Usual complex functions – 2

September 25th, 2020

Complex power functions – 1

Similarly to the real case, for $w \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{0\}$ we want to set $z^w := e^{w \log z}$.

But then it is only defined up to a **factor** $e^{2in\pi w}$, $n \in \mathbb{Z}$, since $\log$ is well defined modulo $2i\pi$.

$$z^w := e^{w \log z} = \left\{ e^{w \operatorname{Log} z} e^{2in\pi w} \ : \ n \in \mathbb{Z} \right\}$$

Example

For instance $\sqrt{z} = z^{\frac{1}{2}} = \left\{ e^{\frac{1}{2} \operatorname{Log} z} e^{in\pi} \ : \ n \in \mathbb{Z} \right\} = \left\{ \pm e^{\frac{1}{2} \operatorname{Log} z} \right\}$.
Indeed, the square root is well-defined only up to a sign.

However there is no indeterminacy when $w \in \mathbb{Z}$ since $e^{2i\pi nw} = 1$ when $nw \in \mathbb{Z}$.

The power functions – 2

Beware

The identity $(z_1 z_2)^w = z_1^w z_2^w$ is only true modulo a factor $e^{2i\pi wn}$, $n \in \mathbb{Z}$ (i.e. as sets/multivalued functions).

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And…As if that wasn't enough…

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BEWARE

The identity $z^{w_1+w_2} = z^{w_1} z^{w_2}$ is generally false ***even as multivalued functions***:

- $z^{w_1+w_2}$ is well-defined up to a factor $e^{2i\pi n(w_1+w_2)}$, $n \in \mathbb{Z}$.
- $z^{w_1} z^{w_2}$ is well-defined up to a factor $e^{2i\pi(nw_1+kw_2)}$, $n, k \in \mathbb{Z}$.

So $z^{w_1+w_2} \subset z^{w_1} z^{w_2}$ (as multivalued functions).

The complex cosine and sine – 1

Definitions

We define $\cos : \mathbb{C} \to \mathbb{C}$ and $\sin : \mathbb{C} \to \mathbb{C}$ respectively by

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin(z) := \frac{e^{iz} - e^{-iz}}{2i}$$

Proposition

- $\forall z \in \mathbb{C},\ \cos^2 z + \sin^2 z = 1$
- $\forall z, w \in \mathbb{C},\ \sin(z + w) = \sin z \cos w + \cos z \sin w$
- $\forall z, w \in \mathbb{C},\ \cos(z + w) = \cos z \cos w - \sin z \sin w$
- $\forall z \in \mathbb{C},\ \sin(-z) = -\sin(z)$
- $\forall z \in \mathbb{C},\ \cos(-z) = \cos(z)$
- $\forall z \in \mathbb{C},\ \sin(z + 2\pi) = \sin(z)$
- $\forall z \in \mathbb{C},\ \cos(z + 2\pi) = \cos(z)$

Homework: prove some of them.

The complex cosine and sine – 2

Proposition

The functions $\cos : \mathbb{C} \to \mathbb{C}$ and $\sin : \mathbb{C} \to \mathbb{C}$ are surjective.

Proof.

Let $w \in \mathbb{C}$, we look for $z \in \mathbb{C}$ such that $\cos(z) = w$, or equivalently $e^{iz} + e^{-iz} = 2w$.
Set $u = e^{iz}$ then the above equation becomes $u + u^{-1} = 2w$ or equivalently $u^2 - 2wu + 1 = 0$.
Take such a u (which is non-zero) then, since the range of $\exp$ is $\mathbb{C} \setminus \{0\}$, there exists $z \in \mathbb{C}$ such that $u = e^{iz}$. ∎

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Homework

What is **wrong** with this proof?

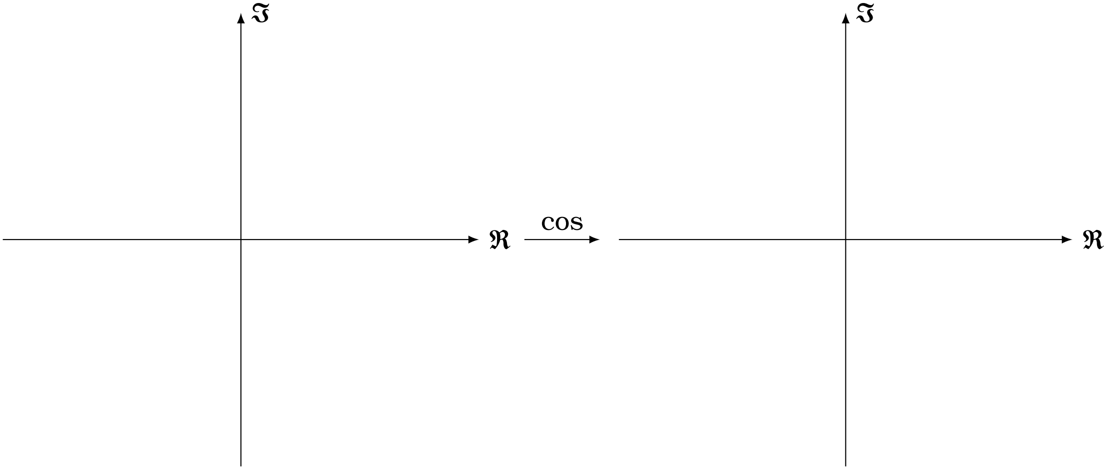
Let $z \in \mathbb{C}$, then $|\cos(z)| = \left|\frac{e^{iz}+e^{-iz}}{2}\right| \leq \left|\frac{e^{iz}}{2}\right| + \left|\frac{e^{-iz}}{2}\right| = \frac{|e^{iz}|}{2} + \frac{|e^{-iz}|}{2} = \frac{1}{2} + \frac{1}{2} = 1$.

Hence $\forall z \in \mathbb{C}, |\cos z| \leq 1$.

*This property is obviously **false** according to the above proposition.*

The complex cosine and sine – 3

The horizontal line $\Im(z) = c$ is mapped by cos to $\left\{ \cos(x)\left(\frac{e^c+e^{-c}}{2}\right) + i\sin(x)\left(\frac{e^{-c}-e^c}{2}\right) \;:\; x \in \mathbb{R} \right\}$ which is an ellipse (possibly flat for $c = 0$).



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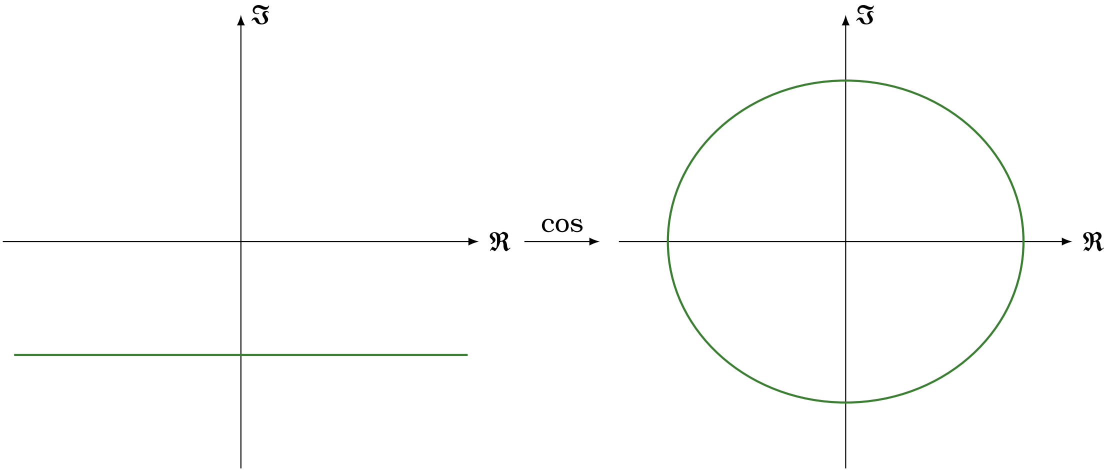

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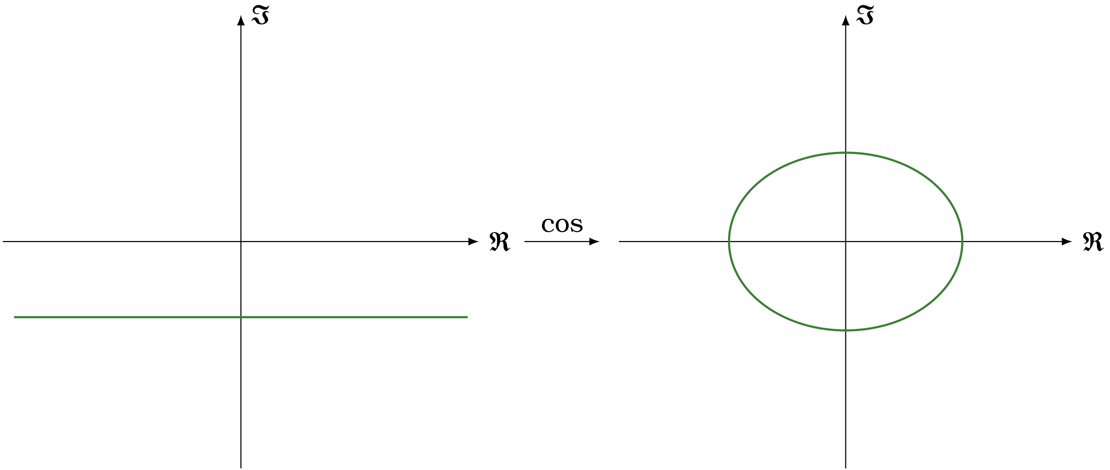

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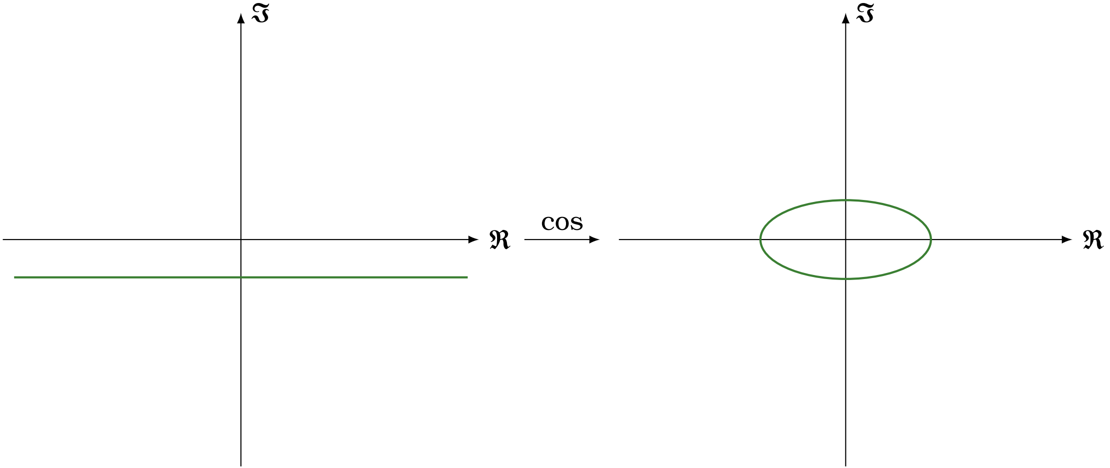

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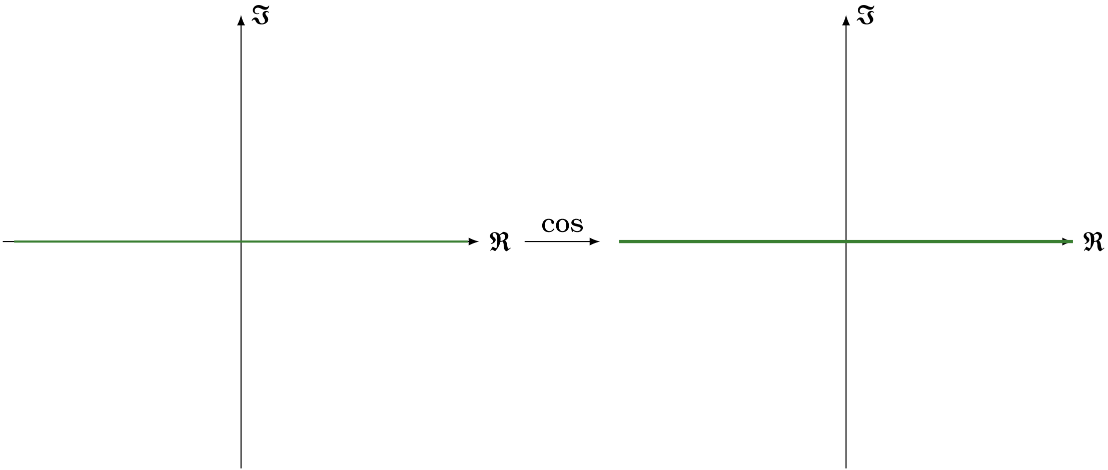

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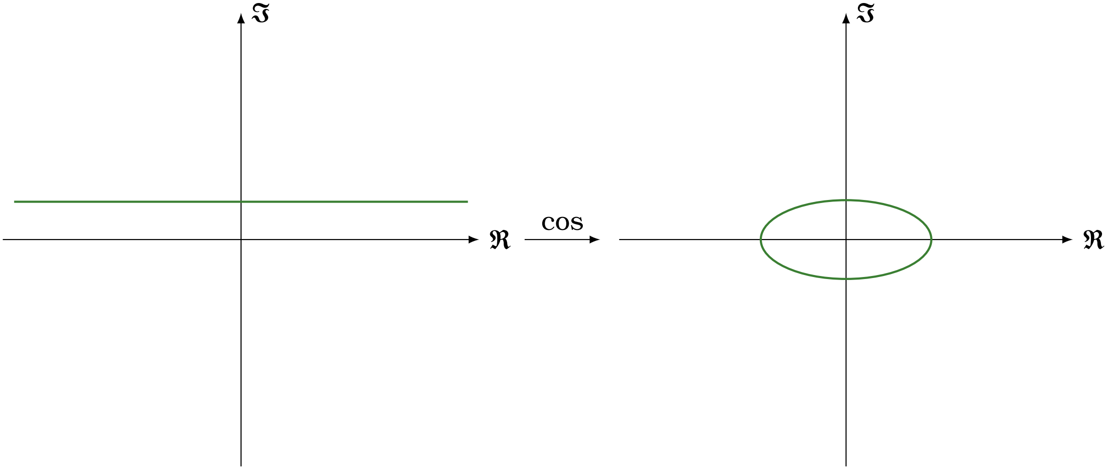

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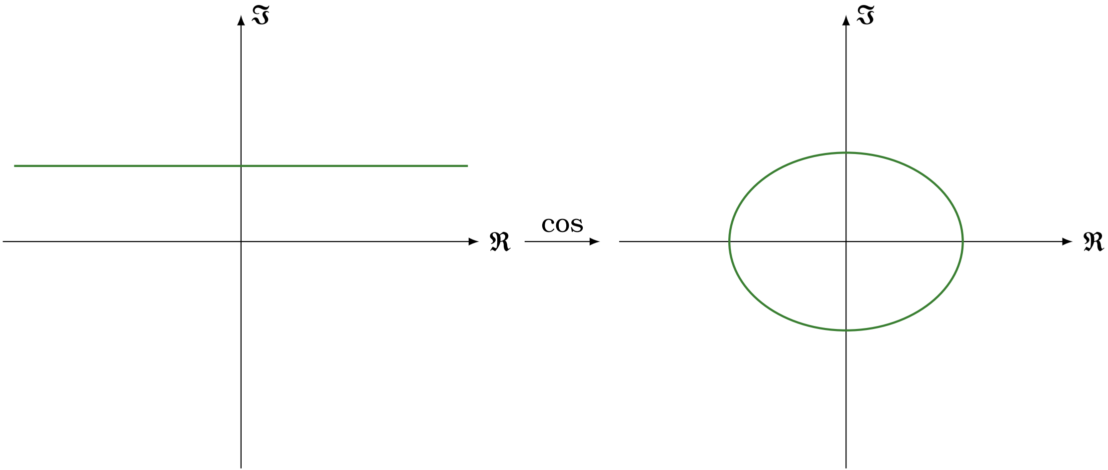

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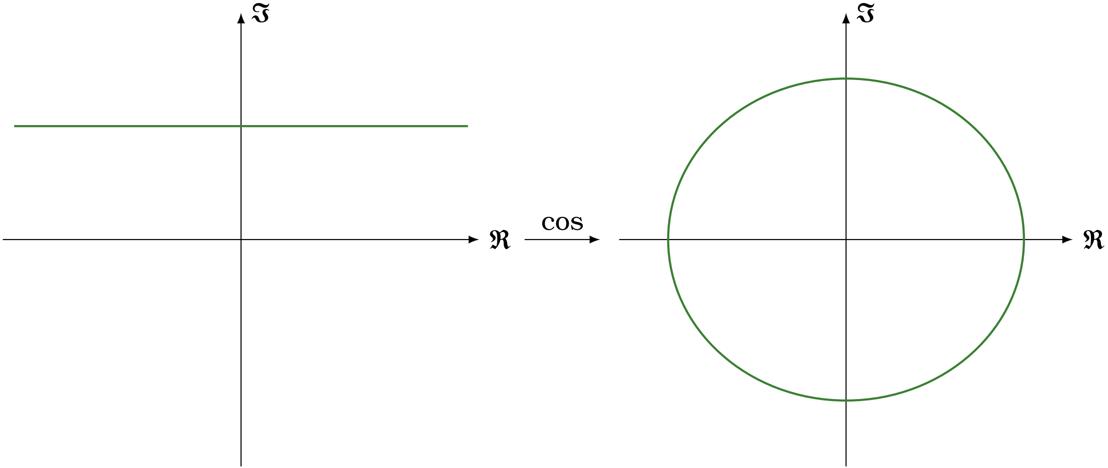

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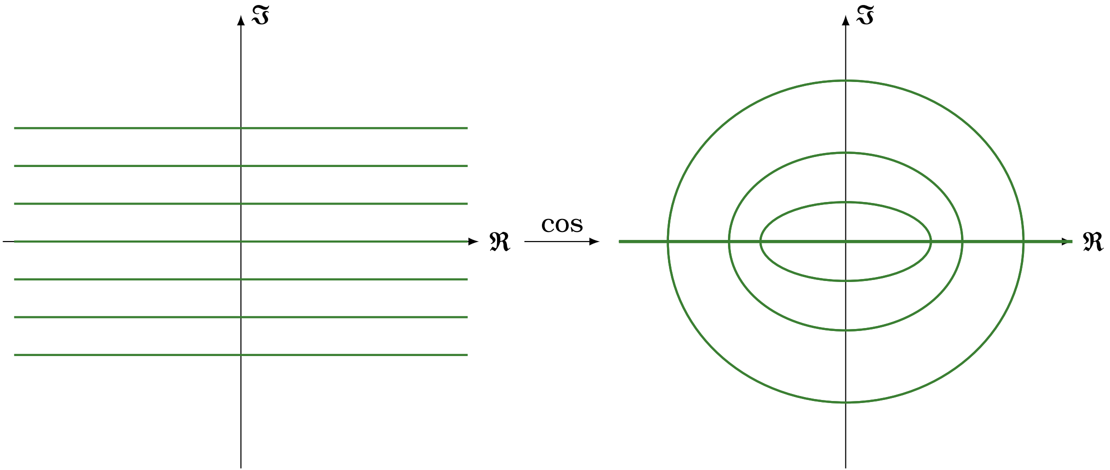
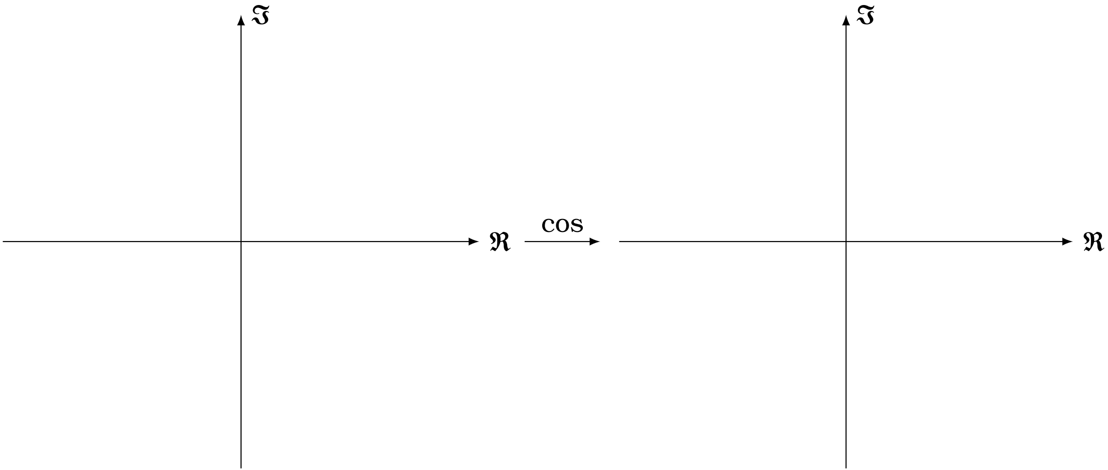


The complex cosine and sine – 4

The vertical line $\Re(z) = c$ is mapped by cos to $\left\{ \cos(c)\left(\frac{e^y+e^{-y}}{2}\right) + i\sin(c)\left(\frac{e^{-y}-e^y}{2}\right) \;:\; y \in \mathbb{R} \right\}$ which is a branch of hyperbola.



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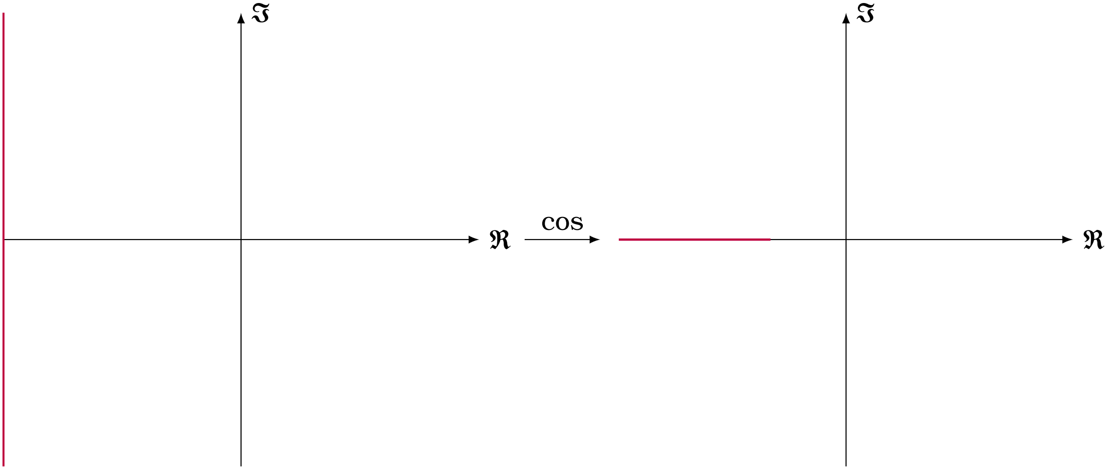

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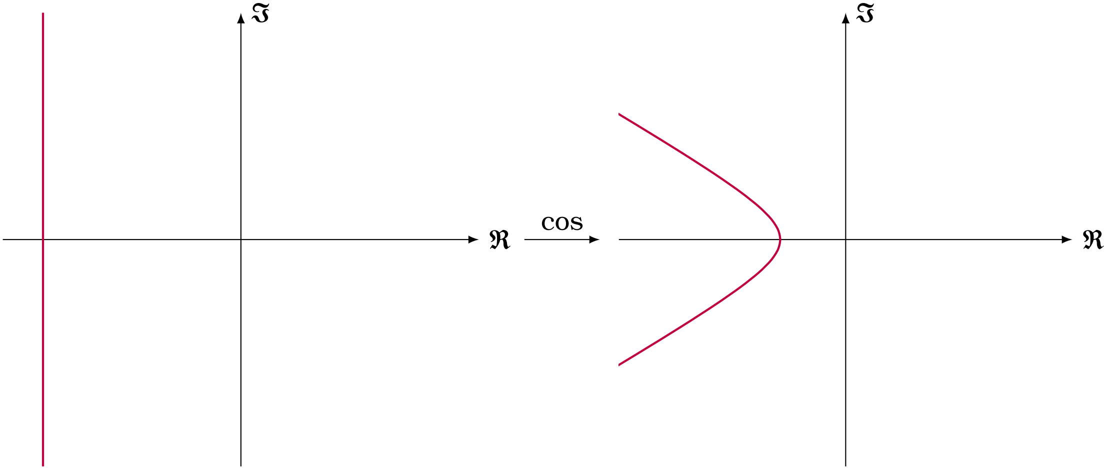

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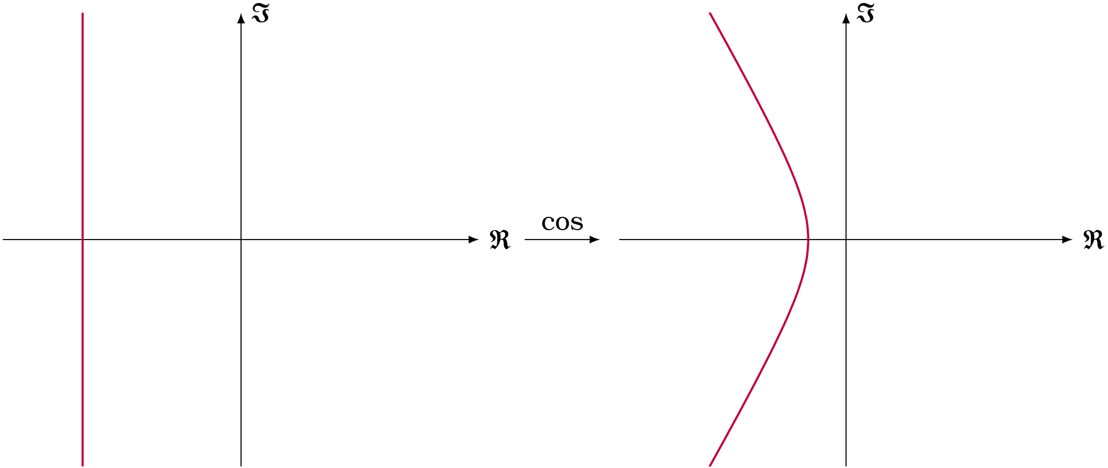

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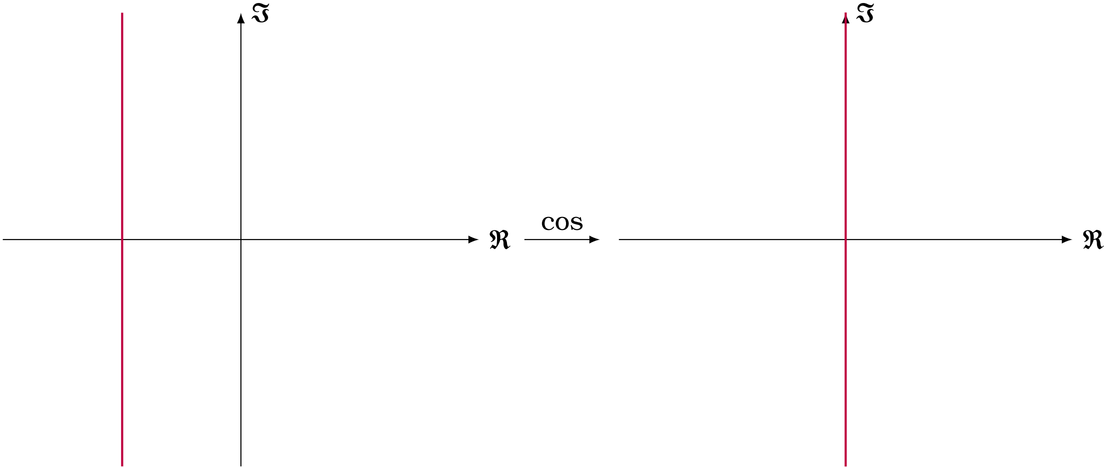

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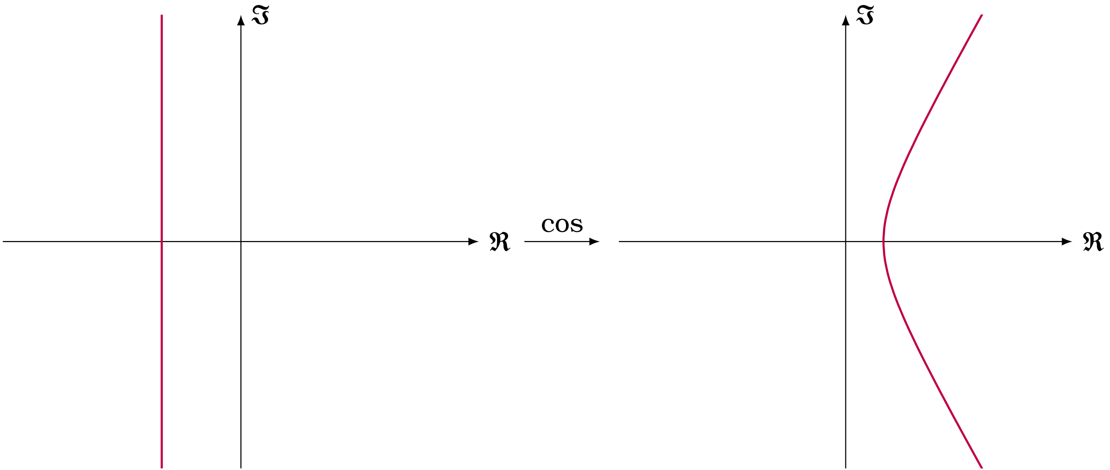

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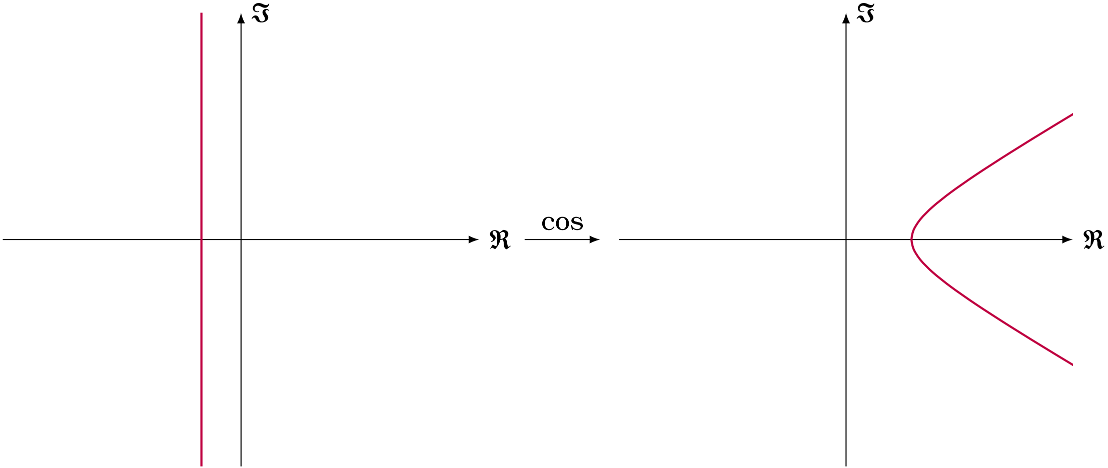

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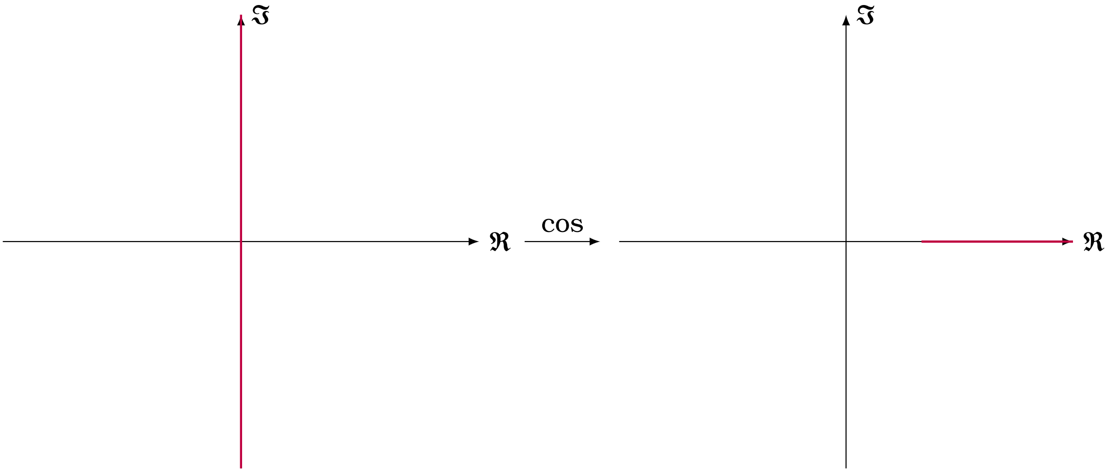

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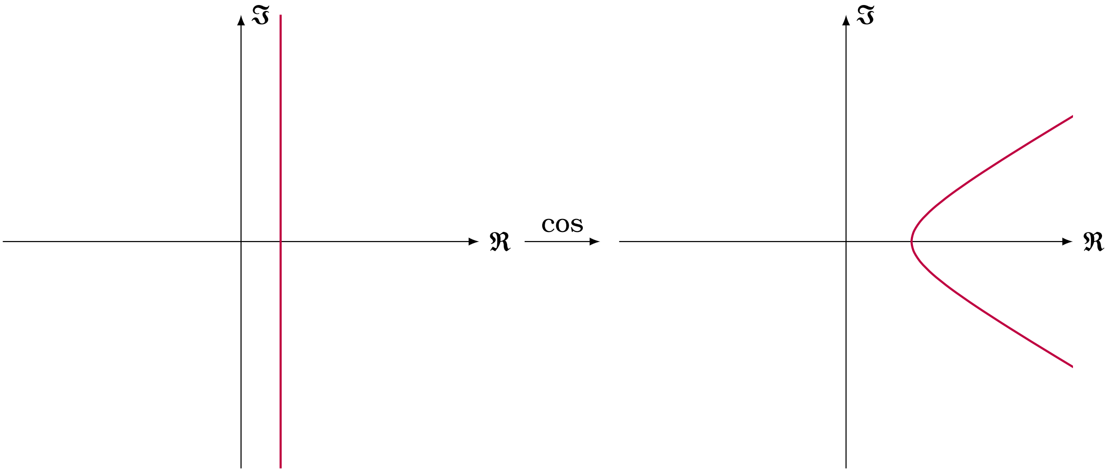

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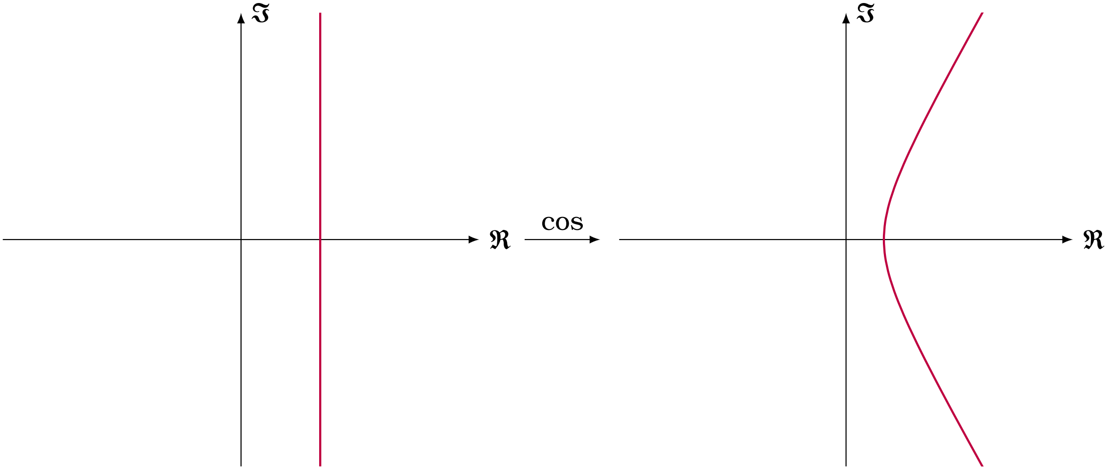

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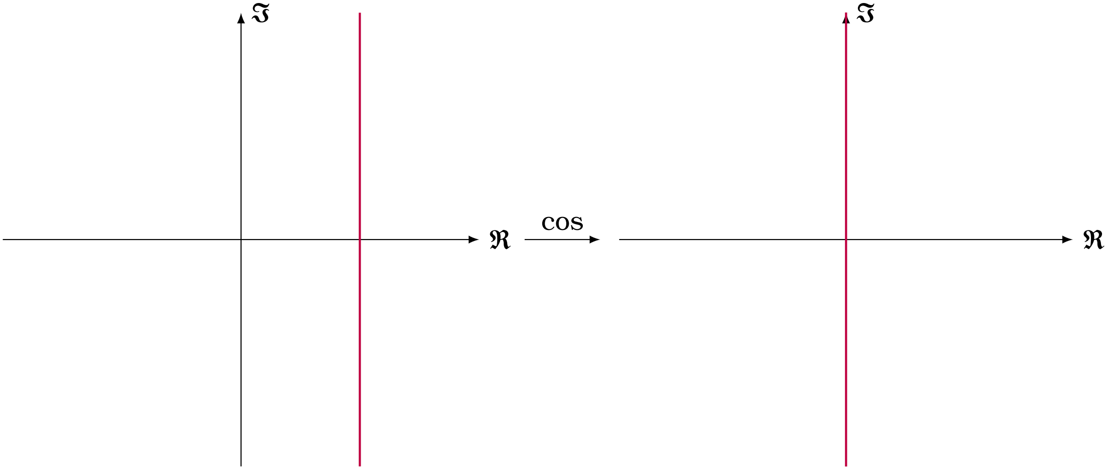

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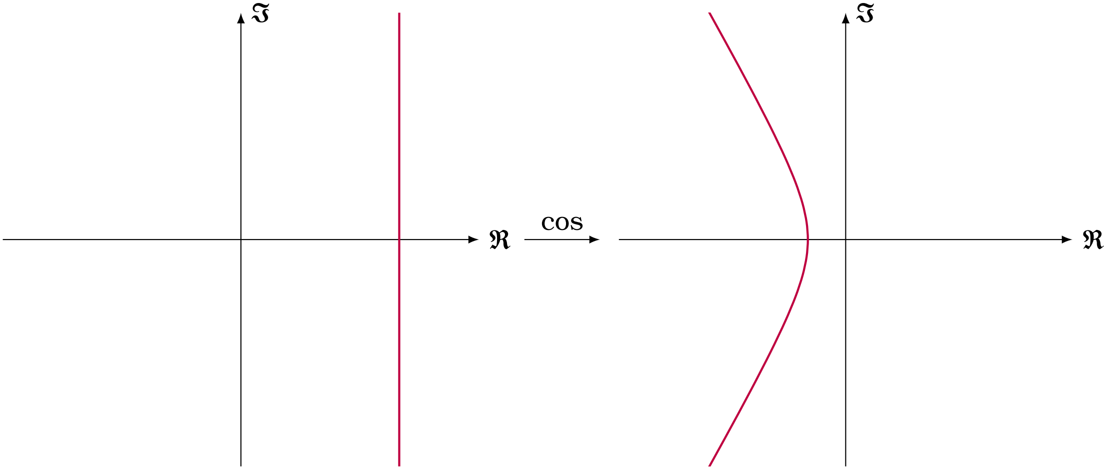

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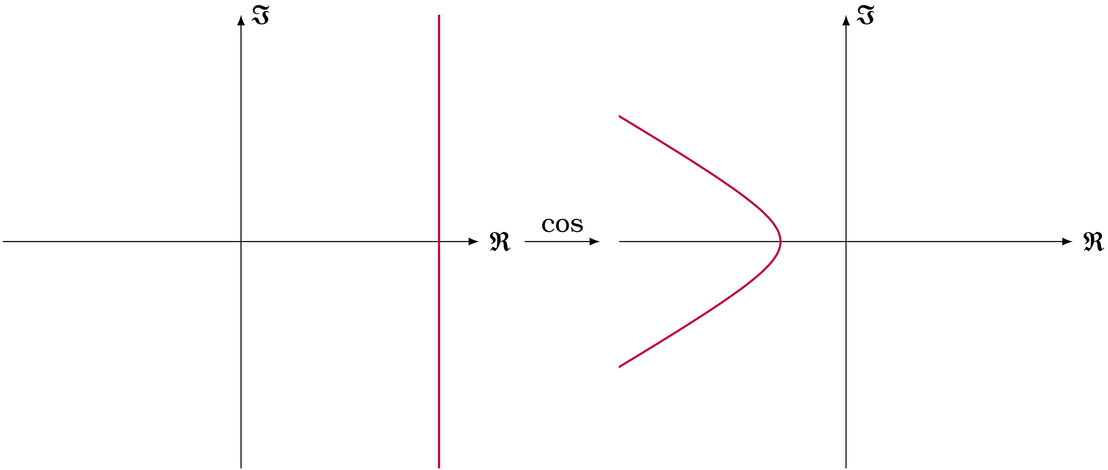

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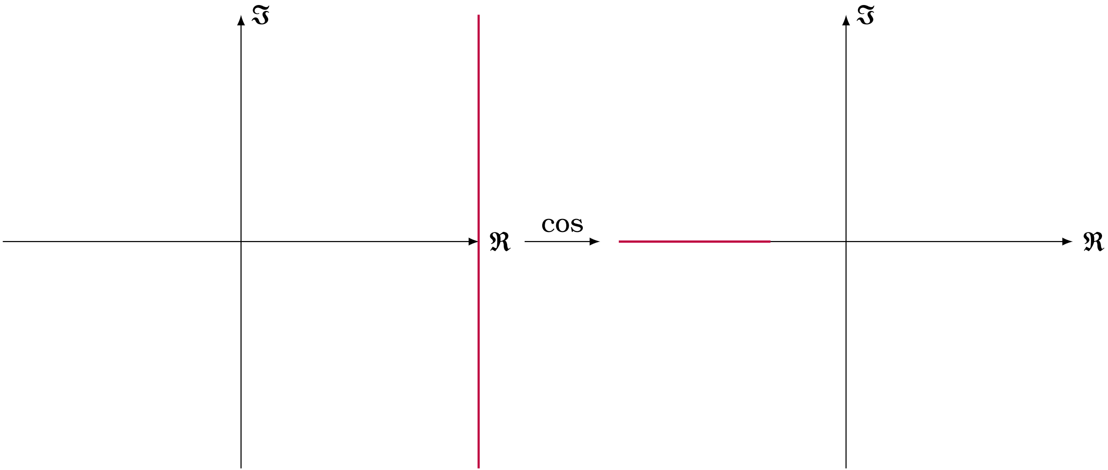

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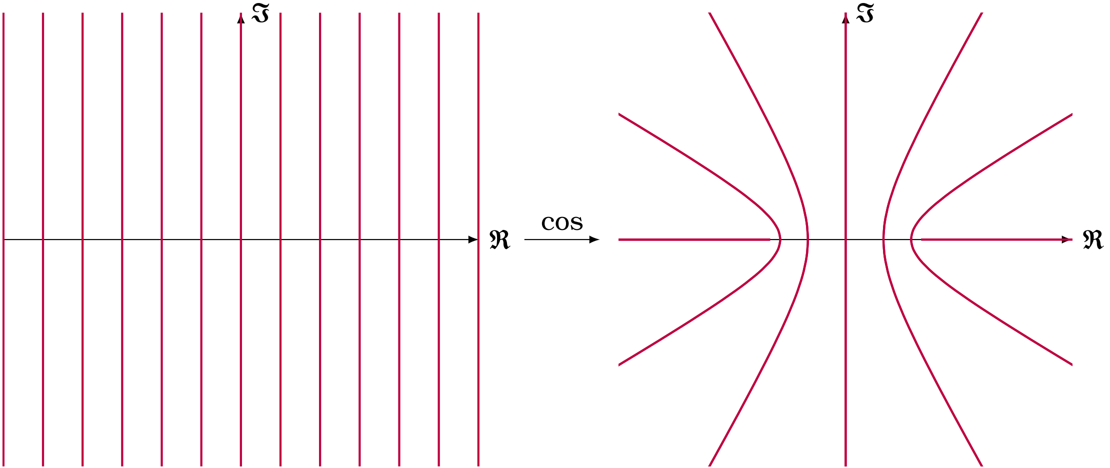
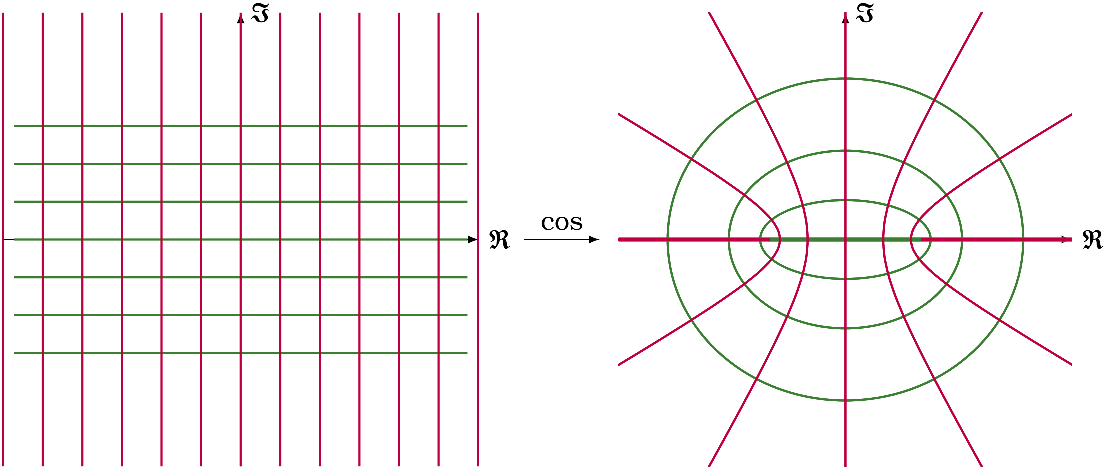


The complex cosine and sine – 5



The complex cosine and sine – 6

Proposition

- $\cos z = 0 \Leftrightarrow \exists n \in \mathbb{Z},\ z = \frac{\pi}{2} + \pi n$
- $\sin z = 0 \Leftrightarrow \exists n \in \mathbb{Z},\ z = \pi n$

Proof.

$$\begin{aligned}\cos z = 0 &\Leftrightarrow e^{iz} + e^{-iz} = 0 \\ &\Leftrightarrow e^{2iz} = -1 \\ &\Leftrightarrow \exists n \in \mathbb{Z},\ 2iz = i\pi + 2i\pi n \\ &\Leftrightarrow \exists n \in \mathbb{Z},\ z = \frac{\pi}{2} + \pi n\end{aligned}$$

$$\begin{aligned}\sin z = 0 &\Leftrightarrow e^{iz} - e^{-iz} = 0 \\ &\Leftrightarrow e^{2iz} = 1 \\ &\Leftrightarrow \exists n \in \mathbb{Z},\ 2iz = 2i\pi n \\ &\Leftrightarrow \exists n \in \mathbb{Z},\ z = \pi n\end{aligned}$$

■

Other complex trigonometric functions

Definition: the complex tangent function

We define $\tan : \mathbb{C} \setminus \left\{ \frac{\pi}{2} + \pi n \ : \ n \in \mathbb{Z} \right\} \to \mathbb{C}$ by

$$\tan z := \frac{\sin z}{\cos z}$$

Definition: the complex cotangent function

We define $\cot : \mathbb{C} \setminus \{ \pi n \ : \ n \in \mathbb{Z} \} \to \mathbb{C}$ by

$$\cot z := \frac{\cos z}{\sin z}$$

Inverse trigonometric functions – 1

We want to find *the* inverse of $\cos$.
($\cos$ *is not injective so we will get a multivalued function as for* $\log$).

$$\cos(w) = z \Leftrightarrow \left(e^{iw}\right)^2 - 2ze^{iw} + 1 = 0$$

Then

$$v^2 - 2zv + 1 = 0 \Leftrightarrow v = z + \sqrt{z^2 - 1}$$

Here $\sqrt{\cdot}$ *is already multivalued: it is only well defined up to its sign.*

Hence

$$\begin{aligned} e^{iw} = z + \sqrt{z^2 - 1} &\Leftrightarrow iw = \log\left(z + \sqrt{z^2 - 1}\right) \\ &\Leftrightarrow w = -i\log\left(z + \sqrt{z^2 - 1}\right) \end{aligned}$$

We may repeat the same thing for $\arcsin$ and $\arctan$.

Inverse trigonometric functions – 2

$$\arccos(z) = -i\log\left(z + \sqrt{z^2 - 1}\right)$$

$$\arcsin(z) = -i\log\left(iz + \sqrt{1 - z^2}\right)$$

$$\arctan(z) = \frac{i}{2}\log\left(\frac{1 - iz}{1 + iz}\right),\ z \neq \pm i$$

Beware

They are **multivalued** functions defined on $\mathbb{C}$.

Beware

By the way, the range of $\tan$ is $\mathbb{C} \setminus \{\pm i\}$.

Inverse trigonometric functions – 3

We define the principal branches of arccos, arcsin and arctan by[1]:

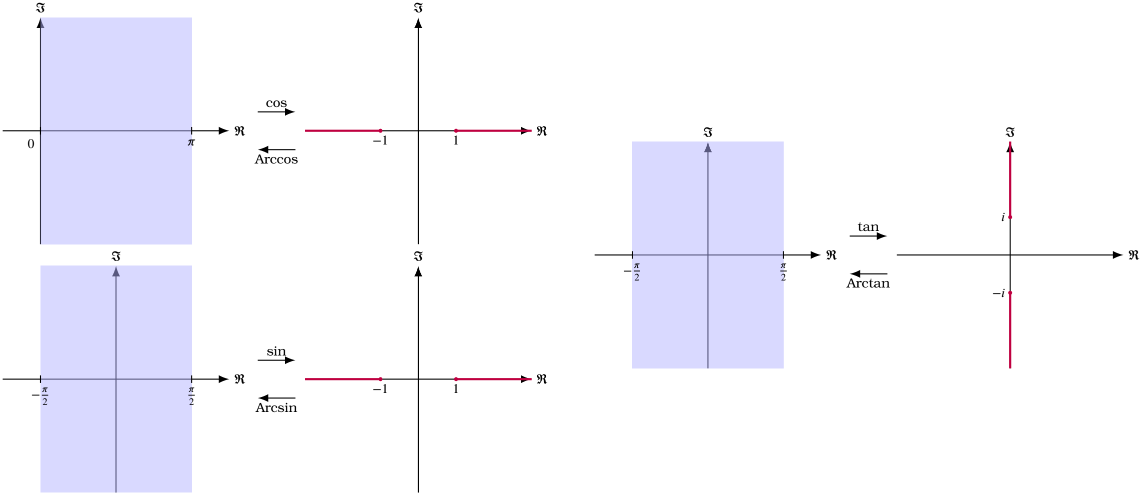
$$\operatorname{Arccos}: \begin{array}{ccc} \mathbb{C}\setminus\left((-\infty,-1]\cup[1,+\infty)\right) & \to & \{z\in\mathbb{C} \,:\, 0<\Re(z)<\pi\} \\ z & \mapsto & -i\operatorname{Log}\left(z+\sqrt{z^2-1}\right) \end{array}$$

$$\operatorname{Arcsin}: \begin{array}{ccc} \mathbb{C}\setminus\left((-\infty,-1]\cup[1,+\infty)\right) & \to & \left\{z\in\mathbb{C} \,:\, -\frac{\pi}{2}<\Re(z)<\frac{\pi}{2}\right\} \\ z & \mapsto & -i\operatorname{Log}\left(iz+\sqrt{1-z^2}\right) \end{array}$$

$$\operatorname{Arctan}: \begin{array}{ccc} \mathbb{C}\setminus\left((-i\infty,-i]\cup[i,+i\infty)\right) & \to & \left\{z\in\mathbb{C} \,:\, -\frac{\pi}{2}<\Re(z)<\frac{\pi}{2}\right\} \\ z & \mapsto & \frac{i}{2}\operatorname{Log}\left(\frac{1-iz}{1+iz}\right) \end{array}$$

[1] We take the square root whose real part is non-negative.

Inverse trigonometric functions – 4



Inverse trigonometric functions – 5

$$\arccos(z) = \{\pm \operatorname{Arccos}(z) + 2\pi n \ : \ n \in \mathbb{Z}\}$$

$$\arcsin(z) = \left\{(-1)^n \operatorname{Arcsin}(z) + \pi n \ : \ n \in \mathbb{Z}\right\}$$

$$\arctan(z) = \{\operatorname{Arctan}(z) + \pi n \ : \ n \in \mathbb{Z}\}$$

The complex hyperbolic functions

Definitions

We define $\cosh : \mathbb{C} \to \mathbb{C}$ and $\sinh : \mathbb{C} \to \mathbb{C}$ respectively by

$$\cosh(z) := \frac{e^z + e^{-z}}{2} \qquad \text{and} \qquad \sinh(z) := \frac{e^z - e^{-z}}{2}$$

Proposition

- $\forall z \in \mathbb{C},\ \cos(z) = \cosh(iz)$
- $\forall z \in \mathbb{C},\ \sin(z) = -i\sinh(iz)$

The above proposition allows us to derive hyperbolic identities from the trigonometric ones.

Homework

- $\forall z \in \mathbb{C},\ \cosh^2(z) - \sinh^2(z) = 1$
- $\forall z, w \in \mathbb{C},\ \cosh(z + w) = \cosh z \cosh w + \sinh z \sinh w$
- $\forall z, w \in \mathbb{C},\ \sinh(z + w) = \sinh z \cosh w + \cosh z \sinh w$
- $\forall x, y \in \mathbb{R},\ \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$
- $\forall x, y \in \mathbb{R},\ \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$