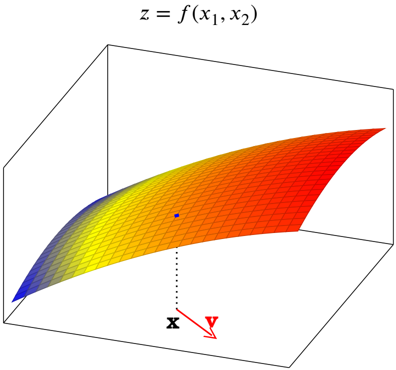
Differentiability of real valued functions: a summary

UNIVERSITY OF TORONTO

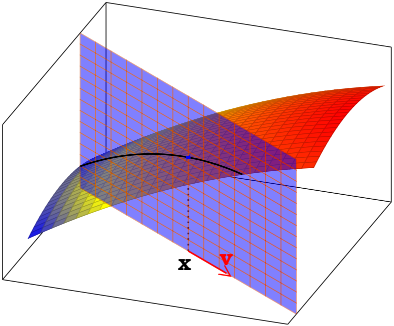
October 10th, 2019

Directional derivatives: geometric interpretation

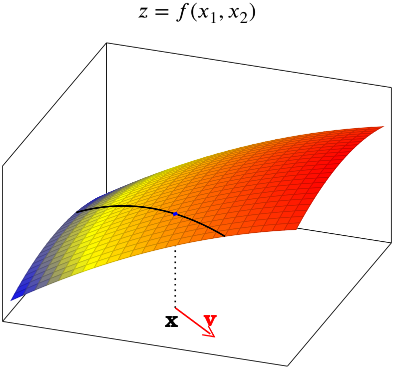


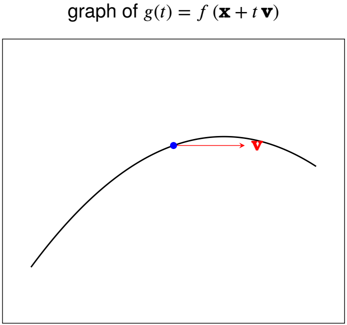
Directional derivatives: geometric interpretation

$z = f(x_1, x_2)$



Directional derivatives: geometric interpretation





Directional derivatives: geometric interpretation

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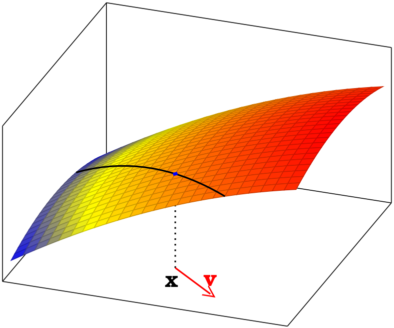
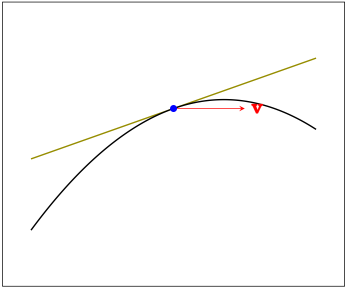
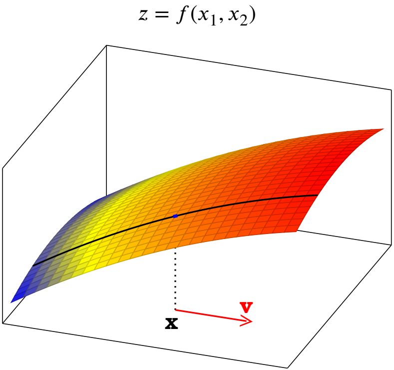


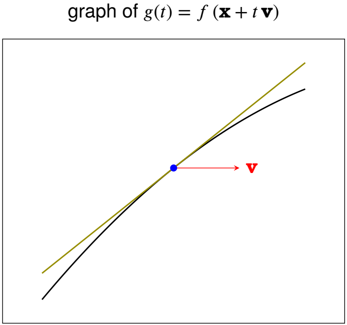
graph of $g(t) = f(\mathbf{x} + t\,\mathbf{v})$



$\partial_{\mathbf{v}} f(\mathbf{x}) = g'(0)$ is the slope of the green tangent line.

Directional derivatives: geometric interpretation





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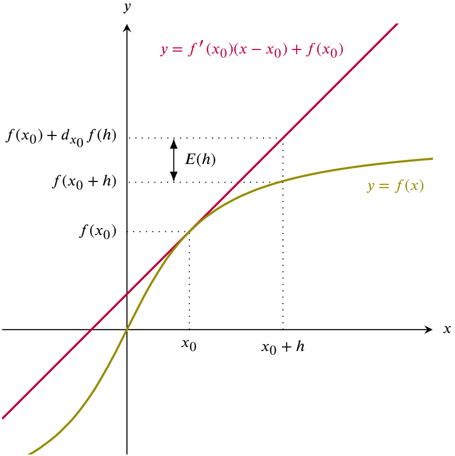
Differentiability: geometric interpretation (1-variable case)

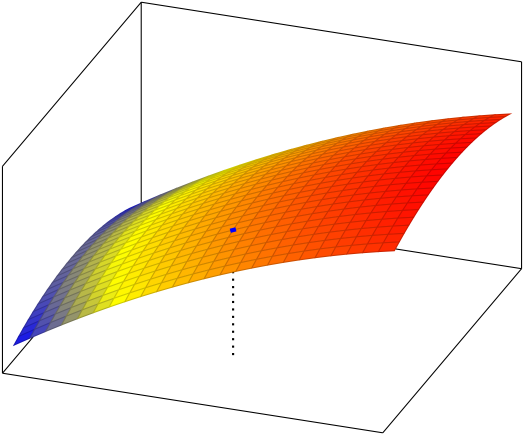


$$f(x_0 + h) = f(x_0) + d_{x_0} f(h) + E(h)$$

where $d_{x_0} f(h) = f'(x_0)h$ is linear and $\lim_{h \to 0} \frac{E(h)}{h} = 0$.

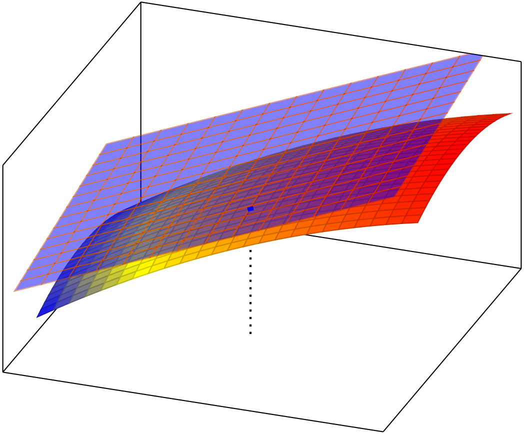
Differentiability: geometric interpretation (2-variable case)

$$z = f(x, y)$$



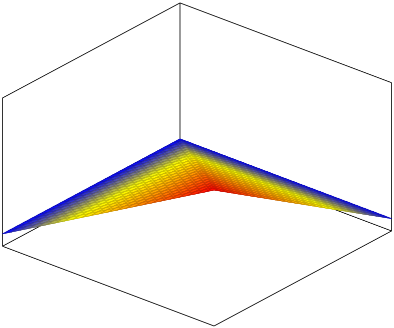
Differentiability: geometric interpretation (2-variable case)

$$z = f(x, y)$$

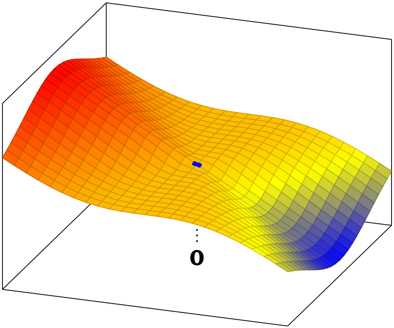


Differentiability: geometric interpretation counter-examples

$z = \min(x, y)$



$z = \frac{x^3}{x^2+y^2}$



Recap – Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$.

Name	Nature	Notation
Directional derivative at $\mathbf{x} \in U$ along $\mathbf{v} \in \mathbb{R}^n$	Real number	$\partial_{\mathbf{v}} f(\mathbf{x})$
i-th partial derivative at $\mathbf{x} \in U$	Real number	$\frac{\partial f}{\partial x_i}(\mathbf{x})$
Gradient at $\mathbf{x} \in U$	Vector in $\mathbb{R}^n$	$\nabla f(\mathbf{x})$
Differential at $\mathbf{x} \in U$ "*f is differentiable at $\mathbf{x}$*"	Linear function $\mathbb{R}^n \to \mathbb{R}$	$d_{\mathbf{x}} f$ $\mathbb{R}^n \ni \mathbf{h} \mapsto d_{\mathbf{x}} f(\mathbf{h}) \in \mathbb{R}$

Relationships – $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, $\mathbf{x} \in U$

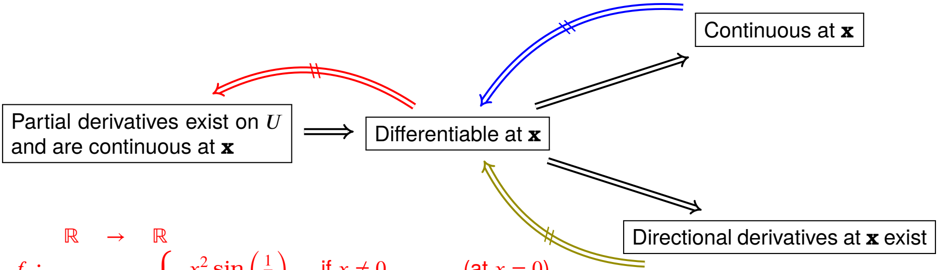
Continuous at $\mathbf{x}$

Partial derivatives exist on U and are continuous at $\mathbf{x}$ $\Longrightarrow$ Differentiable at $\mathbf{x}$ $\Longrightarrow$

Interpretation of $\nabla f(\mathbf{x})$:
direction: dir. of fastest increase through $\mathbf{x}$
magnitude: steepness, instantaneous rate of change in that direction

Directional derivatives at $\mathbf{x}$ exist and moreover

- $\partial_{\mathbf{v}} f(\mathbf{x}) = d_{\mathbf{x}} f(\mathbf{v})$
- $d_{\mathbf{x}} f(\mathbf{h}) = \nabla f(\mathbf{x}) \cdot \mathbf{h}$
- $\partial_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v}$

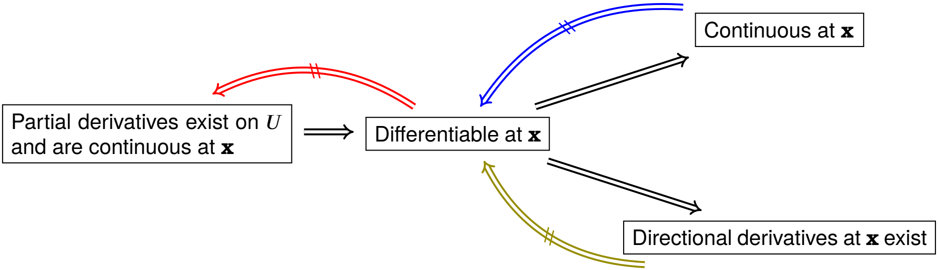


$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases} \end{array} \quad \text{(at } x = 0\text{)}$$

$$g : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & |x| \end{array} \quad \text{(at } x = 0\text{)}$$

$$h : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases} \end{array} \quad \text{(at } \mathbf{x} = (0, 0)\text{)}$$

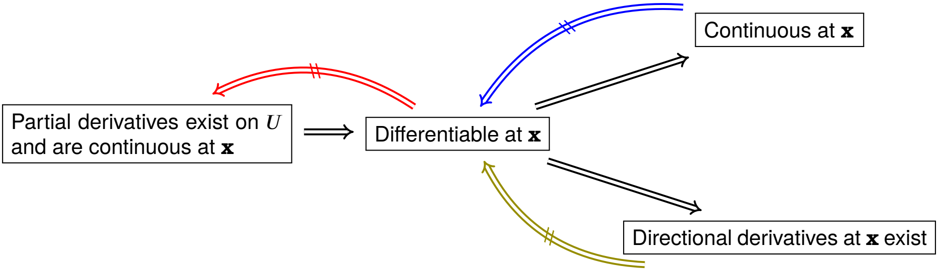
Relationships – $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, $\mathbf{x} \in U$



Partial derivatives at $\mathbf{x}$ exist $\;\not\Longrightarrow\;$ Directional derivatives at $\mathbf{x}$ exist

Counter-example:
$$f : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases} \end{array}$$
(at $\mathbf{x} = (0,0)$).

Relationships – $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, $\mathbf{x} \in U$



All the directional derivatives at $\mathbf{x}$ exist $\not\Longrightarrow$ Continuous at $\mathbf{x}$

Counter-example: $f : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases} \end{array}$ (at $\mathbf{x} = (0,0)$).

Relationships – $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, $\mathbf{x} \in U$

Partial derivatives exist on U and are continuous at $\mathbf{x}$ $\Longrightarrow$ Differentiable at $\mathbf{x}$

Differentiable at $\mathbf{x}$ $\Longrightarrow$ Continuous at $\mathbf{x}$

Differentiable at $\mathbf{x}$ $\Longrightarrow$ Directional derivatives at $\mathbf{x}$ exist and moreover

- $\partial_{\mathbf{v}} f(\mathbf{x}) = d_{\mathbf{x}} f(\mathbf{v})$
- $d_{\mathbf{x}} f(\mathbf{h}) = \nabla f(\mathbf{x}) \cdot \mathbf{h}$
- $\partial_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v}$

Hence, if f is not continuous at $\mathbf{x}$ or if a directional derivative of f at $\mathbf{x}$ doesn't exist, then f is not differentiable at $\mathbf{x}$.

But there is more: notice that if f is differentiable at $\mathbf{x}$ then

$$\partial_{\mathbf{v}_1+\mathbf{v}_2} f(\mathbf{x}) = d_{\mathbf{x}}(\mathbf{v}_1 + \mathbf{v}_2) = d_{\mathbf{x}}(\mathbf{v}_1) + d_{\mathbf{x}}(\mathbf{v}_2) = \partial_{\mathbf{v}_1} f(\mathbf{x}) + \partial_{\mathbf{v}_2} f(\mathbf{x})$$

It may be useful to prove that a function is not differentiable when all its directional derivatives exist. See for instance h with $\mathbf{v}_1 = \mathbf{e}_1$ and $\mathbf{v}_2 = \mathbf{e}_2$.

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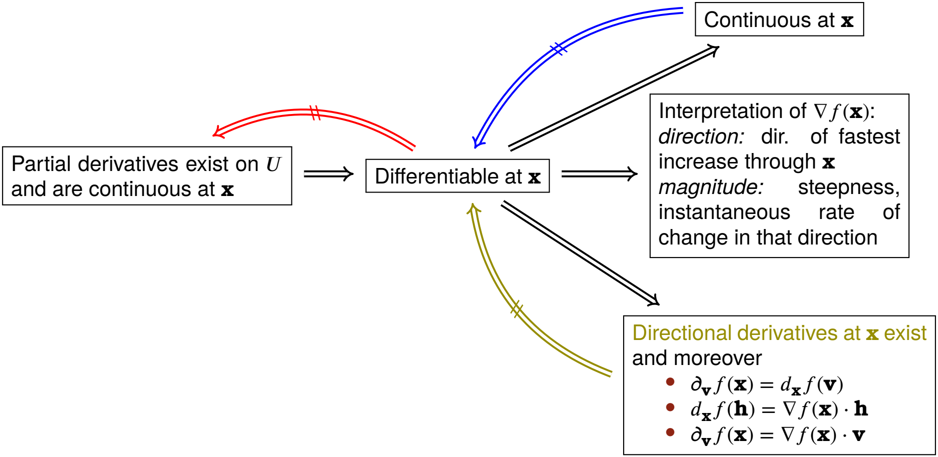

Partial derivatives at $\mathbf{x}$ exist $\not\Longrightarrow$ Directional derivatives at $\mathbf{x}$ exist

All the directional derivatives at $\mathbf{x}$ exist $\not\Longrightarrow$ Continuous at $\mathbf{x}$