MAT334H1-F – LEC0101

Complex Variables

http://uoft.me/MAT334-LEC0101

HOLOMORPHIC/ANALYTIC FUNCTIONS

UNIVERSITY OF TORONTO

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Holomorphic functions

Definition: holomorphic function

Let $\mathscr{U} \subset \mathbb{C}$ be open, $f : \mathscr{U} \to \mathbb{C}$ and $z_0 \in \mathscr{U}$.
We say that f is **holomorphic at** z_0 (or **analytic**[1] **at** z_0, or $\mathbb{C}$**-differentiable at** z_0) if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \text{ exists.}$$

Then we set $f'(z_0) \coloneqq \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$.
We say that f is holomorphic/analytic/$\mathbb{C}$-differentiable if it is everywhere on $\mathscr{U}$.

[1] I don't like the word *analytic* because it means that f can be locally expressed around z_0 as a power series. It is true that $\mathbb{C}$-differentiability is equivalent to analytic but you don't know that yet and analyticity is also well defined over $\mathbb{R}$.
In MAT334, you can use interchangeably *analytic* or *holomorphic* and assume they are synonyms.

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We say that f is holomorphic/analytic/$\mathbb{C}$-differentiable if it is everywhere on $\mathcal{U}$.

Note that $\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ so you can use any of these two limits.

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The Cauchy–Riemann equations – 1

Theorem

Let $\mathcal{U} \subset \mathbb{C}$ be open, $f : \mathcal{U} \to \mathbb{C}$, $z_0 = x_0 + iy_0 \in \mathbb{C}$. We set $\tilde{\mathcal{U}} := \left\{(x, y) \in \mathbb{R}^2 \ : \ x + iy \in \mathcal{U}\right\}$.
Assume that $f(x + iy) = u(x, y) + iv(x, y)$ and define $\tilde{f} : \tilde{\mathcal{U}} \to \mathbb{R}^2$ by $\tilde{f}(x, y) = (u(x, y), v(x, y))$.
Then the following are equivalent:

- f is holomorphic/analytic/$\mathbb{C}$-differentiable at z_0
- $\tilde{f}$ is $\mathbb{R}$-differentiable[2] at (x_0, y_0) and its partial derivatives at (x_0, y_0) satisfy the Cauchy–Riemann equations

$$\begin{cases} \dfrac{\partial u}{\partial x}(x_0, y_0) & = & \dfrac{\partial v}{\partial y}(x_0, y_0) \\ \dfrac{\partial u}{\partial y}(x_0, y_0) & = & -\dfrac{\partial v}{\partial x}(x_0, y_0) \end{cases}$$

In this case, $f'(z_0) = \dfrac{\partial u}{\partial x}(x_0, y_0) + i\dfrac{\partial v}{\partial x}(x_0, y_0)$.

[2] i.e. as a real multivariable function, in the sense of your multivariable course from last year.

The Cauchy–Riemann equations – 2

Comment

You may already notice that *"f is holomorphic/analytic/$\mathbb{C}$-differentiable at $z_0 = x_0 + iy_0$"* is stronger than *"$\tilde{f}$ is $\mathbb{R}$-differentiable at (x_0, y_0)"* since the Cauchy–Riemann equations are required to be satisfied.

It is due to the fact that the complex multiplication plays a role in the definition of *holomorphic/analytic/$\mathbb{C}$-differentiable* (indeed, we divide by h which is a complex number, and not by $|h|$ as in the definition of $\mathbb{R}$-differentiability which relies on the real scalar multiplication).

During this term we will see that *holomorphic/analytic/$\mathbb{C}$-differentiable* functions are very rigid: they satisfy strong properties.

The Cauchy–Riemann equations – 3

The Cauchy–Riemann equations come in various equivalent flavours:

1. $$\begin{cases} \dfrac{\partial u}{\partial x}(x_0, y_0) = \dfrac{\partial v}{\partial y}(x_0, y_0) \\ \dfrac{\partial u}{\partial y}(x_0, y_0) = -\dfrac{\partial v}{\partial x}(x_0, y_0) \end{cases}$$ And in this case, $f'(z_0) = \dfrac{\partial u}{\partial x}(x_0, y_0) + i\dfrac{\partial v}{\partial x}(x_0, y_0)$.

2. $\operatorname{Jac}_{\tilde{f}}(x_0, y_0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$. Then $f'(z_0) = a + ib$.

3. $\dfrac{\partial f}{\partial y}(z_0) = i\dfrac{\partial f}{\partial x}(z_0)$.

4. $\dfrac{\partial f}{\partial \overline{z}}(x_0, y_0) = 0$.

(3, 4) Then $f'(z_0) = \dfrac{\partial f}{\partial x}(z_0) = -i\dfrac{\partial f}{\partial y}(z_0) = \dfrac{\partial f}{\partial z}(z_0)$.

Remember that $\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$ *and* $\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$.

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From the second version of the C-R equations, we obtain the following geometric interpretation:
if f is holomorphic at z_0 and $f'(z_0) \neq 0$ then f is conformal at z_0, i.e. f preserves angles at z_0 (and there is a *converse* result).

Example: $f(z) = \overline{z}$

We define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = \overline{z}$.

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1. *Direct computation:* Let $z_0 \in \mathbb{C}$.

   We want to compute $\displaystyle\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0} \frac{\overline{h}}{h}$.

   Write $h = \rho e^{i\theta}$ where $\rho \in (0,\infty)$ then $\displaystyle\lim_{\rho\to 0^+} \frac{f(z_0+\rho e^{i\theta})-f(z_0)}{\rho e^{i\theta}} = e^{-2i\theta}$ depends on θ.

   Hence $\displaystyle\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ doesn't exist and f is nowhere holomorphic/analytic.

2. *Cauchy–Riemann equations:* $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tilde{f}(x,y) = (x,-y)$ is differentiable everywhere and for all $(x_0,y_0) \in \mathbb{R}^2$,

$$\operatorname{Jac}_{\tilde{f}}(x_0,y_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   Hence the Cauchy–Riemann equations are nowhere satisfied and f is nowhere holomorphic/analytic.

Example: $f(z) = \Re(z)$

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1. *Direct computation:* Let $z_0 \in \mathbb{C}$.

   Pick $h = \rho \in (0, \infty)$, then $\lim\limits_{\rho \to 0^+} \dfrac{f(z_0 + \rho) - f(z_0)}{\rho} = 1$.

   Pick $h = i\rho$, $\rho \in (0, \infty)$, then $\lim\limits_{\rho \to 0^+} \dfrac{f(z_0 + i\rho) - f(z_0)}{i\rho} = 0$.

   Hence $\lim\limits_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$ doesn't exist and f is nowhere holomorphic/analytic.

2. *Cauchy–Riemann equations:* $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tilde{f}(x, y) = (x, 0)$ is differentiable everywhere and for all $(x_0, y_0) \in \mathbb{R}^2$,

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Example: $f(z) = |z|^2$ – part 1

We define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

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1. *Direct computation:*
   - At $z_0 = 0$: $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{|h|^2}{h} = 0.$
     Hence f is holomorphic/analytic at 0 and $f'(0) = 0$.
   - Let $z_0 \in \mathbb{C} \setminus \{0\}$.
     Pick $h = \rho \in (0, +\infty)$, then $\lim_{\rho\to 0^+} \frac{f(z_0+\rho) - f(z_0)}{\rho} = -\overline{z_0} - z_0$.
     Pick $h = i\rho$, $\rho \in (0, +\infty)$, then $\lim_{\rho\to 0^+} \frac{f(z_0+i\rho) - f(z_0)}{i\rho} = -\overline{z_0} + z_0$.
     Hence $\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$ doesn't exist and f is not holomorphic/analytic at z_0.

Therefore f is holomorphic only at 0 and $f'(0) = 0$.

Example: $f(z) = |z|^2$ – part 2

We define $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = |z|^2$.

2. *Cauchy–Riemann equations:*
$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tilde{f}(x, y) = (x^2 + y^2, 0)$ is differentiable everywhere and for all $(x_0, y_0) \in \mathbb{R}^2$,

$$\mathrm{Jac}_{\tilde{f}}(x_0, y_0) = \begin{pmatrix} 2x_0 & 2y_0 \\ 0 & 0 \end{pmatrix}$$

Hence the Cauchy–Riemann equations are satisfied only at $(x_0, y_0) = (0, 0)$.

Therefore f is holomorphic only at 0 and $f'(0) = 0$.

Example: $f(z) = e^z$

We define $\exp : \mathbb{C} \to \mathbb{C}$ by $\exp(z) = e^z$.

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We define $\exp : \mathbb{C} \to \mathbb{C}$ by $\exp(z) = e^z$.

$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tilde{f}(x, y) = \left(e^x \cos y, e^x \sin y\right)$ is differentiable everywhere and for all $(x_0, y_0) \in \mathbb{R}^2$,

$$\operatorname{Jac}_{\tilde{f}}(x_0, y_0) = \begin{pmatrix} e^{x_0} \cos y_0 & -e^{x_0} \sin y_0 \\ e^{x_0} \sin y_0 & e^{x_0} \cos y_0 \end{pmatrix}$$

The Cauchy–Riemann equations are everywhere satisfied, so $\exp$ is everywhere holomorphic/analytic and

$$\exp'(x + iy) = e^x \cos y + ie^x \sin y = e^{x+iy} = \exp(x + iy)$$

i.e. $\exp' = \exp$.

Basic properties of holomorphic functions

Proposition

Let f, g be holomorphic at z_0 and $\lambda \in \mathbb{C}$ then

- λf is holomorphic at z_0 and $(\lambda f)'(z_0) = \lambda f'(z_0)$.
- $f + g$ is holomorphic at z_0 and $(f+g)'(z_0) = f'(z_0) + g'(z_0)$.
- fg is holomorphic at z_0 and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$.
- If additionaly $f(z_0) \neq 0$ then $\frac{1}{f}$ is holomorphic at z_0 and $\left(\frac{1}{f}\right)'(z_0) = -\frac{f'(z_0)}{f(z_0)^2}$

Proposition

Assume that f is holomorphic at z_0 and that g is holomorphic at $f(z_0)$.
Then $g \circ f$ is holomorphic at z_0 and $(g \circ f)'(z_0) = f'(z_0)g'(f(z_0))$

Homework

Prove these properties.

Examples

- A polynomial $f(z) = a_0 + a_1 z + z_2 z^2 + \cdots + a_n z^n$ is holomorphic on $\mathbb{C}$ and $f'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$.
- $f(z) = e^{z^2}$ is holomorphic on $\mathbb{C}$ and $f'(z) = 2ze^{z^2}$.
- $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ is holomorphic on $\mathbb{C}$ and $\cos'(z) = \dfrac{ie^{iz} - ie^{-iz}}{2} = -\dfrac{e^{iz} - e^{-iz}}{2i} = -\sin(z)$.
- $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i}$ is holomorphic on $\mathbb{C}$ and $\sin'(z) = \frac{ie^{iz}+ie^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos(z)$.

Harmonic functions – 1

Definition: Laplacian

Let $\mathscr{U} \subset \mathbb{R}^2$ be open and $f : \mathscr{U} \to \mathbb{R}$ be $\mathscr{C}^2$.

The **Laplacian** of f at $(x_0, y_0) \in \mathscr{U}$ is $\Delta f(x_0, y_0) := \dfrac{\partial^2 f}{\partial x^2}(x_0, y_0) + \dfrac{\partial^2 f}{\partial y^2}(x_0, y_0)$

Definition: Harmonic functions

Let $\mathscr{U} \subset \mathbb{R}^2$ be open and $f : \mathscr{U} \to \mathbb{R}$ be $\mathscr{C}^2$. We say that f is **harmonic** if $\Delta f = 0$ on $\mathscr{U}$.

Definition: Harmonic conjugate functions

Let $\mathscr{U} \subset \mathbb{R}^2$ be open and $u, v : \mathscr{U} \to \mathbb{R}$ be two harmonic functions.

We say that v is **harmonic conjugate** to u when $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$

Check that v is harmonic conjugate to u if and only if $-u$ is harmonic conjugate to v: it is not a symmetric.

Harmonic functions – 2

Theorem

Let $\mathscr{U} \subset \mathbb{C}$ be open and $f : \mathscr{U} \to \mathbb{C}$. Set $\tilde{\mathscr{U}} := \left\{(x, y) \in \mathbb{R}^2 \ : \ x + iy \in \mathscr{U}\right\}$.
Write $f(x + iy) = u(x, y) + iv(x, y)$.
If f is holomorphic/analytic on $\mathscr{U}$ then v is harmonic conjugate to u on $\tilde{\mathscr{U}}$.
Particularly u and v are harmonic functions on $\tilde{\mathscr{U}}$.

Harmonic functions – 2

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If f is holomorphic/analytic on $\mathscr{U}$ then v is harmonic conjugate to u on $\tilde{\mathscr{U}}$.
Particularly u and v are harmonic functions on $\tilde{\mathscr{U}}$.

Proof. We will see later that if f is holomorphic then u and v are $\mathscr{C}^2$. Then

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = 0$$

where the second equality comes from the Cauchy–Riemann equations and the last one from Clairaut's theorem. ∎

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where the second equality comes from the Cauchy–Riemann equations and the last one from Clairaut's theorem. ∎

Hence the real and imaginary parts of a holomorphic/analytic/$\mathbb{C}$-differentiable function are harmonic conjugate functions.
When $\mathscr{U}$ is moreover assumed to be simply connected, then we have the following converse: if u is harmonic on $\tilde{\mathscr{U}}$ then u is the real part of a function holomorphic $\mathscr{U}$ (see Slide 17).

Actually, harmonic functions on $\mathbb{R}^2$ and holomorphic functions on $\mathbb{C}$ share many similar properties so that we may see harmonic functions as the real analogs of holomorphic functions.

Harmonic functions – 3

A natural question is: assuming that $f = u + iv$ is holomorphic/analytic/$\mathbb{C}$-differentiable, up to what extent does u determine f? Or equivalently, up to what extent does u determine v?

Theorem

Let $\mathcal{U} \subset \mathbb{R}^2$ be a domain (i.e. $\mathcal{U}$ is open and **connected**) and $u, v_1, v_2 : \mathcal{U} \to \mathbb{R}$ be harmonic functions.
If v_1 and v_2 are harmonic conjugates to u then v_1 and v_2 differ by a constant, i.e. $v_1 - v_2 = C \in \mathbb{R}$.

Proof. Indeed, $\partial_x(v_1 - v_2) = \partial_x v_1 - \partial_x v_2 = -\partial_y u + \partial_y u = 0$ and similarly $\partial_y(v_1 - v_2) = 0$.
Hence $v_1 - v_2$ is constant since $\mathcal{U}$ is connected. ∎

Another natural question is: does a harmonic function always admit a harmonic conjugate?
Or, equivalently, is a harmonic function always the real part of a holomorphic function?

In general, without additional assumptions, the answer is NO:

Example

Let $u : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined by $u(x,y) = \log(x^2 + y^2)$.
Assume by contradiction that u admits a harmonic conjugate v on $\mathbb{R}^2 \setminus \{(0,0)\}$.
And $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic and $f'(x+iy) = \frac{2x}{x^2+y^2} - i\frac{2y}{x^2+y^2} = \frac{2}{x+iy}$.
Then $\displaystyle\int_{S^1} f'(z)\mathrm{d}z = \int_{S^1} \frac{1}{z}\mathrm{d}z = 4\pi i$.
But by Green's theorem: $\displaystyle\int_{S^1} f'(z)\mathrm{d}z = i \iint_{D_1(0)} 0 = 0$.
Contradiction.

Harmonic functions – 5

Nonetheless, when the assumptions of Poincaré lemma[3] are satisfied, it is possible to use it in order to construct a harmonic conjugate.

Theorem

Let $\mathscr{U} \subset \mathbb{R}^2$ be open and star-shaped[4]. Let $u : \mathscr{U} \to \mathbb{R}$ be a harmonic function.
Then there exists $v : \mathscr{U} \to \mathbb{R}$ a harmonic conjugate to u.

Proof. Note that the vector field $\mathbf{F} = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right)$ satisfies $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ since u is harmonic.
Hence, by Poincaré lemma, there exists $v : \mathscr{U} \to \mathbb{R}$ C^2 such that $\mathbf{F} = \nabla v$,
i.e. $\left(-\dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial x}\right) = \left(\dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}\right)$. Hence v is harmonic conjugate to u. ∎

[3] See for instance Theorem 5 of `http://www.math.toronto.edu/campesat/ens/1920/poincare.pdf`
[4] Actually, we may weaken this assumption and assume that $\mathscr{U}$ is open and simply-connected, i.e. that $\mathscr{U}$ has no hole.
A **hole** of $S \subset \mathbb{C}$ is a bounded connected component of $\mathbb{C} \setminus S$.

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Corollary

A harmonic function on $\mathscr{U} \subset \mathbb{R}^2$ open and star-shaped[4] is the real part of a holomorphic function.

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A **hole** of $S \subset \mathbb{C}$ is a bounded connected component of $\mathbb{C} \setminus S$.

A first example of the *rigidity* of holomorphic functions

Theorem

Let $\mathscr{U} \subset \mathbb{C}$ be a **domain** (i.e. open and path-connected) and $f = u + iv$ be holomorphic on $\mathscr{U}$.
If either u, or v, or $u^2 + v^2$ is constant on $\mathscr{U}$ then f is also constant on $\mathscr{U}$.

This theorem tells us that if the range of a holomorphic function defined on a **domain** lies on a horizontal line, or on a vertical line, or on a circle, then this function is actually constant.

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Proof.

- Assume that u is constant (similar proof for v).
  Then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. By the Cauchy–Riemann equations we also have that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$.
  Since $\tilde{\mathscr{U}}$ is connected, we get that v is constant. Therefore $f = u + iv$ is constant.

∎

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  Since $\tilde{\mathscr{U}}$ is connected, we get that v is constant. Therefore $f = u + iv$ is constant.

- Assume that $|f|^2 = u^2 + v^2 = c$ is constant. If $c = 0$ then $f = 0$ so we may assume that $c > 0$.
  From $c = |f|^2 = f\overline{f}$ we get that $\overline{f} = \frac{c}{f}$ is holomorphic since f doesn't vanish.
  Hence $\mathfrak{R}(f) = \frac{f+\overline{f}}{2}$ is holomorphic with constant imaginary part equal to 0.
  Then, by the previous point, $\mathfrak{R}(f)$ is constant.
  Finally, since the real part of f is constant, so is f (still by the previous point).

∎

A few words about simple connectedness

Definition: hole

A **hole** of $S \subset \mathbb{C}$ is a bounded connected component of $\mathbb{C} \setminus S$.

Definition: simple connectedness

We say that $S \subset \mathbb{C}$ is **simply connected** if it is path-connected and has no hole.

Figure: A simply connected set

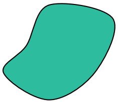


Figure: A set NOT simply connected

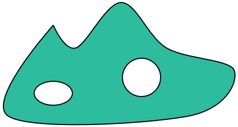
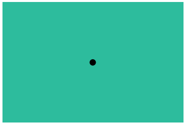


Figure: A set NOT simply connected



$\mathbb{C} \setminus \{0\}$

Figure: A set NOT simply connected