begin{aligned} S_N &> \int_1^{N+1} \frac{1}{x}\,dx \\ &= [\ln x]_1^{N+1} \\ \Rightarrow \lim_{N\to\infty} S_N &\geq \lim_{N\to\infty} \ln(N+1) \\ &= \infty \end{aligned}$$

Termwise Addition and Multiplication of Infinite Series

If

$$A = \sum_{n=0}^{\infty} a_n \text{ and } B = \sum_{n=0}^{\infty} b_n$$

are both convergent series; then both

$$\sum_{n=0}^{\infty} (a_n + b_n) \text{ and } \sum_{n=0}^{\infty} c a_n$$

are also convergent, with

$$\sum_{n=0}^{\infty} (a_n + b_n) = A + B \text{ and } \sum_{n=0}^{\infty} c\, a_n = c\, A.$$

Example 3

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{3}{4^n} - \frac{2}{5^{n-1}} \right) &= \sum_{n=1}^{\infty} \frac{3}{4^n} - \sum_{n=1}^{\infty} \frac{2}{5^{n-1}} \\
&= \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{4^{n-1}} - 2 \sum_{n=1}^{\infty} \frac{1}{5^{n-1}} \\
(\text{let } k = n-1) \quad &= \frac{3}{4} \sum_{k=0}^{\infty} \frac{1}{4^k} - 2 \sum_{k=0}^{\infty} \frac{1}{5^k} \\
&= \frac{3}{4} \left(\frac{1}{1-1/4} \right) - 2 \left(\frac{1}{1-1/5} \right) = -\frac{3}{2}
\end{aligned}$$

Statement of the Integral Test

Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing, positive term sequence. That is,

$$a_n > a_{n+1} \text{ and } a_n > 0.$$

Let $f(x)$ be a decreasing $(f'(x) < 0)$ positive function $(f(x) > 0)$ such that

$$f(n) = a_n.$$

Then

$$\sum_{n=1}^{\infty} a_n \text{ converges } \Leftrightarrow \int_1^{\infty} f(x)\,dx \text{ converges.}$$

Example 4

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, because

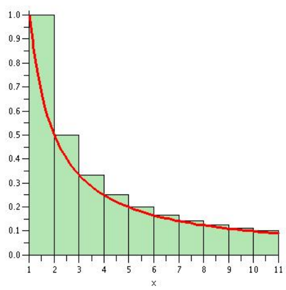
$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1.$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, because

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \left[2\sqrt{x}\right]_1^b = \lim_{b\to\infty} \left(2\sqrt{b} - 2\right) = \infty.$$

Proof of Integral Test: Suppose $\int_1^{\infty} f(x)\,dx$ Diverges

Consider a regular partition of $[1, n+1]$ into n subintervals; take x_i^* to be the left endpoint of each subinterval. The Riemann sum is $f(1) + f(2) + \cdots + f(n) = a_1 + a_2 + \cdots + a_n = S_n$.

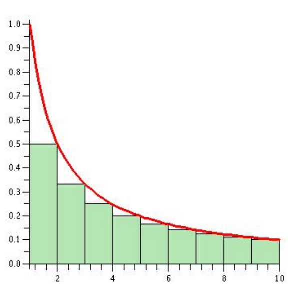


$$\begin{aligned} S_n &> \int_1^{n+1} f(x)\,dx \\ \Rightarrow \lim_{n\to\infty} S_n &\geq \int_1^{\infty} f(x)\,dx = \infty \end{aligned}$$

So $\sum_{n=1}^{\infty} a_n$ diverges.

Proof of Integral Test: Suppose $\int_1^{\infty} f(x)\,dx$ Converges

Consider a regular partition of $[1, n]$ into $n-1$ subintervals; take x_i^* to be the right endpoint of each subinterval. The Riemann sum is $f(2) + f(3) + \cdots + f(n) = a_2 + a_3 + \cdots + a_n = S_n - a_1$.



$$\begin{aligned} S_n - a_1 &< \int_1^{n} f(x)\,dx \\ \Rightarrow S_n &< a_1 + \int_1^{\infty} f(x)\,dx \end{aligned}$$

So the sequence $\{S_n\}_{n=1}^{\infty}$ is bounded, and increasing. (Why?) So the sequence of partial sums converges.

p-Series

The two series in Example 1 are both examples of what are called p-series. A p-series is

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

It converges if and only if $p > 1$. Why? If $p > 1$, then

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^b = 0 + \frac{1}{p-1}.$$

The harmonic series, which diverges, is the special case when $p = 1$. If $p < 1$ then the p-series diverges as well, by the integral test, as you can check.

Examples of p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \text{ is not known}$$

Integral Test Remainder Form

Suppose the series $S = \sum_{n=1}^{\infty} a_n$ converges by the integral test. Let

$$R_n = S - S_n = a_{n+1} + a_{n+2} + \cdots .$$

R_n is called the remainder term, or error term. It can be shown (see book) that

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx.$$

Example 5

Consider the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. How many terms must be added up to approximate the sum correctly to within 10^{-4}? **Solution:**

$$\begin{aligned} R_n &\le \int_n^{\infty} \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_n^b = \frac{1}{n} \\ &\le 10^{-4} \Rightarrow n \ge 10^4 \end{aligned}$$

So you have to add up 10,000 terms.

Approximating π

Let's use the formula from above:

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945}.$$

Let $f(x) = 1/x^6$, and take $n = 10$. We know

$$\int_{11}^{\infty} \frac{1}{x^6}\, dx \le \frac{\pi^6}{945} - S_{10} \le \int_{10}^{\infty} \frac{1}{x^6}\, dx.$$

On the next slide I will fill in the values of all these expressions, and then use them to approximate π.

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$$\int_{11}^{\infty} \frac{1}{x^6}\, dx \le \frac{\pi^6}{945} - S_{10} \le \int_{10}^{\infty} \frac{1}{x^6}\, dx$$

$$\Leftrightarrow \quad \lim_{b\to\infty} \left[-\frac{x^{-5}}{5}\right]_{11}^{b} \le \frac{\pi^6}{945} - S_{10} \le \lim_{b\to\infty} \left[-\frac{x^{-5}}{5}\right]_{10}^{b}$$

$$\Leftrightarrow \quad \frac{1}{805\,255} \le \frac{\pi^6}{945} - S_{10} \le \frac{1}{500\,000}$$

$$\Leftrightarrow \quad \frac{1}{805\,255} + S_{10} \le \frac{\pi^6}{945} \le \frac{1}{500\,000} + S_{10}$$

$$\Leftrightarrow \quad \frac{945}{805\,255} + 945 \cdot S_{10} \le \pi^6 \le \frac{945}{500\,000} + 945 \cdot S_{10}$$

$$\Leftrightarrow \quad 3.141592496\cdots \le \pi \le 3.141592886\ldots$$

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The Ratio Test, for Positive Term Series

The ratio test is one of the most useful convergence tests.

Let $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

1. *If* $\rho < 1$, *then* $\sum a_n$ *converges.*
2. *If* $\rho > 1$, *then* $\sum a_n$ *diverges.*
3. *If* $\rho = 1$, *then the ratio test is inconclusive.*

Sketch of Proof: Since $a_{n+1} \simeq \rho\, a_n$, the series $\sum a_n$ is basically a geometric series with constant ratio ρ.

Example 1

Does the series $\sum_{n=2}^{\infty} \frac{3^n}{n^2 \ln n}$ converge or diverge? **Solution:** Let $a_n = \frac{3^n}{n^2 \ln n}$. Try the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{3^{n+1}\, n^2\, \ln n}{3^n\,(n+1)^2\, \ln(n+1)} \\
&= 3 \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \\
&= 3\cdot 1\cdot 1 = 3 > 1
\end{aligned}$$

So the original series diverges, by the ratio test.

Example 2

Does the series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

converge or diverge? **Solution:** Let $a_n = \dfrac{n!}{n^n}$. Try the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{n^n\,(n+1)!}{(n+1)^{(n+1)}\,n!} = \lim_{n\to\infty} \frac{n^n\,(n+1)\,n!}{(n+1)^n\,(n+1)\,n!} \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e} < 1
\end{aligned}$$

So the original series converges, by the ratio test.

The Root Test, for Positive Term Series

The root test is very similar to the ratio test.

Let $\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$.

1. *If* $\rho < 1$, *then* $\sum a_n$ *converges.*
2. *If* $\rho > 1$, *then* $\sum a_n$ *diverges.*
3. *If* $\rho = 1$, *then the root test is inconclusive.*

Sketch of Proof: Since $a_n \simeq \rho^n$, the series $\sum a_n$ is basically a geometric series with constant ratio ρ.

Example 3

Does the series

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n-1} \right)^n$$

converge or diverge? **Solution:** Let $a_n = \left(\frac{n}{2n-1} \right)^n$. Try the root test:

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \frac{n}{2n-1} \\ &= \frac{1}{2} < 1 \end{aligned}$$

So the original series converges, by the root test.

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Example 4

Does the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{\tan^{-1} n} \right)^n$$

converge or diverge? **Solution:** Let $a_n = \left(\frac{1}{\tan^{-1} n} \right)^n$. Try the root test:

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{a_n} &= \lim_{n\to\infty} \frac{1}{\tan^{-1} n} \\ &= \frac{2}{\pi} < 1 \end{aligned}$$

So the original series converges, by the root test.

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The Comparison Test

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be positive term sequences. That is $a_n > 0$ and $b_n > 0$, Then:

1. If $a_n \leq b_n$ for "all n" and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges too.
2. If $a_n \geq b_n$ for "all n" and $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges too.

Note: for "all n" means that there can be finitely many exceptions.

Proof of Part 1

Let $S_N = a_1 + a_2 + \cdots + a_N$ be the N-th partial sum of $\sum a_n$. Then $\{S_N\}_{N=1}^{\infty}$ is a bounded, increasing sequence. Why?

1. It is increasing since $a_{n+1} > 0 \Rightarrow S_{N+1} = S_N + a_{N+1} > S_N$.
2. It is bounded by $\displaystyle\sum_{n=1}^{\infty} b_n$ since

$$S_N < b_1 + b_2 + \cdots + b_N < \sum_{n=1}^{\infty} b_n.$$

Thus $\displaystyle\lim_{N\to\infty} S_N$ exists; that is, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Example 5

Does the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge or diverge? Let

$$a_n = \frac{1}{n^2+1};\, b_n = \frac{1}{n^2}.$$

We have

$$a_n < b_n, \text{ since } \frac{1}{n^2+1} < \frac{1}{n^2} \Leftrightarrow n^2 < n^2+1.$$

Since $\sum b_n$ converges, (why?) and $a_n < b_n$, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges too, by the comparison test.

Example 6

Does the series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$ converge or diverge? Let

$$a_n = \frac{1}{\sqrt{n^2-1}};\, b_n = \frac{1}{n}.$$

We have

$$a_n > b_n, \text{ since } \frac{1}{\sqrt{n^2-1}} > \frac{1}{n} \Leftrightarrow n^2 > n^2-1.$$

Since $\sum b_n$ diverges, (why?) and $a_n > b_n$, $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$ diverges too, by the comparison test.

Comments About the Comparison Test

1. We almost always take $\sum b_n$ to be a p-series, since it is easy to tell if a p-series converges or diverges.
2. The problem with the comparison test is to find an inequality such that $a_n < b_n$, or $a_n > b_n$. Things won't always be as easy as in Examples 1 and 2.
3. The most inconvenient thing about the comparison test is if the inequality goes the "wrong way." For example, $\displaystyle\sum_{n=1} \frac{1}{n+1}$ diverges by the integral test, but if you try to compare it with the p-series, $\displaystyle\sum_{n=1} \frac{1}{n}$, the inequality $\dfrac{1}{n+1} < \dfrac{1}{n}$ is useless!

The Limit Comparison Test

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be positive term sequences. Let

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

If $0 < L < \infty$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges } \Leftrightarrow \sum_{n=1}^{\infty} b_n \text{ converges.}$$

Sketch of proof: For very large values of n

$$a_n \simeq L\,b_n \Rightarrow \sum a_n \simeq L \sum b_n.$$

Thus one series converges if and only if the other one does.

Example 7

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^{3/2}+n}{5n^{5/2}-n+14}$ converge or diverge? Let

$$a_n = \frac{n^{3/2}+n}{5n^{5/2}-n+14};\ b_n = \frac{1}{n}.$$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^{3/2}+n}{5n^{5/2}-n+14}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{5/2}+n^2}{5n^{5/2}-n+14} = \frac{1}{5}.$$

Since $\sum b_n$ diverges, (why?) $\displaystyle\sum_{n=1}^{\infty} \frac{n^{3/2}+n}{5n^{5/2}-n+14}$ diverges too, by the limit comparison test.

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Example 8

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2}$ converge or diverge? Let

$$a_n = \frac{\tan^{-1} n}{n^2};\ b_n = \frac{1}{n^2}.$$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\tan^{-1} n}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \tan^{-1} n = \frac{\pi}{2}.$$

Since $\sum b_n$ converges, (why?) $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^2}$ converges too, by the limit comparison test.

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What About Sequences With Negative Terms?

The previous convergence tests, the integral test, the comparison tests, the ratio test, and the root test, all assume that $a_n > 0$. What if some of the terms a_n are negative? One type of series that includes negative terms, and for which there is a convergence test, is an alternating series. Let $a_n > 0$; the series

$$\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots$$

is called an alternating series. Its the positive and negative signs that alternate.

Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots$$

converges if

1. the sequence $\{a_n\}_{n=0}^{\infty}$ is nonincreasing; that is, $a_{n+1} \leq a_n$.
2. $\lim_{n\to\infty} a_n = 0$.

Proof: see book.

Example 1

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$$

converges by the alternating series test:

$$a_n=\frac{1}{n}$$

and it is pretty obvious that

1. $a_{n+1}<a_n \Leftrightarrow \frac{1}{n+1}<\frac{1}{n} \Leftrightarrow n<n+1.$
2. $\lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty}\dfrac{1}{n}=0.$

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Example 2

$$\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{n\cdot\pi}{n+1}\right)$$

converges by the alternating series test.

$$a_n=\sin\left(\frac{n\cdot\pi}{n+1}\right)>0 \text{ since } n\geq 1\Rightarrow\frac{\pi}{2}\leq\frac{n\cdot\pi}{n+1}<\pi;$$

and we have

1. $a_{n+1}<a_n$ since $\sin x$ is decreasing on $[\pi/2,\pi]$.
2. $\lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty}\sin\left(\dfrac{n\cdot\pi}{n+1}\right)=\sin\pi=0.$

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Alternating Series Remainder Term

Suppose the alternating series

$$\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots$$

converges to S. Let $R_n = S - S_n$. which is called the remainder term, or the error term. Then

$$0 < |R_n| < a_{n+1},$$

which is the simplest approximation for an error term that you'll ever see.

Example 3

Later we'll see that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}.$$

How many terms of this alternating series have to be added to approximate $\frac{\pi}{4}$ correctly to within 10^{-4}? **Solution:** $a_n = \frac{1}{2n+1}$.

$$|R_n| < a_{n+1} = \frac{1}{2n+3} < 10^{-4} \Rightarrow 2n+3 > 10^4 \Rightarrow n > 4\,998.5$$

So you need to add up the first 4,999 terms.

Connection Between $\sum a_n$ and $\sum |a_n|$

$|a_n|$ is always positive. So the question arises: what can we say about the convergence of $\sum a_n$ if we know whether the series $\sum |a_n|$ converges or diverges? The full answer to this question is very tricky! It's at the heart of what makes infinite series a hard topic. To begin:

$$-|a_n| \leq a_n \leq |a_n| \quad \Rightarrow \quad 0 \leq a_n + |a_n| \leq 2|a_n|$$

This means that the series $\sum(a_n + |a_n|)$ is a positive term series. Moreover, if $\sum |a_n|$ converges, then so does $\sum(a_n + |a_n|)$, by the comparison test. Then $\sum a_n = \sum(a_n + |a_n|) - \sum |a_n|$ also converges. So one way to show $\sum a_n$ converges, is to show $\sum |a_n|$ converges.

Absolute and Conditional Convergence

For any series $\sum a_n$, one of three possibilities occurs:

1. $\sum |a_n|$ and $\sum a_n$ both converge. In this case the series $\sum a_n$ is said to converge absolutely.
2. $\sum |a_n|$ diverges but $\sum a_n$ converges. In this case the series $\sum a_n$ is said to converge conditionally.
3. $\sum |a_n|$ and $\sum a_n$ both diverge. In this case the series $\sum a_n$ is said to diverge.

Infinite series would be easy if Case 2 didn't exist! but it does.

Example 4

Let $b_n = \dfrac{(-1)^{n+1}}{n}$. As we saw in Example 1,

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the alternating series test. But

$$\sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} \left|(-1)^{n+1}\frac{1}{n}\right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges. (Why?) Thus $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

Example 5

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

converges absolutely. To show this all we have to check is that

$$\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n^2}\right|$$

converges. We have $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a p-series, with $p = 2 > 1$, and so converges.

The Ratio Test for Absolute Convergence

A variation of the ratio test can be applied to an arbitrary series, as long as $a_n \neq 0$.

Let $\rho = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$.

1. *If* $\rho < 1$, *then* $\sum a_n$ *converges absolutely.*
2. *If* $\rho > 1$, *then* $\sum a_n$ *diverges.*
3. *If* $\rho = 1$, *then the ratio test is inconclusive.*

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The Root Test for Absolute Convergence

Similarly, a variation of the root test can be applied to an arbitrary series, as long as $a_n \neq 0$.

Let $\rho = \lim\limits_{n\to\infty} \sqrt[n]{|a_n|}$.

1. *If* $\rho < 1$, *then* $\sum a_n$ *converges absolutely.*
2. *If* $\rho > 1$, *then* $\sum a_n$ *diverges.*
3. *If* $\rho = 1$, *then the root test is inconclusive.*

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Example 6

Does the series

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$$

converge or diverge? **Solution:** Let $a_n = \dfrac{(-2)^n}{n!}$; which is negative if n is odd, and positive if n is even. Try the ratio test for absolute convergence:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2^{n+1}\, n!}{2^n\,(n+1)!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

So the original series converges, by the ratio test for absolute convergence.

Example 7

Does the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n\cos n}{2n-1}\right)^n$ converge or diverge? **Solution:** Let $a_n = \left(\dfrac{n\cos n}{2n-1}\right)^n$, which is not always positive. Use the comparison test and the root test for positive-term series:

$$|a_n| = \left| \left(\frac{n\,\cos n}{2n-1}\right)^n \right| = |\cos n|^n \left(\frac{n}{2n-1}\right)^n \leq \left(\frac{n}{2n-1}\right)^n.$$

By Example 3 in Section 9.5 the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n-1}\right)^n$ converges, so the original series converges absolutely, by comparison.

Example 7 Is Very Tricky!

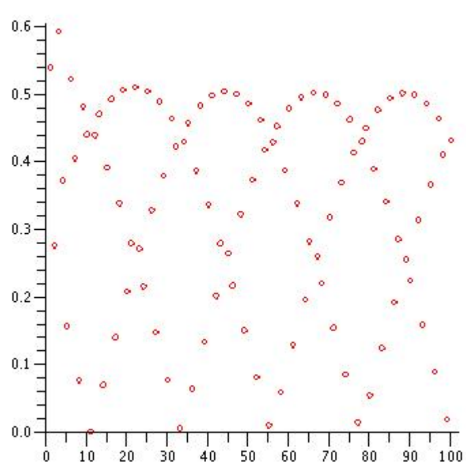
Does the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n\cos n}{2n-1}\right)^n$ converge or diverge?

If you simply started with the root test for absolute convergence, you would have a problem. Let $a_n = \left(\dfrac{n\cos n}{2n-1}\right)^n$.

$$\sqrt[n]{|a_n|} = \frac{n\,|\cos n|}{2n-1} \le \frac{n}{2n-1} \Rightarrow \lim_{n\to\infty}\sqrt[n]{|a_n|} \le \frac{1}{2} < 1.$$

This would imply that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n\cos n}{2n-1}\right)^n$ converges **if** $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|}$ exists.

In fact, $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{n\,|\cos n|}{2n-1}$ does not exist. Here's a plot of the first 100 terms in the sequence $\{\sqrt[n]{|a_n|}\}_{n=1}^{\infty}$:



Notation for Higher Order Derivatives

The n-th derivative of $f(x)$ at $x = a$ is denoted by $f^{(n)}(a)$. So

$$f^{(0)}(a) = f(a), f^{(1)}(a) = f'(a), f^{(2)}(a) = f''(a), f^{(3)}(a) = f'''(a).$$

For fourth derivatives or higher, we use $f^{(n)}(a)$ exclusively. Besides, would it be

$$f''''(a) \text{ or } f^{iv}(a)??$$

It gets tiresome to write all those primes, and switching into Roman numerals for derivatives would be folly!

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What Are Maclaurin and Taylor Polynomials?

The n-th degree Taylor Polynomial of $f(x)$ at $x = a$ is defined to be the n-th degree – or possibly lower – polynomial $P_n(x)$ such that

$$P_n^{(k)}(a) = f^{(k)}(a), \text{ for } k = 0, 1, 2, \ldots, n.$$

That is, P_n must satisfy $n + 1$ conditions:

$$P_n(a) = f(a), P_n'(a) = f'(a), P_n''(a) = f''(a), \ldots, P_n^{(n)}(a) = f^{(n)}(a).$$

If $a = 0$ then $P_n(x)$ is called the n-th degree Maclaurin polynomial.

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$P_1(x)$ is the Tangent Line Approximation to $f(x)$ at $x = a$

$P_1(x)$ is just the tangent line approximation to $f(x)$ at $x = a$: namely

$$P_1(x) = f(a) + f'(a)(x - a),$$

because

$$P_1(a) = f(a) \text{ and } P_1'(a) = f'(a).$$

Higher degree Taylor polynomials can also be used to approximate $f(x)$; hopefully the higher the degree, the better the approximation. First we have to find the formula for $P_n(x)$.

Formula for $P_2(x)$

Let $P_2(x) = P_1(x) + c(x - a)^2$, for some constant c. Then

$$\begin{aligned} P_2(a) &= P_1(a) + c(a - a)^2 = f(a) + 0 = f(a); \\ P_2'(a) &= P_1'(a) + 2c(a - a) = f'(a) + 0 = f'(a). \end{aligned}$$

Finally,

$$P_2^{(2)}(a) = f^{(2)}(a) \Leftrightarrow 2c = f^{(2)}(a) \Leftrightarrow c = \frac{f^{(2)}(a)}{2}.$$

So the formula for $P_2(x)$ is

$$P_2(x) = f(a) + f'(a)(x - a) + \frac{f^{(2)}(a)}{2}(x - a)^2.$$

Formula for $P_3(x)$

Let $P_3(x) = P_2(x) + d\,(x-a)^3$. Then automatically,

$$P_3(a) = f(a), P_3'(a) = f'(a) \text{ and } P_3^{(2)}(a) = f^{(2)}(a).$$

To find d, we use

$$P_3^{(3)}(a) = f^{(3)}(a) \Leftrightarrow 6d = f^{(3)}(a) \Leftrightarrow d = \frac{f^{(3)}(a)}{6} = \frac{f^{(3)}(a)}{3!}$$

So the formula for $P_3(x)$ is

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3.$$

Formula for $P_n(x)$

By the now the pattern is pretty clear:

$$\begin{aligned} & P_n(x) \\ = \; & f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ = \; & \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \end{aligned}$$

The formula for the n-th degree Maclaurin polynomial is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k.$$

Example 1

To compute Taylor Polynomials you need to compute lots of derivatives. Let's start with $f(x) = e^x$, at $a = 0$ for which

$$f^{(n)}(x) = e^x \text{ and } f^{(n)}(0) = e^0 = 1.$$

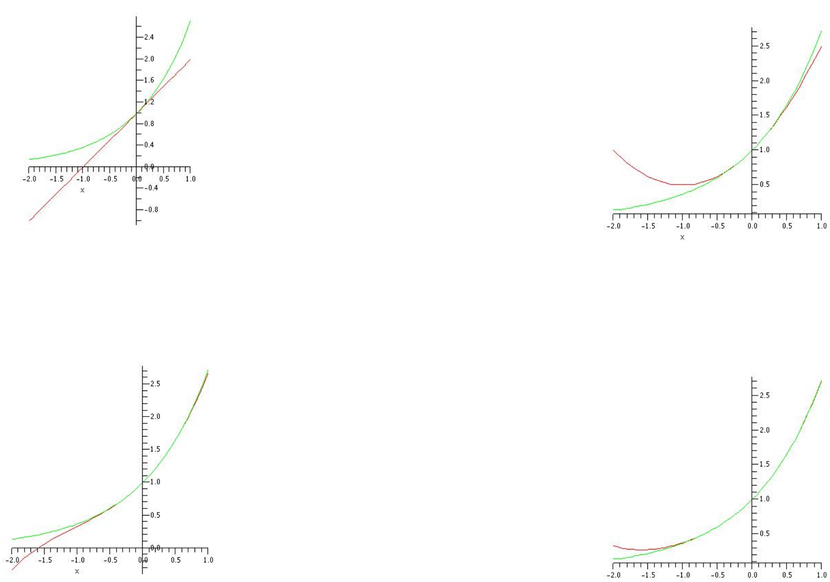
So

$$P_1(x) = 1 + x; P_2(x) = 1 + x + \frac{x^2}{2}, P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!};$$

and in general

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}.$$

The First Four Maclaurin Polynomials for $f(x) = e^x$



Example 2: Maclaurin Polynomials for $f(x) = \sin x$

$$f'(x) = \cos x;\, f^{(2)}(x) = -\sin x;\, f^{(3)}(x) = -\cos x;\, f^{(4)}(x) = \sin x.$$

$$f'(0) = 1;\, f^{(2)}(0) = 0;\, f^{(3)}(0) = -1;\, f^{(4)}(0) = 0.$$

The pattern is: $f^{(4k+1)}(0) = 1;\, f^{(2k)}(0) = 0;\, f^{(4k+3)}(0) = -1$. So

$$P_1(x) = P_2(x) = x;\, P_3(x) = P_4(x) = x - \frac{x^3}{3!},$$

and

$$P_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

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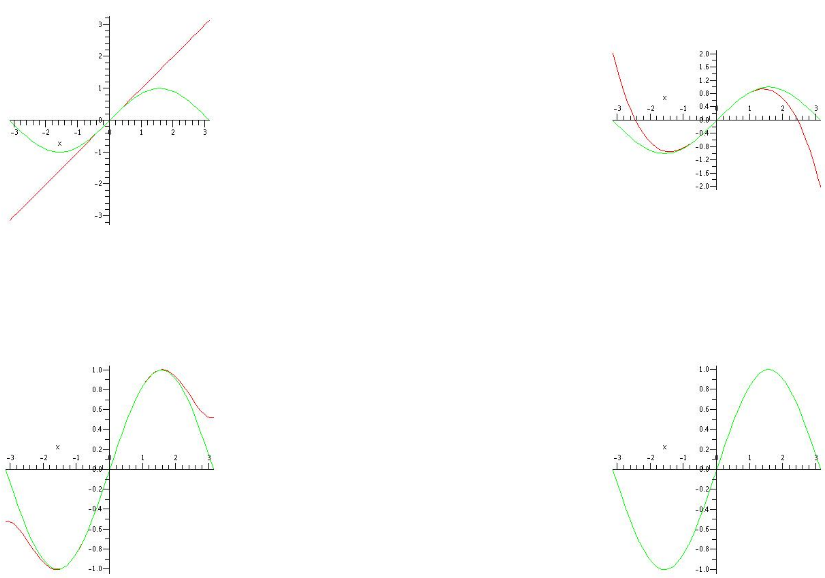
Approximating $\sin x$ with $P_1(x), P_3(x), P_5(x), P_{13}(x)$



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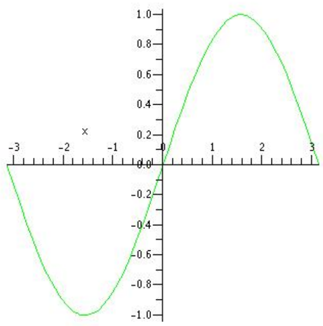
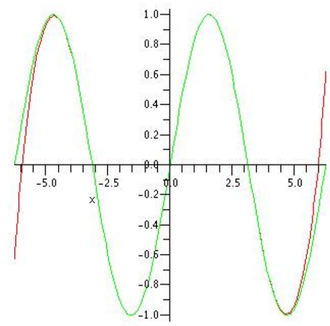
Approximating $\sin x$ with $P_{13}(x)$ on Different Intervals.

$$P_{13}(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \frac{x^{11}}{39916800} + \frac{x^{13}}{6227020800}$$





Example 3: Maclaurin Polynomials for $f(x) = \cos x$

$$f'(x) = -\sin x;\ f^{(2)}(x) = -\cos x;\ f^{(3)}(x) = \sin x;\ f^{(4)}(x) = \cos x.$$

$$f'(0) = 0;\ f^{(2)}(0) = -1;\ f^{(3)}(0) = 0;\ f^{(4)}(0) = 1.$$

The pattern is: $f^{(4k)}(0) = 1; f^{(2k+1)}(0) = 0; f^{(4k+2)}(0) = -1$. So

$$P_2(x) = P_3(x) = 1 - \frac{x^2}{2};\ P_4(x) = P_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

and

$$P_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Example 4: Maclaurin Polynomials for $f(x) = \sqrt{1+x}$

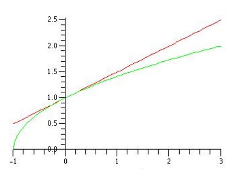
$$f'(x) = \frac{1}{2}(1+x)^{-1/2};\ f^{(2)}(x) = -\frac{1}{4}(1+x)^{-3/2};\ f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2}.$$

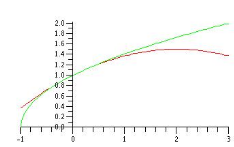
$$f'(0) = \frac{1}{2};\ f^{(2)}(0) = -\frac{1}{4};\ f^{(3)}(0) = \frac{3}{8};\ f^{(4)}(0) = -\frac{15}{16};\ f^{(5)}(0) = \frac{105}{32}.$$

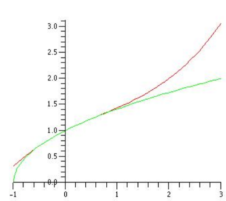
$$P_1(x) = 1 + \frac{x}{2};\ P_2(x) = 1 + \frac{x}{2} - \frac{x^2}{8},\ P_3(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16},$$

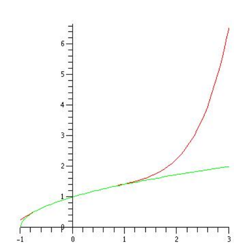
$$P_5(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256}.$$

Approximating $\sqrt{1+x}$ with $P_1(x), P_2(x), P_3(x), P_5(x)$









Determining How Well $P_n(x)$ Approximates $f(x)$

1. Based on Examples 1 and 2, and the graphical evidence, it seems that for $f(x) = e^x$ and $f(x) = \sin x$ the Taylor polynomials $P_n(x)$ at $a = 0$ can be made to approximate $f(x)$ as close as you want, for any value of x that you want.
2. However, based on Example 4, the Taylor Polynomials $P_n(x)$ of $f(x) = \sqrt{1+x}$ at $a = 0$ can only be made to approximate $f(x)$ closely, if $-1 < x < 1$.

What we need is some kind of general result that clarifies how good an approximation $P_n(x)$ actually is to $f(x)$. One such result is Taylor's Theorem.

Taylor's Theorem

Let f be continuous on an interval containing x and a; suppose all of $f', f'', \ldots, f^{(n)}, f^{(n+1)}$ are defined on the same interval. Then

$$f(x) = P_n(x) + R_n(x),$$

with

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

and

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1},$$

for some z between a and x. $R_n(x)$ is called the remainder term.

Example 5: Approximating e to within 10^{-8}

Let $f(x) = e^x$, $a = 0$. Then

$$e \simeq P_n(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!},\ R_n(1) = \frac{e^z}{(n+1)!},$$

for some number z between 0 and 1. $0 < z < 1 \Rightarrow e^z < e < 3$. So

$$\begin{aligned} R_n(1) < \frac{3}{(n+1)!} < 10^{-8} \quad &\Rightarrow \quad (n+1)! > 3 \cdot 10^8 \\ &\Rightarrow \quad n \geq 11, \text{ by trial and error.} \end{aligned}$$

Thus
$e \simeq P_{11}(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{11!} = 2.718281826\ldots$;
compare with the actual value: $e = 2.7182818284590452354\ldots$

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Example 6: Approximating $\sin x$ to Within 10^{-4} with $P_3(x)$

From Example 2, $P_3(x) = x - \dfrac{x^3}{3!} = P_4(x)$; so

$$\begin{aligned} |\sin x - P_4(x)| = |R_4(x)| \quad &= \quad \left| \frac{\sin^{(5)}(z)\, x^5}{5!} \right|, \text{ for some } z \\ &\leq \quad |x|^5/120, \text{ since } \sin^{(5)}(z) = \cos z \\ &\leq \quad 10^{-4} \Leftrightarrow |x|^5 \leq 120 \cdot 10^{-4} = 0.012 \\ \Rightarrow |x| \quad &\leq \quad \sqrt[5]{0.012} = 0.412892\ldots \end{aligned}$$

Compare: $P_3(.36) = 0.352224$; $\sin .36 = .352274233\ldots$.
But $P_3(.42) = 0.407652$; $\sin .42 = 0.407760453\ldots$

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What is a Power Series?

Any infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

is called a power series in $x - c$. The infinite series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is called a power series in x.

The Radius of Convergence of a Power Series

Let $u_n = a_n(x-c)^n$. Then the power series $\sum_{n=0}^{\infty} a_n(x-c)^n = \sum_{n=0}^{\infty} u_n$ converges, by the ratio test, if

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \quad &\Leftrightarrow \quad \lim_{n\to\infty}\left|\frac{a_{n+1}(x-c)^{n+1}}{a_n(x-c)^n}\right| < 1 \\
&\Leftrightarrow \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| |x-c| < 1 \\
&\Leftrightarrow \quad |x-c| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = R.
\end{aligned}$$

R is called the radius of convergence.

The Interval of Convergence of a Power Series

The significance of the radius of convergence of a power series is that the power series

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

converges if $|x-c|<R$, but diverges if $|x-c|>R$. The interval

$$(c-R, c+R)=\{x \in \mathbb{R} \mid |x-c|<R\}$$

is called the open interval of convergence of the power series. A power series may, or may not, converge at the endpoints. The interval of convergence of a power series is the open interval together with any endpoints at which the series converges.

Example 1

Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}x^n$, for which $a_n=\dfrac{1}{n\,3^n}$. Then

1. $R=\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{n+1}{n}\frac{3^{n+1}}{3^n}=3.$
2. At $x=3$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{n\,3^n}=\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges.
3. At $x=-3$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-3)^n}{n\,3^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges.

So the interval of convergence is $[-3,3)$.

Example 2

Consider the power series

$$\sum_{n=0}^{\infty}(-1)^n\frac{n^2}{4^n}x^{2n}.$$

Let $z=x^2$, let $a_n=(-1)^n\frac{n^2}{4^n}$. Then

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{n^2}{(n+1)^2}\frac{4^{n+1}}{4^n}=4$$

is the radius of convergence for $\sum_{n=0}^{\infty}(-1)^n\frac{n^2}{4^n}z^n$.

Example 2, Continued

Thus the original power series converges for

$$|z|<4\Leftrightarrow x^2<4\Leftrightarrow |x|<2.$$

At $x=\pm 2$, the original series is

$$\sum_{n=0}^{\infty}(-1)^n\frac{n^2}{4^n}(\pm 2)^{2n}=\sum_{n=0}^{\infty}(-1)^n n^2,$$

which diverges. So the interval of convergence is $(-2,2)$.

Addition of Power Series and Multiplication by Powers of x

If both $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for x in the interval I, then

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n,$$

and

$$x^m \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} a_n x^{n+m}$$

also converge absolutely on the interval I.

Example 3

If $f(x) = \sum_{n=0}^{\infty} x^n$ and $g(x) = \sum_{n=0}^{\infty} x^{2k}$, then for $|x| < 1$,

$$x^5 f(x) = x^5 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+5} \Leftrightarrow \frac{x^5}{1-x} = x^5 + x^6 + x^7 + \cdots$$

and

$$f(x) + g(x) = 2 + x + 2x^2 + x^3 + 2x^4 + x^5 + 2x^6 + \cdots$$

$$\Leftrightarrow \frac{1}{1-x} + \frac{1}{1-x^2} = \frac{2+x}{1-x^2}.$$

Differentiating and Integrating Power Series

Suppose the radius of convergence of the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

is R. Then for a, x in $(c-R, c+R)$,

1. $f'(x) = \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x-c)^n = \sum_{n=1}^{\infty} a_n\, n\,(x-c)^{n-1}$
2. $\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} a_n \int_a^x (t-c)^n\,dt = \sum_{n=0}^{\infty} a_n \left[\frac{(t-c)^{n+1}}{n+1}\right]_a^x$

and both these power series have radius of convergence R.

Example 4

Find the exact value of

$$\sum_{n=1}^{\infty} n \left(\frac{3}{4}\right)^n.$$

This is not a power series, but it is a special case of the power series

$$\sum_{n=1}^{\infty} nx^n, \text{ namely with } x = \frac{3}{4}.$$

Now use properties of power series to get a formula for $\sum_{n=1}^{\infty} nx^n$.

Example 4, Continued

$$\begin{aligned}
\sum_{n=1}^{\infty} nx^n &= x\sum_{n=1}^{\infty} nx^{n-1} = x\sum_{n=1}^{\infty} \frac{dx^n}{dx} \\
&= x\frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right) = x\frac{d}{dx}\left(\frac{1}{1-x} - 1\right) \\
&= x\left(\frac{1}{(1-x)^2}\right) = \frac{x}{(1-x)^2} \\
\Rightarrow \sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n &= \frac{3}{4}\frac{1}{(1-3/4)^2} = \frac{3}{4}\frac{1}{(1/4)^2} = 12.
\end{aligned}$$

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Example 5; Power Series for $\ln(1+x)$

$$\begin{aligned}
\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt &= \int_0^x \left(1 - t + t^2 - t^3 + \cdots\right)\,dt, \text{ if } |x| < 1 \\
&= \left[t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \cdots\right]_0^x \\
&= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots
\end{aligned}$$

This series converges at $x = 1$ by the alternating series test. So:

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

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Example 6; Power Series for $\tan^{-1} x$

Here is a less formal way, compared to Example 5:

$$\begin{aligned} \tan^{-1} x &= \int \frac{1}{1+x^2}\, dx \\ &= \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx, \text{ if } |x| < 1 \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + C \end{aligned}$$

To find C, substitute $x = 0 : \tan^{-1} 0 = 0 + C \Leftrightarrow C = 0$. So

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$$

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Leibniz's Series for π

The interval of convergence of

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

is $[-1, 1]$ since at both $x = \pm 1$ the infinite series converges by the alternating series test. In particular, at $x = 1$:

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This is known as Leibniz's formula, and it can be used to approximate π. However, today many better formulas for approximating π are known.

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Maclaurin Series

If the first n derivatives of f exist at $x = 0$, then

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k = f(0)+f'(0)x+\frac{f^{(2)}(0)}{2}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n$$

is called the n-th degree Maclaurin polynomial of f. If f has derivatives of all orders at $x = 0$, then

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f^{(2)}(0)}{2}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

is called the Maclaurin series of f.

Taylor Series

If the first n derivatives of f exist at $x = a$, then

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the n-th degreeTaylor polynomial of f at $x = a$. If f has derivatives of all orders at $x = a$, then

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f^{(2)}(a)}{2}(x-a)^2+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k+\cdots$$

is called the Taylor series of f at $x = a$.

Example 1

The n-th degree Maclaurin polynomial for $f(x) = e^x$ is

$$P_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}.$$

So the Maclaurin series for $f(x) = e^x$ is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^k}{k!} + \cdots .$$

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Example 2

The $2n+1$-st degree Maclaurin polynomial for $f(x) = \sin x$ is

$$P_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

So the Maclaurin series for $f(x) = \sin x$ is

$$\sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots .$$

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Example 3

The $2n$-th degree Maclaurin polynomial for $f(x) = \cos x$ is

$$P_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

So the Maclaurin series for $f(x) = \cos x$ is

$$\sum_{k=0}^{\infty} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots .$$

Example 4

It is not always possible to write down a nice simple formula for the Maclaurin or Taylor series of a function. For example, the 5-th degree Maclaurin polynomial of $f(x) = \sqrt{1+x}$ is

$$P_5(x) = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256}.$$

There is no obvious pattern to these coefficients, so we write the Maclaurin series of $f(x) = \sqrt{1+x}$ as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{(k)!} x^k = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} + \cdots .$$

When Does the Taylor Series of $f(x)$ Converge to $f(x)$?

Taylor's Theorem states that $f(x) = P_n(x) + R_n(x)$. Then: $f(x) =$

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} (P_n(x)+R_n(x)) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k + \lim_{n\to\infty} R_n(x).$$

Hence

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \Leftrightarrow \lim_{n\to\infty} R_n(x) = 0.$$

It can be quite difficult to show that $\lim_{n\to\infty} R_n(x) = 0$. One useful fact in this regard is: for any fixed x,

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0.$$

If $|x| \leq 1$, this is obvious. If $|x| > 1$, then there is an integer N such that $|x| \in (N, N+1]$. In this case, for $n > N+1$,

$$\begin{aligned}
\frac{|x|^n}{n!} &= \frac{|x|}{1}\frac{|x|}{2}\cdots\frac{|x|}{(N+1)}\cdot\frac{|x|}{(N+2)}\cdot\frac{|x|}{(N+3)}\cdots\frac{|x|}{n} \\
&\leq |x|^{N+1}\cdot\left(\frac{|x|}{N+2}\right)^{n-(N+1)} \\
&\to 0, \text{ as } n\to\infty, \text{ since } \frac{|x|}{N+2} < 1.
\end{aligned}$$

Example 5: Maclaurin Series for e^x Converges for all x.

$\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ is the Maclaurin series for e^x. For some z between 0 and x

$$|R_n(x)| = \left| \frac{e^z}{(n+1)!} x^{n+1} \right| \leq \begin{cases} \dfrac{|x|^{n+1}}{(n+1)!}, & \text{if } x < z < 0 \\ e^x \dfrac{|x|^{n+1}}{(n+1)!}, & \text{if } 0 < z < x \end{cases}$$

In either case, $R_n(x) \to 0$ as $n \to \infty$. Thus for all x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

Example 6: Maclaurin Series for $\sin x$ and $\cos x$

Similar calculations to the above show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$

for all x.

Example 7; Infinite Geometric Series Revisited

Not all Taylor series converge for all x. Let $f(x) = \dfrac{1}{1-x}$. Check the following:

$$f'(x) = \frac{1}{(1-x)^2}, f^{(2)}(x) = \frac{2}{(1-x)^3}, f^{(3)}(x) = \frac{6}{(1-x)^4}, \dots$$

$$f(0) = 1, f'(0) = 1, f^{(2)}(0) = 2, f^{(3)}(0) = 6, \dots, f^{(n)}(0) = n!$$

So the Maclaurin series of $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

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From the previous slide, the Maclaurin series of $1/(1-x)$ is

$$1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

which is the infinite geometric series. We already know that this series converges to $\dfrac{1}{1-x} = f(x)$ if $|x| < 1$. Try proving it using Taylor's remainder formula. It's not easy.

$$|R_n(x)| = \left| \frac{(n+1)!}{(1-z)^{n+1}(n+1)!} \right| |x|^{n+1} = \frac{|x|^{n+1}}{|1-z|^{n+1}},$$

for some z between 0 and x. Now try showing $|R_n(x)| \to 0$ as $n \to \infty$, for $|x| < 1$.

An Attempt at Showing $R_n(x) \to 0$ if $|x| < 1$

Suppose $-1 < x < 0$. Then z is some number in $(x, 0)$. We have

$$\begin{aligned} -1 < x < z < 0 \quad &\Rightarrow \quad 1 > -x > -z > 0 \\ &\Rightarrow \quad 2 > 1 - x > 1 - z > 1 \\ &\Rightarrow \quad 0 < \frac{1}{1-z} < 1 \\ &\Rightarrow \quad \frac{|x|}{|1-z|} = \frac{|x|}{1-z} < |x| \cdot 1 = |x| < 1 \end{aligned}$$

Thus

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|1-z|^{n+1}} \leq \lim_{n\to\infty} |x|^{n+1} = 0, \text{since} |x| < 1.$$

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The Case $0 < x < 1$

This is the hard case, because z could be close to 1, making $\frac{1}{1-z}$ very large. The following only shows $R_n(x) \to 0$ if $0 < x \leq 1/2$:

$$\begin{aligned} 0 < z < x \leq 1/2 \quad &\Rightarrow \quad -z > -x \geq -1/2 \\ &\Rightarrow \quad 1 - z > 1 - x \geq 1/2 \\ &\Rightarrow \quad 0 < \frac{1}{1-z} < 2 \\ &\Rightarrow \quad \frac{|x|}{|1-z|} = \frac{x}{1-z} < \frac{1}{2} \cdot 2 = 1 \end{aligned}$$

$$\text{Thus } \lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|1-z|^{n+1}} = 0, \text{ since } \frac{|x|}{|1-z|} < 1.$$

Power Series Representations Must Be Taylor Series

Suppose the radius of convergence of the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

is $R > 0$. Then the power series is the Taylor series for f at $x = c$.
Proof: $f(c) = a_0$.
$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots \Rightarrow f'(c) = a_1$.
$f''(x) = 2a_2 + 6a_3(x-c) + \cdots \Rightarrow f''(c) = 2a_2$. In general:

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

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Finding The Interval of Convergence of a Taylor Series

By the previous slide, a power series must converge to a Taylor Series. So instead of using Taylor's remainder formula to determine for which x a Taylor Series for $f(x)$ converges to $f(x)$, it is much easier to find the interval of convergence of the Taylor series considered as a power series. That is, once you have calculated

$$a_n = \frac{f^{(n)}(c)}{n!},$$

you can calculate the radius of convergence of $\sum a_n(x-c)^n$ using

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

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Interval of Convergence for Well-Known Power Series

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for all x, since $a_n = \frac{1}{n!}$, and

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

Similarly, the power series

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1} \text{ and } \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$$

converge for all x.

The Binomial Series

$$f(x) = (1+x)^\alpha \Rightarrow f'(x) = \alpha(1+x)^{\alpha-1}, f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2},$$

$$f^{(n)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))(1+x)^{\alpha-n}, \text{ for } n \geq 2,$$

and

$$f^{(n)}(0) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1)), \text{ for } n > 2.$$

The Maclaurin series for f, called the binomial series, is

$$1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

Interval of Convergence for Binomial Series

with

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!}.$$

The radius of convergence is

$$\begin{aligned} R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{\alpha(\alpha-1)\cdots(\alpha-(n-1))(\alpha-n)}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{\alpha-n}\right| = 1 \end{aligned}$$

Thus

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

if $|x| < 1$. Depending on α the binomial series may or may not converge at $x = \pm 1$. Aside: if α is a positive whole number then the binomial series is finite and converges for all x. Indeed, the binomial series becomes the binomial formula, that you may have seen in high school.

Example 8

The infinite geometric series,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ for } |x| < 1,$$

is a special case of the binomial series. Take $\alpha = -1$ and replace x by $-x$:

$$\begin{aligned} & (1+(-x))^{-1} \\ = \; & 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \cdots \\ = \; & 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

Example 9, From Special Relativity, for $v < c$.

$$\begin{aligned} E = mc^2 \; & = \frac{m_0 c^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = m_0 c^2 \left(1 - \left(\frac{v}{c}\right)^2\right)^{-1/2} \\ & = m_0 c^2 \left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\left(\frac{v}{c}\right)^2\right)^2 + \cdots\right) \\ & = \underbrace{m_0 c^2}_{\text{rest energy}} + \underbrace{\frac{1}{2} m_0 v^2}_{\text{kinetic energy}} + \underbrace{\frac{3}{8} m_0 \frac{v^4}{c^2}}_{\text{correction term}} + \cdots \end{aligned}$$

Example 1: Limits with Power Series

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x - \tan^{-1} x}{x^3} &= \lim_{x\to 0}\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)}{x^3} \\ &= \lim_{x\to 0}\frac{\frac{1}{6}x^3 - \frac{23}{120}x^5 + \cdots}{x^3} \\ &= \lim_{x\to 0}\left(\frac{1}{6} - \frac{23}{120}x^2 + \cdots\right) \\ &= \frac{1}{6}\end{aligned}$$

Example 2: Power Series for $f(x) = 1/(2+3x)$

This kind of division is just a variation of the infinite geometric series:

$$\begin{aligned}\frac{1}{2+3x} &= \frac{1}{2}\frac{1}{1+\frac{3}{2}x} \\ &= \frac{1}{2}\frac{1}{1-\left(-\frac{3}{2}x\right)} \\ &= \frac{1}{2}\left(1 - \frac{3}{2}x + \frac{9}{4}x^2 - \frac{27}{8}x^3 + \cdots\right), \text{ if } \left|-\frac{3}{2}x\right| < 1 \\ &= \frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \cdots, \text{ if } |x| < \frac{2}{3}\end{aligned}$$

Example 3: Prove $\sin'(x) = \cos x$

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}x^{2n+1}\right) \\
&= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \frac{dx}{dx} - \frac{1}{3!}\frac{dx^3}{dx} + \frac{1}{5!}\frac{dx^5}{dx} - \cdots \\
&= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \cdots \\
&= \cos x
\end{aligned}$$

Example 4: $f'(x) = f(x) \Rightarrow f(x) = ae^x$, for some a

Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$. $f'(x) = f(x) \Rightarrow$
$a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$

$$\begin{aligned}
&\Rightarrow\quad a_1 = a_0 \\
2a_2 = a_1 &\Rightarrow\quad a_2 = \frac{1}{2}a_1 = \frac{1}{2}a_0 \\
3a_3 = a_2 &\Rightarrow\quad a_3 = \frac{1}{3}a_2 = \frac{1}{6}a_0 \\
4a_4 = a_3 &\Rightarrow\quad a_4 = \frac{1}{4}a_3 = \frac{1}{24}a_0, \text{ etc.}
\end{aligned}$$

So $f(x) = a_0 + a_0x + a_0\frac{x^2}{2!} + a_0\frac{x^3}{3!} + a_0\frac{x^4}{4!} + \cdots = a_0e^x$.

Example 5

For $|x| < 1$,

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \\
\Rightarrow \frac{d}{dx}(1-x)^{-1} &= 0 + 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots \\
\Rightarrow \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots
\end{aligned}$$

In series notation:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \Rightarrow \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

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Example 6: Power Series For $\sin^{-1} x$

For $|x| < 1$,

$$\begin{aligned}
\sin^{-1} x &= \int_0^x \frac{1}{\sqrt{1-t^2}}\, dt = \int_0^x (1-t^2)^{-1/2}\, dt \\
&= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{(-1/2)(-3/2)}{2}(-t^2)^2 + \cdots\right) dt \\
&= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \frac{5}{16}t^6 + \cdots\right) dt \\
&= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{108}x^7 + \cdots
\end{aligned}$$

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Addition and Multiplication of Power Series

If both $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for x in the interval I,then for all x in I

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n,$$

and

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n,$$

with $c_n = \sum_{j=0}^{n} a_j b_{n-j} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots a_n b_0.$

Example 7: Prove $\sin(2x) = 2\sin x \cos x$

$$\begin{aligned}
2\sin x \cos x &= 2\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right) \\
&= 2\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right) \\
&= 2\left(x + \left(-\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{120} + \frac{1}{12} + \frac{1}{24}\right)x^5 + \cdots\right) \\
&= 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \cdots \\
&= \sin(2x)
\end{aligned}$$

Example 8: Power Series for $\tan x$

$\tan x = \sin x / \cos x \Leftrightarrow \cos x \tan x = \sin x$. Switching to series:

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)(a_0 + a_1 x + a_2 x^2 + \cdots) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\Leftrightarrow \quad a_0 + a_1 x + \left(a_2 - \frac{1}{2}a_0\right)x^2 + \left(a_3 - \frac{1}{2}a_1\right)x^3 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$
$$\Rightarrow \quad a_0 = 0, a_1 = 1, a_2 = 0, a_3 = 1/2 - 1/6 = 1/3, \text{ etc.}$$

So
$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

Example 9: Euler's Formula, with $i = \sqrt{-1}$, so $i^2 = -1$.

$$\begin{aligned} e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} &= 1 + ix + i^2\frac{x^2}{2!} + i^3\frac{x^3}{3!} + i^4\frac{x^4}{4!} + i^5\frac{x^5}{5!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

If $x = \pi$ then
$$e^{i\pi} = -1 + 0 \Leftrightarrow e^{i\pi} + 1 = 0.$$

Example 10

$$\begin{aligned} x^2 e^{-x} &= x^2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\ &= x^2 \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots \right) \\ &= x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \frac{x^6}{4!} - \cdots, \text{ for all } x \end{aligned}$$

Example 10, Continued

This may seem trivial, but it is not. For instance the above result means that the $6th$ degree Taylor polynomial of $f(x) = x^2 e^{-x}$ at $c = 0$ is

$$P_6(x) = x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \frac{x^6}{4!}.$$

Try calculating it directly! Moreover, from the polynomial you can read off that

$$\frac{f^{(5)}(0)}{5!} = -\frac{1}{3!} \Leftrightarrow f^{(5)}(0) = -20;$$

hardly obvious.

Numerical Approximations with Power Series

Numerical approximation is one of the most important applications of infinite series. If you have ever wondered how your calculator works – where do all those decimals come from? – now you know. Here are three examples that I will do in the following slides:

1. Approximate $\sqrt{105}$ correct to five decimal places.
2. Approximate $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx$ correct to seven decimal places.
3. For which values of x is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

correct to within five decimal places?

Example 11, Using Binomial Series

$$\begin{aligned}
\sqrt{105} = \sqrt{100(1.05)} &= 10\sqrt{1.05} = 10(1+.05)^{1/2} \\
&= 10\left(1+\frac{1}{2}(.05)-\frac{1}{8}(.05)^2+\frac{1}{16}(.05)^3-\frac{5}{128}(.05)^4+\cdots\right) \\
&= 10+.25-.003125+.000078125-.0000024414+\cdots \\
&= 10.24695312, \text{ correct to within } .0000024414
\end{aligned}$$

So $\sqrt{105} = 10.24695$, correct to five decimal places; as you can check with your calculator.

Example 12

$$\begin{aligned}
\int_0^1 \frac{\sin x}{x}\,dx &= \int_0^1 \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots\right) dx \\
&= \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \cdots\right) dx \\
&= \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \frac{x^9}{9\cdot 9!} - \frac{x^{11}}{11\cdot 11!} + \cdots\right]_0^1 \\
&= 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \frac{1}{9\cdot 9!},\ \text{to within } \frac{1}{11\cdot 11!} \\
&= 0.9460830726,\ \text{to within } 2.2774\times 10^{-9}
\end{aligned}$$

Example 13

For which values of x does $P_5(x)$ approximate $\sin x$ correct to five decimals? **Solution:**

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
&= P_5(x),\ \text{correct to within } \frac{|x|^7}{7!}
\end{aligned}$$

$$\frac{|x|^7}{7!} < .000005 \Leftrightarrow |x^7| < .0252 \Leftrightarrow |x| < .591056$$