Terminology: open and closed sets

Def: $S \subset \mathbb{R}^n$ is open if $\mathring{S} = S$

Theorem: S is open $\Leftrightarrow S \cap \partial S = \emptyset$

Δ $\Rightarrow$ Assume that S is open, then $S \cap \partial S = \mathring{S} \cap \partial S = \emptyset$

$\Leftarrow$: We are going to prove the contrapositive: S is not open $\Rightarrow S \cap \partial S \neq \emptyset$
Assume that S is not open then $\mathring{S} \subsetneq S$ and there exists $x \in S \setminus \mathring{S}$
but $S \setminus \mathring{S} \subset \bar{S} \setminus \mathring{S} = \partial S$
so $x \in S \cap \partial S \neq \emptyset$ □

→ that's why we will like to have open sets as domains in calculus.

Theorem: S is open $\Leftrightarrow \forall x \in S, \exists \varepsilon > 0, B(x,\varepsilon) \subset S$

Δ $\Rightarrow$: let $x \in S$, then $x \in \mathring{S}$ since $\mathring{S} = S$.
so $\exists \varepsilon > 0$, $B(x,\varepsilon) \subset S$

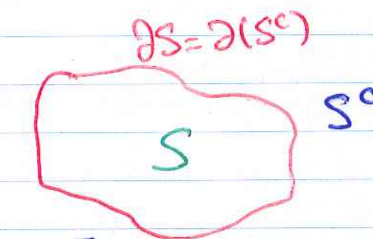
$\Leftarrow$: we already know that $\mathring{S} \subset S$. Let's prove that $S \subset \mathring{S}$.
Let $x \in S$, then $\exists \varepsilon > 0$ s.t $B(x,\varepsilon) \subset S$
hence $x \in \mathring{S}$ and therefore $S \subset \mathring{S}$ □

Ex $B(a,r)$ is open $\quad \bar{B}(a,r)$ is not open $\quad S$ is not open

Def: $S \subset \mathbb{R}^n$ is closed if $\bar{S} = S$

Theorem: S is closed $\Leftrightarrow \partial S \subset S$

Δ $\Rightarrow$ $\partial S \subset \bar{S} = S$
$\Leftarrow$ $\bar{S} = S \cup \partial S = S$ □



Theorem: S is closed $\Leftrightarrow S^c$ is open

Δ Recall that $\partial(S^c) = \partial S$ and look at the boundary □

Question: Find a subset of $\mathbb{R}$ which is

1. Open but not closed
2. Closed but not open
3. Neither closed nor open
4. Both open and closed

Question: Find all the subsets of $\mathbb{R}^n$ that are both open and closed

Question: Prove that $\mathring{S}$ is open and that $\overline{S}$ is closed

Question: Prove $(\mathring{S})^c = \overline{S^c}$

$(\overline{S})^c = \mathring{S^c}$

Do the questions at the end of section 1.1 !

Advanced questions:

1. Let $(O_i)_{i\in I}$ be a (possibly infinite) family of open subsets of $\mathbb{R}^n$ prove that $O := \bigcup_{i\in I} O_i = \{x \in \mathbb{R}^n : \exists i \in I, x \in O_i\}$ is open
2. Prove that if $U, V \subset \mathbb{R}^n$ are open then $U \cap V$ is open
3. Find an (infinite) family of open sets whose intersection is not open
4. Using that "S closed ⇔ S^c open" obtain results about closed sets

Limits of multivariable functions

In class activity: start with the definition of $\ell = \lim_{x\to a} f(x)$ for $f: I \to \mathbb{R}$ a one variable function defined on an interval I containing a.
Generalize the above definition and check that we need to restrict to limit points

Def: let $S \subset \mathbb{R}^n$. We say that $a \in \mathbb{R}^n$ is a limit point of S if

$$\forall \delta > 0, \exists x \in S, \ 0 < \|x - a\| < \delta$$

or geometrically: $\forall \delta > 0, \ (B(a,\delta) \cap S) \setminus \{a\} \neq \emptyset$

Theorem: a is a limit point of $S \Leftrightarrow a \in \overline{S \setminus \{a\}}$

Δ Notice that $(B(a,\delta) \cap S) \setminus \{a\} = B(a,\delta) \cap (S \setminus \{a\})$ $\square$

Ex:
① 0 is a limit point of $\{\frac{1}{n} : n \in \mathbb{N}_{>0}\} \subset \mathbb{R}$

② 0 is a limit point of $[0,3)$ or of $(0,3)$

③ 0 is NOT a limit point of $\{0\} \cup [1,2)$

Intuition: a limit point is a closure point which is not isolated

Remark: if $a \in \mathring{S}$ then a is a limit point of S

Def: let $S \subset \mathbb{R}^n$, a be a limit point of S, $f: S \to \mathbb{R}^k$ and $L \in \mathbb{R}^k$.
We say that L is the limit of f at a, denoted $\lim_{x\to a} f(x) = L$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, \ 0 < \|x - a\| < \delta \Rightarrow \|f(x) - L\| < \varepsilon$$

Proposition: let $f,g: S \to \mathbb{R}$ (Δ $k=1$) and a be a limit point of S then

$$\lim_{x\to a} f(x) = L, \ \lim g(x) = M \Rightarrow \begin{cases} \lim (f+g) = M+L \\ \lim fg = ML \end{cases}$$

Proposition: $f,g,h: S \to \mathbb{R}$ (Δ $k=1$) and a limit point of S

$$\begin{cases} f \leq g \leq h \\ \lim_a f = \lim_a h = L \end{cases} \Rightarrow \lim_{x\to a} g = L$$