University of Toronto – MAT334H1-F – LEC0101

Complex Variables

10 - Consequences of Cauchy's integral formula

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Contents

1 Holomorphic functions are analytic

Theorem 1. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic/$\mathbb{C}$-differentiable. Then f can be* ***locally*** *expressed as a power series in a neighborhood of any point of U.*

More precisely, if $\overline{D_r}(z_0) \subset U$ then

$$\forall z \in D_r(z_0),\ f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w$$

and $\gamma : [0, 1] \to \mathbb{C}$ is defined by $\gamma(t) = z_0 + re^{2i\pi t}$ and the radius of convergence of this power series is greater than or equal to r.

Proof. By Cauchy's integral formula

$$\begin{aligned}
f(z) &= \frac{1}{2i\pi}\int_\gamma \frac{f(w)}{w-z}\mathrm{d}w \\
&= \frac{1}{2i\pi}\int_\gamma \frac{f(w)}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}}\mathrm{d}w \\
&= \frac{1}{2i\pi}\int_\gamma \frac{f(w)}{w-z_0}\sum_{n=0}^{+\infty}\left(\frac{z-z_0}{w-z_0}\right)^n \mathrm{d}w && \text{since } |z-z_0|<|w-z_0|=r \\
&= \sum_{n=0}^{+\infty}\left(\frac{1}{2i\pi}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w\right)(z-z_0)^n
\end{aligned}$$

We need to justify the last equality (i.e. the permutation $\int - \sum$).

Let $\varepsilon > 0$. There exists N such that if $k \geq N$ then the remainder satisfies $\left|\sum_{n=k}^{+\infty}\left(\frac{z-z_0}{w-z_0}\right)^n\right| \leq \varepsilon$, so that

$$\begin{aligned}
\left|f(z) - \sum_{n=0}^{k}\left(\frac{1}{2i\pi}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w\right)(z-z_0)^n\right| &\leq \left|\frac{1}{2i\pi}\int_\gamma \frac{f(w)}{w-z_0}\sum_{n=k}^{+\infty}\left(\frac{z-z_0}{w-z_0}\right)^n\mathrm{d}w\right| \\
&\leq \frac{1}{2\pi}\frac{\max_{|w-z_0|=r}|f|}{r}\varepsilon\,\mathrm{Length}(\gamma) \\
&\leq \varepsilon \max_{|w-z_0|=r}|f|
\end{aligned}$$

■

Corollary 2. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be a function.*
Then f is holomorphic/analytic/$\mathbb{C}$-differentiable if and only if f can be ***locally*** *expressed as a power series in a neighborhood of any point of U.*

Proof. $\Rightarrow$: by Theorem 1.

$\Leftarrow$: if $f(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ in $D_r(z_0)$ then f is holomorphic on $D_r(z_0)$ from last week lecture. ■

Corollary 3. *A holomorphic/analytic/$\mathbb{C}$-differentiable function is infinitely many times $\mathbb{C}$-differentiable.*

Remark 4. The previous results are false for $\mathbb{R}$-differentiability.

- Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$

Then f is $\mathbb{R}$-differentiable but not C^1.

- Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$

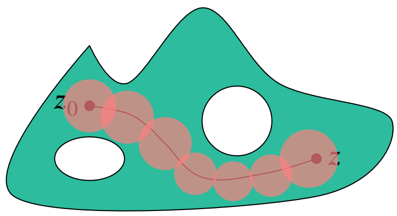
Then g is $\mathbb{R}$-differentiable, even C^∞, but not analytic at 0,
i.e. it can't be expressed as a power series around 0:
Indeed $\forall n \in \mathbb{N}_{\geq 0}$, $g^{(n)}(0) = 0$, so if g were equal to its Taylor series around 0 then it would be constant equal to 0 but g is non-zero in any neighborhood of 0.

• By a theorem of Borel[⋆], given a real sequence $(a_n)_{n\in\mathbb{N}_{\geq 0}}$, there exists a C^∞ function defined in a neighborhood of 0 in $\mathbb{R}$ such that $\forall n \in \mathbb{N}_{\geq 0}$, $f^{(n)}(0) = a_n$.
Otherwise stated, any real power series is the Taylor expansion of a C^∞ function.
If we take $a_n = (n!)^2$ then we obtain a power series whose radius of convergence is $R = 0$, hence a function with such a Taylor expansion can't be analytic.

2 Continuation of analytic functions

Theorem 5. *Let $U \subset \mathbb{C}$ be a **domain** and $f : U \to \mathbb{C}$ be a holomorphic/analytic function.*
If there exists $z_0 \in U$ such that $\forall n \in \mathbb{N}_{\geq 0}$, $f^{(n)}(z_0) = 0$ then $f \equiv 0$ on U.

Proof. Let $z \in U$. Since U is path connected, there exists a curve $\gamma : [a,b] \to \mathbb{C}$ with $\gamma(a) = z_0$ and $\gamma(b) = z$.
Since U is open, for every $w \in \gamma([a,b])$ there exists $r_w > 0$ such that $D_{r_w}(w) \subset U$. Since $\gamma([a,b])$ is compact we may assume that it is covered by finitely many of these disks $D_{r_1}(w_1), \dots, D_{r_k}(w_k)$.



By Theorem 1, if there exists $v \in D_{r_i}(w_i)$ such that $\forall n \in \mathbb{N}_{\geq 0}$, $f^{(n)}(v) = 0$ then $f \equiv 0$ on $D_{r_i}(w_i)$.
Two consecutive disks intersect (since they cover γ), so we conclude using the previous remark disk by disk from z_0 to z. ■

Remark 6. If you attend MAT327, another proof consists in showing that $\{z \in U \ : \ \forall n \in \mathbb{N}_{\geq 0},\ f^{(n)}(z) = 0\}$ is open, closed, and non-empty, hence equals to U by connectedness.

Corollary 7. *Let $U \subset \mathbb{C}$ be a **domain** and $f, g : U \to \mathbb{C}$ be holomorphic/analytic functions.*
If f and g coincide in the neighborhood of a point,

$$\text{i.e. } \exists z_0 \in U,\ \exists r > 0,\ \forall z \in D_r(z_0) \cap U,\ f(z) = g(z),$$

then they coincide on U,

$$\textit{i.e. } \forall z \in U,\ f(z) = g(z).$$

Proof. Then $\forall n \in \mathbb{N}_{\geq 0}$, $(f-g)^{(n)}(z_0) = 0$ (since $f - g \equiv 0$ on $D_r(z_0) \cap U$).
Hence $f - g \equiv 0$ on U by the previous theorem. ■

Remark 8. The previous results are false for $\mathbb{R}$-differentiability. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

then f is $\mathbb{R}$-differentiable (even C^∞). And

• $\forall n \in \mathbb{N}_{\geq 0}$, $f^{(n)}(0) = 0$ but $f \not\equiv 0$.

• $\forall x \in (-\infty, 0)$, $f(x) = 0$ but $f \not\equiv 0$.

[⋆] It generalizes to multivariable functions.

A common way to construct an analytic function consists in defining it in a "small" domain and then to extend it to an analytic function with a bigger domain.
By the above result, this analytic continuation, if it exists, is unique.
That's a very powerful tool: knowing a function on a "small" domain determines the function everywhere else[⋆]. *Holomorphic/analytic functions are very rigid!*

⚠ The maximal domain may not be $\mathbb{C}$: for instance, if we try to extend Log, we won't be able to do a full turn around the origin since we won't recover the same values (it increases by $2i\pi$).

Example 9. Let $f : D_1(0) \to \mathbb{C}$ be defined by $f(z) = \sum_{n=0}^{+\infty} z^n$.
Then f coincides with $\frac{1}{1-z}$ on $D_1(0)$.
Hence we may extend f with $F : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ defined by $F(z) = \frac{1}{1-z}$.

3 Order of a zero

Definition 10. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic.
Let $z_0 \in U$ be such that $f(z_0) = 0$. We define the **order of vanishing of f at z_0** by $m_f(z_0) := \min \left\{ n \in \mathbb{N} \; : \; f^{(n)}(z_0) \neq 0 \right\}$.

Remark 11. Note that $m_f(z_0) > 0$ since $f(z_0) = f^{(0)}(z_0) = 0$.

Proposition 12. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic. Let $z_0 \in U$ be such that $f(z_0) = 0$. Denote the power series expansion of f at z_0 by $f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$. Then $m_f(z_0) = \min \left\{ n \in \mathbb{N} \; : \; a_n \neq 0 \right\}$.*

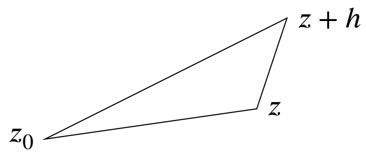
Proof. $a_n = \frac{f^{(n)}(z_0)}{n!}$ ■

Proposition 13. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic.
Then z_0 is a zero of order $n \in \mathbb{N}_{>0}$ of f if and only if there exists $g : U \to \mathbb{C}$ holomorphic such that $f(z) = (z-z_0)^n g(z)$ and $g(z_0) \neq 0$.*

4 Morera's theorem

Theorem 14 (Morera's theorem)**.** *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous.
If for every (full) triangle T lying in U we have $\int_{\partial T} f = 0$ then f is holomorphic/analytic on U.*

Proof. Let $z_0 \in U$ and $r > 0$ be such that $D_r(z_0) \subset U$.
We define $F : D_r(z_0) \to \mathbb{C}$ by $F(z) = \int_{[z_0, z]} f$.
Let $z, h \in \mathbb{C}$ be such that $z, z + h \in D_r(z_0)$ then, considering the triangle whose vertices are z_0, z and $z + h$, we obtain $\int_{[z_0,z]} f + \int_{[z,z+h]} f + \int_{[z+h,z_0]} f = 0$, i.e. $F(z + h) - F(z) = \int_{[z,z+h]} f$.



⋆ And we will even weaken the assumptions later this term: it is enough to know the function on a set with a limit point.

Then

$$\begin{aligned}
\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| &= \left|\frac{\left(\int_{[z,z+h]} f\right)-hf(z)}{h}\right| \\
&= \frac{1}{|h|}\left|\int_{[z,z+h]}(f(w)-f(z)\mathrm{d}w\right| \\
&\leq \frac{\sup\limits_{w\in[z,z+h]}|f(w)-f(z)|}{|h|}\,\mathrm{Length}([z,z+h]) \\
&= \sup_{w\in[z,z+h]}|f(w)-f(z)| \xrightarrow[h\to 0]{} 0
\end{aligned}$$

Hence F is holomorphic on $D_r(z_0)$ and $F'=f$. Furthermore, f is holomorphic on $D_r(z_0)$ (and hence at z_0) as the complex derivative of a holomorphic function. ■

5 Characterizations of holomorphicity/analyticity

Theorem 15. *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$. Then the following are equivalent:*

1. *f is holomorphic/analytic/$\mathbb{C}$-differentiable, i.e. $\forall z_0 \in U$, $\lim\limits_{h\to 0}\dfrac{f(z_0+h)-f(z_0)}{h}$ exists.*
2. *$\tilde{f} : \tilde{U} \to \mathbb{R}^2$ is $\mathbb{R}$-differentiable and satisfies the Cauchy–Riemann equations on $\tilde{U}$.*
3. *f may be written as a power series $f(z)=\displaystyle\sum_{n=0}^{+\infty}a_n(z-z_0)^n$ on a neighborhood of every $z_0 \in U$.*
4. *f is continuous and for every (full) triangle T lying in U we have $\displaystyle\int_{\partial T} f = 0$.*
5. *f is continuous and for every simple closed curve γ on U whose inside is also included in U, we have $\displaystyle\int_\gamma f = 0$.*
6. *f admits local primitives/antiderivatives: for every $z_0 \in U$ there exists $F : D_r(z_0)\cap U \to \mathbb{C}$ holomorphic for some $r>0$ such that $F'=f$ on $D_r(z_0)\cap U$.*

Remark 16. When U is simply-connected, we may drop the assumption that the inside of the triangle/curve is included in U in (4) and (5). Furthermore we may also replace "local primitives/antiderivatives" by "a primitive/antiderivative" in (6) i.e. there exists $F : U \to \mathbb{C}$ holomorphic s.t. $F'=f$.

6 Liouville's theorem

Definition 17. We say that a function $f : \mathbb{C} \to \mathbb{C}$ is **entire** if it is holomorphic (everywhere) on $\mathbb{C}$.

Theorem 18 (Liouville's theorem)**.** *A bounded entire function is constant.*

Lemma 19 (Cauchy's inequalities)**.** *Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic.*
Let $r>0$. If $D_r(z_0) \subset U$ then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{r^n}\max_{|z-z_0|=r}|f(z)|$$

Proof.

$$\left|\frac{f^{(n)}(z_0)}{n!}\right| = \left|\frac{1}{2i\pi}\int_{|z-z_0|=r}\frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w\right| \leq \frac{\max\limits_{|z-z_0|=r}|f(z)|}{r^{n+1}}\frac{\mathrm{Length}\left(|z-z_0|=r\right)}{2\pi} = \frac{\max_{|z-z_0|=r}|f(z)|}{r^n}$$

For the first equality: we know that the n-th coefficient of the power expansion of f at z_0 is $a_n = \frac{f^{(n)}(z_0)}{n!}$ and by Theorem 1 that it is also equal to $a_n = \frac{1}{2i\pi}\int_{|z-z_0|=r} \frac{f(w)}{(w-z_0)^{n+1}}\mathrm{d}w$. ∎

Proof of Liouville's theorem. Assume that there exists $M > 0$ such that $\forall z \in \mathbb{C},\ |f(z)| \leq M$.
Let $z \in \mathbb{C}$. For $r > 0$, by Cauchy's inequality with $n = 1$, we have $\left|f'(z)\right| \leq \frac{M}{r} \xrightarrow[r\to+\infty]{} 0$.
Hence, $\forall z \in \mathbb{C},\ f'(z) = 0$. Therefore f is constant on $\mathbb{C}$. ∎

Theorem 20 (d'Alembert–Gauss theorem or the Fundamental Theorem of Algebra)**.**
A non-constant polynomial with coefficients in $\mathbb{C}$ admits a root/zero in $\mathbb{C}$.

Proof. Assume that P is a complex polynomial with no root in $\mathbb{C}$.
Then $Q = \frac{1}{P}$ is a entire function and Q is bounded since $\lim\limits_{z\to+\infty} Q = 0$.
Therefore, by Liouville's theorem, Q is constant, and so is P. ∎

7 Analytic logarithm

Theorem 21. *Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ a holomorphic/analytic function which doesn't vanish, i.e. $\forall z \in U,\ f(z) \neq 0$.*
Then there exists $g : U \to \mathbb{C}$ holomorphic/analytic such that $e^g = f$.

Proof. Since f doesn't vanish, $\frac{f'}{f}$ is holomorphic on U.
Then, since U is simply connected, $\frac{f'}{f}$ admits a complex primitive/antiderivative, i.e. there exists $\tilde{g} : U \to \mathbb{C}$ holomorphic/analytic such that $\tilde{g}' = \frac{f'}{f}$.
Then $(fe^{-\tilde{g}})' = (f' - \tilde{g}'f)e^{-\tilde{g}} = 0$ and therefore $fe^{-\tilde{g}} = K$ is constant since U is connected.
Since $K \neq 0$, there exists $w \in \mathbb{C}$ such that $e^w = K$.
Then, for $g = \tilde{g} + w$, we have $e^g = f$. ∎

Remark 22. Such a function g is not unique! For instance $g + 2i\pi$ is another suitable function.

8 Multiplication of complex power series

We are going to prove the following theorem, but using results related to analytic functions.

Theorem 23. *Let $S_A(z) = \sum_{n=0}^{+\infty} a_n(z-z_0)^n$ and $S_B(z) = \sum_{n=0}^{+\infty} b_n(z-z_0)^n$ be two power series of radii R_A and R_B.*
We define $S_{AB}(z) \coloneqq \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(z-z_0)^n$ and denote its radius of convergence by R_{AB}.
Then

1. *$R_{AB} \geq \min(R_A, R_B)$.*
2. *If $|z-z_0| < \min(R_A, R_B)$ then $S_{AB}(z) = S_A(z)S_B(z)$,*
   i.e. $\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)(z-z_0)^n = \left(\sum_{n=0}^{+\infty} a_n(z-z_0)^n\right)\left(\sum_{n=0}^{+\infty} b_n(z-z_0)^n\right)$.

Proof. We know that S_A and S_B are holomorphic on $z \in \mathbb{C}$ such that $|z-z_0| < \min(R_A, R_B)$.
Then $f(z) = S_A(z)S_B(z)$ is holomorphic on $|z-z_0| < \min(R_A, R_B)$, so it can be written as a power series

$f(z) = \sum_{n=0}^{+\infty} c_n(z-z_0)^n$ on $|z-z_0| < \min(R_A, R_B)$ (particularly its radius of convergence is at least $\min(R_A, R_B)$).
Then

$$\begin{aligned} c_n = \frac{f^{(n)}(z_0)}{n!} &= \frac{1}{n!}\sum_{k=0}^{n} \frac{n!}{(n-k)!k!} S_A^{(n-k)}(z_0) S_B^{(k)}(z_0) \text{ by Leibniz rule} \\ &= \sum_{k=0}^{n} \frac{S_A^{(n-k)}(z_0)}{(n-k)!} \frac{S_B^{(k)}(z_0)}{k!} \\ &= \sum_{k=0}^{n} a_{n-k} b_k = \sum_{k=0}^{n} a_k b_{n-k} \end{aligned}$$

■