University of Toronto – MAT237Y1 – LEC5201

Multivariable calculus!

The Gamma function and the Beta function

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The following questions are NOT part of the material of MAT237 but I think that these results are quite interesting, so, if you have time, you can have a look at them.
The Gamma and the Beta functions are functions defined by improper integrals which appear in various areas of mathematics. In these questions we study a few of their properties and some applications.
The questions numbered in red are a little bit more difficult.

We admit the following theorem which will be useful for 1.3.(a).

Theorem. Let I be an open interval and J be an interval. Let $F : \begin{array}{ccc} I\times J & \to & \mathbb{R} \\ (x,t) & \mapsto & F(x,t) \end{array}$ be a continuous function.
Assume that

1. $\forall x \in I$, $\int_J F(x,t)\mathrm{d}t$ is absolutely convergent.
2. $\frac{\partial F}{\partial x}(x,t)$ exists and is continuous on $I \times J$.
3. For all $K \subset I$ compact, there exists $\varphi_K : J \to \mathbb{R}$ integrable on J such that $\forall (x,t) \in K \times J$, $\left|\frac{\partial F}{\partial x}(x,t)\right| \le \varphi_K(t)$.

Then $f : I \to \mathbb{R}$ defined by $f(x) = \int_J F(x,t)\mathrm{d}t$ is C^1 and $f'(x) = \int_J \frac{\partial F}{\partial x}(x,t)\mathrm{d}t$ where this last integral is absolutely convergent for every $x \in I$.

1 The Gamma function

Definition. We define $\Gamma : \mathbb{R}_{>0} \to \mathbb{R}$ by $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}\mathrm{d}t$.

1.1. Prove that Γ is well-defined (i.e. that the integral is convergent for any $x > 0$).

1.2. (a) Prove that $\forall x \in \mathbb{R}_{>0}$, $\Gamma(x+1) = x\Gamma(x)$ (*Hint: integration by parts*).
(b) Deduce that $\forall n \in \mathbb{N}_{\ge 0}$, $\Gamma(n+1) = n!$.

1.3. (a) Prove that Γ is C^∞ and that $\forall n \in \mathbb{N}_{\ge 0}$, $\forall x \in \mathbb{R}_{>0}$, $\Gamma^{(n)}(x) = \int_0^{+\infty} (\ln t)^n t^{x-1}e^{-t}\mathrm{d}t$.
(b) Prove that Γ is convex.
(c) Prove that $\Gamma(x) \underset{0^+}{\sim} \frac{1}{x}$ (i.e. $\lim_{x\to 0^+} x\Gamma(x) = 1$).
(d) Study the monotonicity of Γ, compute $\lim_{x\to+\infty} \Gamma(x)$ and $\lim_{x\to+\infty} \frac{\Gamma(x)}{x}$, then sketch the graph of Γ.

1.4. Application 1: the Gaussian/Euler–Poisson integral.
(a) Prove that $\forall x \in \mathbb{R}_{>0}$, $\Gamma(x) = \int_0^{+\infty} 2e^{-u^2}u^{2x-1}\mathrm{d}u$.
(b) Prove that $\Gamma(1/2) = \int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x$.
(c) For $r, s > 0$, prove that $I_{r,s} = \int_{\mathbb{R}^2_{>0}} 4e^{-u^2-v^2}u^{2r-1}v^{2s-1}\mathrm{d}u\mathrm{d}v$ is well defined and that $I_{r,s} = \Gamma(r)\Gamma(s)$.
(d) Prove that $\Gamma(r)\Gamma(s) = 2\Gamma(r+s)\int_0^{\frac{\pi}{2}} \cos^{2r-1}(\theta)\sin^{2s-1}(\theta)\mathrm{d}\theta$ (*Hint: polar coordinates*).
(e) Compute the value of the Gaussian/Euler–Poisson integral[⋆] : $\int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$.

⋆ That's the third proof we met in MAT237, I really start to believe it is true...
For the first 2 proofs, see p75 and p85 of http://www.math.toronto.edu/campesat/ens/1920/winter-notes.pdf.

1.5. (a) Prove that for any $c \in \mathbb{R}_{>0}$, $x \mapsto c^x\Gamma(x)$ is convex on $\mathbb{R}_{>0}$ (*Hint: study the integrand first*).
(b) (*log-convexity*[⋆]) Using a suitable c, deduce that $\forall x, y \in \mathbb{R}_{>0}$, $\forall \lambda \in [0,1]$, $\Gamma(\lambda x+(1-\lambda)y) \leq \Gamma(x)^\lambda \Gamma(y)^{1-\lambda}$.
(c) (*Gautschi's inequality*) Prove that $\forall x \in \mathbb{R}_{>0}$, $\forall s \in [0,1]$, $x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+1)^{1-s}$ (*Hint: use (b) twice*).

2 The Beta function

Definition. We define $\mathrm{B} : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}$ by $\mathrm{B}(r,s) = \displaystyle\int_0^1 t^{r-1}(1-t)^{s-1}\mathrm{d}t$.

2.1. Prove that B is well-defined (i.e. that the integral is convergent for any $(r,s) \in \mathbb{R}^2_{>0}$).

2.2. Connection with the Gamma function.

(a) Prove that $\forall (r,s) \in \mathbb{R}^2_{>0}$, $B(r,s) = B(s,r) = 2\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2r-1}(\theta)\sin^{2s-1}(\theta)\mathrm{d}\theta$.
(*Hint for the second equality: set* $t = \sin^2\theta$)

(b) Prove that $\forall (r,s) \in \mathbb{R}^2_{>0}$, $\mathrm{B}(r,s) = \dfrac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$.

2.3. Application 2: Wallis' integrals[†], the Stirling formula and the Wallis product.
For $n \in \mathbb{N}_{\geq 0}$, we define Wallis' integrals[‡] by $W_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n(t)\mathrm{d}t$.

(a) Prove that $W_n = \frac{1}{2}\mathrm{B}\left(\frac{n+1}{2}, \frac{1}{2}\right)$.
(b) Prove that $W_n \underset{+\infty}{\sim} \sqrt{\frac{\pi}{2n}}$ (*Hint: use Gautschi's inequality*).
(c) Prove that $\forall x \in \mathbb{R}_{>0}$, $B(x,x) = 2^{-2x+1}B(1/2, x)$.
(d) Prove Legendre's duplication formula: $\forall x \in \mathbb{R}_{>0}$, $\Gamma(x)\Gamma(x+1/2) = \dfrac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)$.
(e) Prove that $\forall n \in \mathbb{N}_{\geq 0}$, $\Gamma(n+1/2) = \dfrac{(2n)!\sqrt{\pi}}{2^{2n}n!}$.
(f) Prove that $\forall p \in \mathbb{N}_{\geq 0}$, $W_{2p} = \dfrac{\pi}{2}\dfrac{(2p)!}{(2^p p!)^2}$ and $W_{2p+1} = \dfrac{(2^p p!)^2}{(2p+1)!}$.
(g) (*Stirling formula*[§]). We assume that there exists $C \in \mathbb{R} \setminus \{0\}$ such that $n! \underset{+\infty}{\sim} C\sqrt{n}\left(\frac{n}{e}\right)^n$. Find C.
(h) (*Wallis product*) Prove that $\dfrac{\pi}{2} = \displaystyle\prod_{k=1}^{+\infty} \frac{4k^2}{4k^2-1}$.

2.4. Application 3: volume and surface area of an n-dimensional ball.
For $n \in \mathbb{N}_{\geq 1}$ we denote by $V_n(r)$ the *volume* of $\overline{B}(\mathbf{0}, r) \subset \mathbb{R}^n$ and by $A_n(r)$ its *surface area*.

(a) Prove that $\forall n \in \mathbb{N}_{\geq 1}$, $\forall r > 0$, $V_n(r) = r^n V_n(1)$.
(b) Prove that $\forall n \in \mathbb{N}_{\geq 1}$, $V_{n+1}(1) = 2V_n(1)\displaystyle\int_0^1 \left(1-x^2\right)^{\frac{n}{2}}\mathrm{d}x$.
(c) Prove that $\forall n \in \mathbb{N}_{\geq 1}$, $V_{n+1}(1) = V_n(1)B\left(\dfrac{1}{2}, \dfrac{n}{2}+1\right)$.
(d) Prove[‖] that $\forall n \in \mathbb{N}_{\geq 1}$, $V_n(1) = \dfrac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} = \dfrac{2\pi^{\frac{n}{2}}}{n\Gamma(n/2)}$.
(e) Give a formula for $V_n(r)$.
(f) Prove that $A_n(r) = V_n'(r)$ and then give a formula for $A_n(r)$.

[⋆] We usually prove the log-convexity of Γ using Cauchy–Schwarz inequality for integrals or even faster Hölder inequality. By the Bohr–Mollerup theorem, Γ is the only function $\mathbb{R}_{>0} \to \mathbb{R}$ such that $\Gamma(1) = 1$, $\Gamma(x+1) = x\Gamma(x)$ and Γ is log-convex (i.e. $\ln \circ \Gamma$ is convex).
[†] Questions (b) and (f) admit alternative elementary proofs: you can use an induction relying on a double integration by parts and the monotonicity of $(W_n)_n$.
[‡] Notice that by setting $u = t - \frac{\pi}{2}$, we may replace cos by sin in the definition of W_n.
[§] Moivre proved the formula up to the constant C which was subsequently determined by Stirling.
[‖] There is an alternative formula with an elementary proof which doesn't involve the Gamma function: using generalized cylindrical coordinates, one may prove that $V_{n+2}(1) = \frac{2\pi}{n+2}V_n(1)$ and then conclude by induction, however this formula depends on the parity of n.