Cauchy's integral formula

October 14th, 2020

Simple connectedness – 1

Definition: hole

A **hole** of $S \subset \mathbb{C}$ is a bounded connected component of $\mathbb{C} \setminus S$.

Definition: simple connectedness *(that's a formal definition with the above definition of a hole)*

We say that $S \subset \mathbb{C}$ is **simply connected** if it is path connected and has no hole.

Figure: A simply connected set

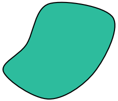


Figure: A set NOT simply connected

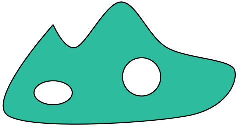
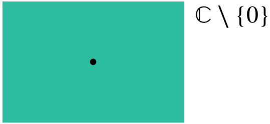


Figure: A set NOT simply connected



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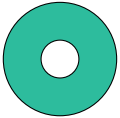

Simple connectedness – 2

Theorem

$S \subset \mathbb{C}$ is simply connected if and only if it is path-connected and for any simple closed curve included in S, its inside[1] is also included in S.

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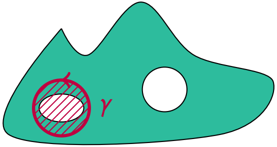

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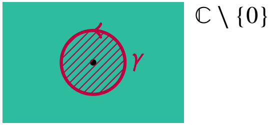

[1] See Jordan curve theorem from September 28.

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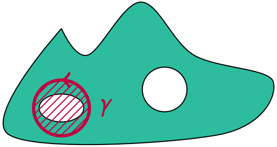

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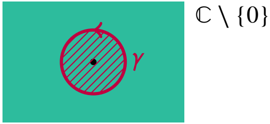
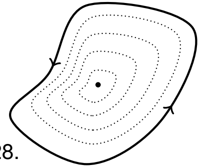


Figure: A set NOT simply connected

An important property[2] of simply connected sets is that any closed curve on it can be *continuously deformed* to a constant curve without leaving it:



[1] See Jordan curve theorem from September 28.

[2] That's actually the usual definition of simple connectedness: a path connected set is simply connected if any closed curve on it is homotopic to a point.

Cauchy's integral theorem – 1

Theorem: Cauchy's integral theorem – version 1

Let $U \subset \mathbb{C}$ be an open subset and $f : U \to \mathbb{C}$ be holomorphic/analytic.
Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth simple closed curve on U whose inside is also entirely included in U, then

$$\int_\gamma f(z)\mathrm{d}z = 0$$

Cauchy's integral theorem – 1

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Proof. There is an easy proof with the extra assumption that f' is continuous[3]:

$$\begin{aligned}\int_\gamma f &= i \iint_{\gamma \cup \mathrm{Inside}} \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) && \text{by Green's theorem}\\ &= i \iint 0 && \text{by the Cauchy–Riemann equations}\\ &= 0\end{aligned}$$

■

[3] It is always the case: the derivative of a holomorphic/analytic function is always continuous but you don't know that yet. To avoid circular arguments, it would be better to give a proof without this assumption. There is such a proof (due to Goursat), but it is far more technical. So I am cheating a little bit here.

Cauchy's integral theorem – 2

Corollary

Let $U \subset \mathbb{C}$ be an open **simply connected** subset and $f : U \to \mathbb{C}$ be holomorphic/analytic.
Then there exists $F : U \to \mathbb{C}$ holomorphic/analytic such that $F' = f$.

We say that F is a **(complex) antiderivative/primitive** of f on U.

The simple connectedness assumption ensures that for any simple closed curve on U, its inside is included in U, so that we can use Cauchy's integral theorem.
Hence we will assume that the domains are simply connected in the next corollaries.

Cauchy's integral theorem – 2

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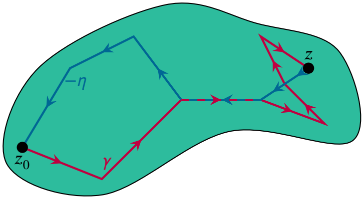
Proof. Fix $z_0 \in U$ and for $z \in U$ set $F(z) = \int_\gamma f(z)\mathrm{d}z$ where γ is a polygonal curve from z_0 to z.

F doesn't depend on the choice of γ, indeed if η is another such curve then $\int_\gamma f(z)\mathrm{d}z - \int_\eta f(z)\mathrm{d}z$

$= \int_{\gamma-\eta} f(z)\mathrm{d}z = 0$ (the integrals cancel each other on the common edges with reversed orientation, and by the previous theorem, each integral around a simple closed polygonal curve is 0).



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Proof. Fix $z_0 \in U$ and for $z \in U$ set $F(z) = \displaystyle\int_\gamma f(z)\mathrm{d}z$ where γ is a polygonal curve from z_0 to z.

Then

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{\int_{[z,z+h]} f(w)\mathrm{d}w}{h} - f(z)\right| = \left|\int_{[z,z+h]} \frac{f(w)-f(z)}{h}\mathrm{d}w\right|$$
$$\leq \left(\max_{w\in[z,z+h]} |f(w)-f(z)|\right) \frac{\mathscr{L}([z,z+h])}{|h|}$$
$$= \left(\max_{w\in[z,z+h]} |f(w)-f(z)|\right) \xrightarrow[h\to 0]{} 0$$

∎

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For the second equality, I used that
$hf(z) = \int_{[z,z+h]} f(w)\mathrm{d}w$.

∎

Cauchy's integral theorem – 3

Corollary

Let $U \subset \mathbb{C}$ be an open **simply connected** subset and $f : U \to \mathbb{C}$ be holomorphic/analytic.

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth closed curve included[4] in U, then $\int_\gamma f(z)\mathrm{d}z = 0$.

[4] i.e. $\forall t \in [a, b],\ \gamma(t) \in U$.

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Corollary

Let $U \subset \mathbb{C}$ be an open **simply connected** subset and $f : U \to \mathbb{C}$ be holomorphic/analytic.
Let $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ be two piecewise smooth curves included[4] in U with same endpoints [a], then $\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z$.

[a] i.e. $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$.

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Proof of both corollaries. Let F be a primitive of f on U then

$$\int_\gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t = \int_a^b (F \circ \gamma)'(t)\mathrm{d}t = F(\gamma(b)) - F(\gamma(a))$$

∎

[4] i.e. $\forall t \in [a, b],\ \gamma(t) \in U$.

Cauchy's integral theorem – 4

When the domain is **simply connected**, we proved:

Theorem: Cauchy's integral theorem – version 2

Let $U \subset \mathbb{C}$ be an open simply connected subset and $f : U \to \mathbb{C}$ be holomorphic/analytic. Then

- For $\gamma : [a,b] \to \mathbb{C}$ a piecewise smooth closed curve included in U,

$$\int_\gamma f(z)\mathrm{d}z = 0$$

- For $\gamma_i : [a_i, b_i] \to \mathbb{C}$, $i = 1,2$ two piecewise smooth curves included in U with same endpoints,

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z$$

- f admits a primitive/antiderivative[5] F on U,
  i.e. there exists $F : U \to \mathbb{C}$ holomorphic/analytic such that $F' = f$.

[5] *Actually a function $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is simply connected, is holomorphic if and only if it admits a (complex) antiderivative: indeed, we will see soon that the derivative of a holomorphic function is holomorphic too.*

Careful

The simple connectedness assumption of the domain is essential.

Indeed $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = 1/z$ is holomorphic, but :

- For $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) = e^{it}$, we have $\displaystyle\int_\gamma \frac{1}{z} \mathrm{d}z = 2i\pi \neq 0$
- f has no antiderivative on $\mathbb{C} \setminus \{0\}$.

An application of Cauchy's integral theorem: Fresnel's integrals

The following (improper) integrals are difficult to compute using only "real" methods.

Frenel's integrals

$$\int_0^{+\infty} \cos\left(t^2\right) \mathrm{d}t = \int_0^{+\infty} \sin\left(t^2\right) \mathrm{d}t = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

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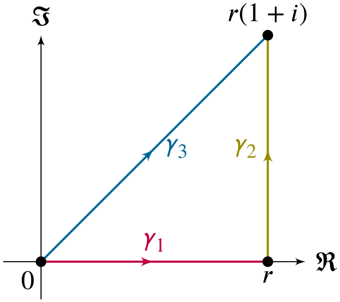

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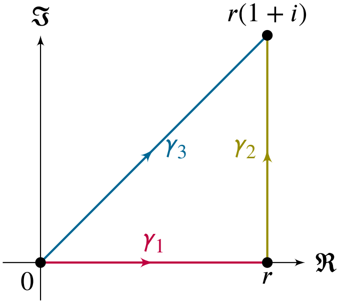

- $\displaystyle\int_{\gamma_1} e^{-z^2} \mathrm{d}z = \int_0^r e^{-t^2} \mathrm{d}t \xrightarrow[r \to +\infty]{} \frac{\sqrt{\pi}}{2}.$

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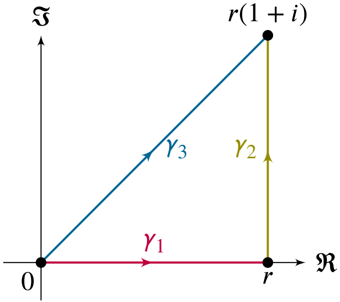

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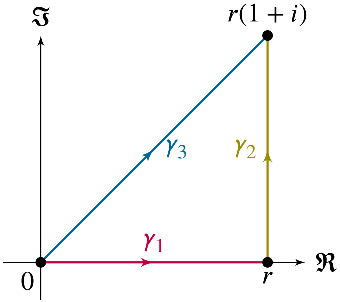

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- $\displaystyle\int_{\gamma_3} e^{-z^2} \mathrm{d}z = \int_0^r (1+i) e^{((1+i)t)^2} \mathrm{d}t = (1+i) \int_0^r e^{-2it^2} \mathrm{d}t = e^{i\frac{\pi}{4}} \int_0^{\frac{r}{\sqrt{2}}} e^{-it^2} \mathrm{d}t$

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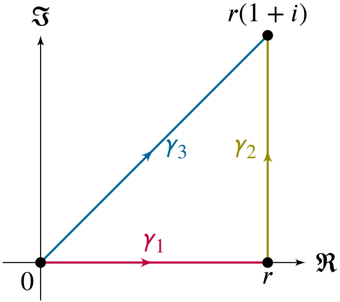

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By Cauchy's integral theorem $\displaystyle 0 = \int_{\gamma_1+\gamma_2-\gamma_3} e^{-z^2} \mathrm{d}z = \int_{\gamma_1} e^{-z^2} \mathrm{d}z + \int_{\gamma_2} e^{-z^2} \mathrm{d}z - \int_{\gamma_3} e^{-z^2} \mathrm{d}z.$

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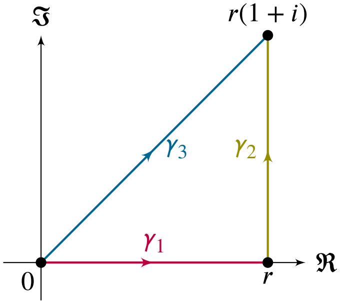

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- $\displaystyle\int_{\gamma_3} e^{-z^2} \mathrm{d}z = \int_0^r (1+i)e^{((1+i)t)^2} \mathrm{d}t = (1+i)\int_0^r e^{-2it^2} \mathrm{d}t = e^{i\frac{\pi}{4}} \int_0^{\frac{r}{\sqrt{2}}} e^{-it^2} \mathrm{d}t$

By Cauchy's integral theorem $\displaystyle 0 = \int_{\gamma_1+\gamma_2-\gamma_3} e^{-z^2} \mathrm{d}z = \int_{\gamma_1} e^{-z^2} \mathrm{d}z + \int_{\gamma_2} e^{-z^2} \mathrm{d}z - \int_{\gamma_3} e^{-z^2} \mathrm{d}z.$

So $\displaystyle\int_0^{+\infty} e^{-it^2} \mathrm{d}t = e^{-i\frac{\pi}{4}} \frac{\sqrt{\pi}}{2}$ and finally we identify the real and imaginary parts. ∎

Cauchy's integral formula – 1

Theorem: Cauchy's integral formula

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic.
Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth positively oriented simple closed curve on U whose inside $\Omega \coloneqq \mathrm{Inside}(\gamma)$ is also included in U then

$$\forall z \in \Omega,\ f(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z} \mathrm{d}w$$

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Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth positively oriented simple closed curve on U whose inside $\Omega := \operatorname{Inside}(\gamma)$ is also included in U then

$$\forall z \in \Omega,\ f(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z} \mathrm{d}w$$

Proof.
Define $g : U \to \mathbb{C}$ by $g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z \end{cases}$.

Then g is holomorphic on $U \setminus \{z\}$ and continuous on U.

By Cauchy's integral theorem $\int_\gamma g(w)\mathrm{d}w = 0$, thence $\int_\gamma \frac{f(w)}{w-z}\mathrm{d}w = \int_\gamma \frac{f(z)}{w-z}\mathrm{d}w = f(z)\int_\gamma \frac{1}{w-z}\mathrm{d}w$.

We conclude using the next lemma from which $\int_\gamma \frac{1}{w-z}\mathrm{d}w = 2i\pi$. ∎

Cauchy's integral formula – 2

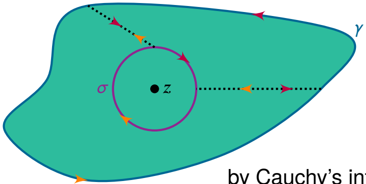
Lemma

Let $U \subset \mathbb{C}$ be open. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth positively oriented simple closed curve on U whose inside $\Omega := \mathrm{Inside}(\gamma)$ is also included in U then

$$\forall z \in \Omega, \int_\gamma \frac{1}{w-z}\mathrm{d}w = 2i\pi$$

Proof. Let $z \in \Omega$. There exists $\varepsilon > 0$ such that $\overline{D_\varepsilon(z)} \subset \Omega$.

We can't directly apply Cauchy's integral theorem because $w \mapsto \frac{1}{w-z}$ is not defined at z.
So we divide $\Omega \setminus D_\varepsilon(z)$ into two simply connected pieces.



$$0 = 0 + 0 = \int_{\text{orange}} \frac{1}{w-z}\mathrm{d}w + \int_{\text{red}} \frac{1}{w-z}\mathrm{d}w$$ by Cauchy's integral theorem

$$= \int_\gamma \frac{1}{w-z}\mathrm{d}w + \int_\sigma \frac{1}{w-z}\mathrm{d}w$$ since the segment lines are counted twice with reversed orientation

$$= \int_\gamma \frac{1}{w-z}\mathrm{d}w - 2i\pi$$ since $\sigma(t) = z + e^{-it}$, $t \in [0, 2\pi]$ ∎

Corollary

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic.
Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise smooth positively oriented simple closed curve on U whose inside is also included in U.

- If $z \in U$ is in the inside of γ then

$$\int_\gamma \frac{f(w)}{w-z} \mathrm{d}w = 2i\pi f(z)$$

- If $z \in U$ is in the outside of γ then

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Careful: we don't say anything when $z \in \gamma$ (the integrand $w \mapsto \frac{f(w)}{w-z}$ is not defined at z).

The above corollary could be improved using "winding numbers".

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Computing a difficult (real) integral with Cauchy's integral formula

$$\int_{-\infty}^{+\infty} \frac{\cos(t)}{1+t^2} \mathrm{d}t = \frac{\pi}{e}$$

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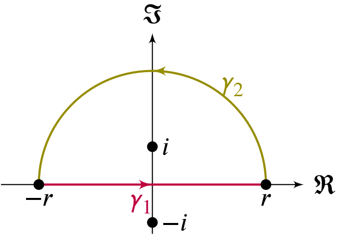

- $$\begin{aligned}\int_{\gamma_1+\gamma_2} \frac{e^{iz}}{1+z^2}\mathrm{d}z &= \frac{1}{2i}\left(\int_{\gamma_1+\gamma_2} \frac{e^{iz}}{z-i}\mathrm{d}z - \int_{\gamma_1+\gamma_2} \frac{e^{iz}}{z+i}\mathrm{d}z\right) \quad \text{since } \frac{1}{1+z^2} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) \\ &= \frac{1}{2i}\left(2i\pi e^{i^2} - 0\right) \quad \text{by Cauchy's integral formula (resp. theorem) if } r > 1 \\ &= \frac{\pi}{e}\end{aligned}$$

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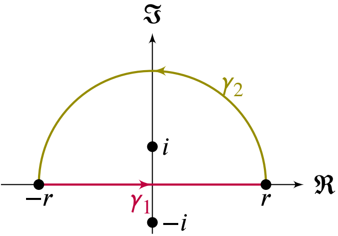

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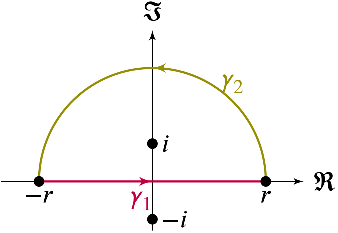

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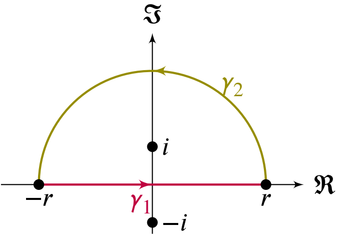

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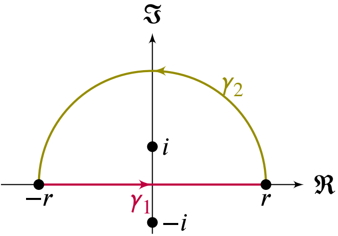

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Therefore $\displaystyle\frac{\pi}{e} = \lim_{r\to+\infty}\int_{\gamma_1+\gamma_2}\frac{e^{iz}}{1+z^2}\mathrm{d}z = \int_{-\infty}^{+\infty}\frac{\cos(t)}{1+t^2}\mathrm{d}t + 0.$

Next lecture, we will see that Cauchy's integral formula has deep consequences concerning properties of holomorphic functions!