Cauchy's residue theorem

October 30^{th}, 2020 and November 2^{nd}, 2020

Cauchy's residue theorem – 1

Theorem: the residue theorem

Let $U \subset \mathbb{C}$ be open. Let $S \subset U$ be finite. Assume that $f : U \setminus S \to \mathbb{C}$ is holomorphic/analytic.
Let $\gamma : [a, b] \to \mathbb{C}$ be a positively oriented piecewise smooth simple closed curve on $U \setminus S$[a] whose inside[b] is entirely included in U. Then[c]

$$\int_\gamma f(z)\mathrm{d}z = 2i\pi \sum_{z \in \mathrm{Inside}(\gamma)} \mathrm{Res}(f, z)$$

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[a] i.e. γ doesn't pass through any point of S.
[b] See Jordan's curve theorem, September 28.
[c] The following sum is finite since $\mathrm{Res}(f, z) \neq 0$ only for $z \in S$.

Cauchy's residue theorem – 1

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[b] See Jordan's curve theorem, September 28.
[c] The following sum is finite since $\mathrm{Res}(f, z) \neq 0$ only for $z \in S$.

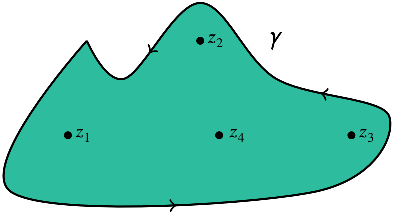
Corollary

Let $U \subset \mathbb{C}$ be open and simply-connected. Let $S \subset U$ be finite.
Assume that $f : U \setminus S \to \mathbb{C}$ is holomorphic/analytic.
Let $\gamma : [a, b] \to \mathbb{C}$ be a positively oriented piecewise smooth simple closed curve on $U \setminus S$. Then

$$\int_\gamma f(z)\mathrm{d}z = 2i\pi \sum_{z \in \mathrm{Inside}(\gamma)} \mathrm{Res}(f, z)$$

Cauchy's residue theorem – 2

Proof.



$\{z_1, \dots, z_n\}$ are the points of S enclosed in γ.

Cauchy's residue theorem – 2

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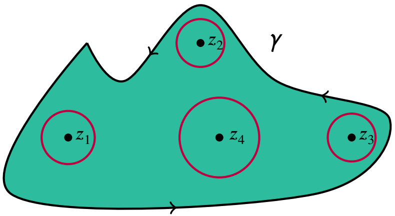
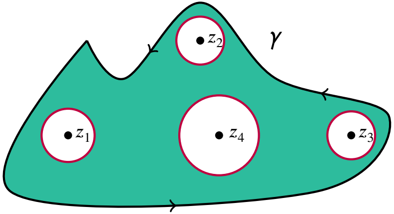


We may find pairwise disjoints disks $\overline{D_{r_k}(z_k)} \subset U$ where $\{z_1, \ldots, z_n\}$ are the points of S enclosed in γ.

Proof.



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We apply Green's theorem to $T = \overline{\mathrm{Inside}(\gamma)} \setminus \left(\bigcup_{i=1}^{n} D_{r_k}(z_k) \right)$

Cauchy's residue theorem – 2

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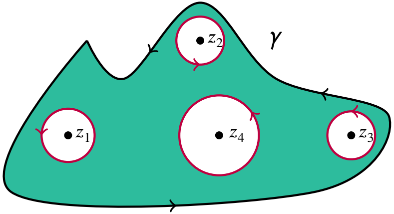

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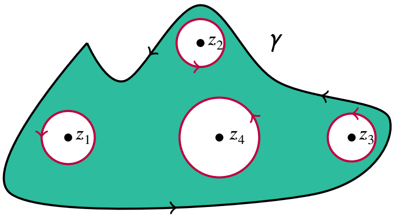
$$\int_\gamma f(z)\mathrm{d}z - \sum_{k=1}^{n} \int_{\gamma_k} f(z)\mathrm{d}z = i \iint_T \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

where $\gamma_k : [0,1] \to \mathbb{C}$ is defined by $\gamma_k(t) = z_k + r_k e^{2i\pi t}$.

Proof.



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Cauchy's residue theorem – 2

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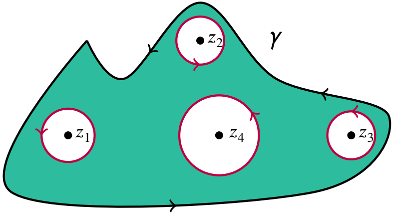

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Then $\displaystyle\int_\gamma f(z)\mathrm{d}z = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\mathrm{d}z = 2i\pi \sum_{k=1}^{n} \mathrm{Res}(f, z_k)$ ■

Cauchy's residue theorem – 3

Corollary

Let $S \subset \mathbb{C}$ be finite. Assume that $f : \mathbb{C} \setminus S \to \mathbb{C}$ is holomorphic/analytic. Then

$$\operatorname{Res}(f, \infty) + \sum_{z \in S} \operatorname{Res}(f, z) = 0$$

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Proof. Take $r > 0$ such that $S \subset D_r(0)$ and define $\gamma : [0,1] \to \mathbb{C}$ by $\gamma(t) = re^{2i\pi t}$. Then

$$\begin{aligned}\sum_{z\in S} \operatorname{Res}(f,z) &= \frac{1}{2i\pi}\int_\gamma f(z)\mathrm{d}z \quad \text{by Cauchy's residue theorem} \\ &= -\operatorname{Res}(f,\infty)\end{aligned}$$

∎

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We may rewrite the above conclusion as $\displaystyle\sum_{z \in \widehat{\mathbb{C}}} \operatorname{Res}(f, z) = 0$.

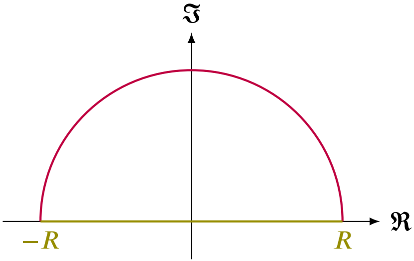
Example: $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\mathrm{d}x$ (rational function) – 1

Assume that P and Q are two polynomials. Set $f(z) = \frac{P(z)}{Q(z)}$.

We know that $\displaystyle\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\mathrm{d}x$ is convergent if and only if $\deg Q \geq \deg P + 2$, let's assume the latter.

Then $\displaystyle\lim_{|z|\to\infty} zf(z) = 0$.

Assume that Q has no real root.



Define $\gamma_R : [0,1] \to \mathbb{C}$ by $\gamma_R(t) = Re^{i\pi t}$.

For $R > 0$ big enough, all the poles of f whose imaginary part is positive are included within the upper-half disk centered at 0 and of radius R.

Then $\displaystyle\int_{-R}^{R} f(z)\mathrm{d}z + \int_{\gamma_R} f(z)\mathrm{d}z = 2i\pi \sum_{z \text{ s.t. } \mathfrak{I}(z)>0} \operatorname{Res}(f,z)$.

But $\displaystyle\left|\int_{\gamma_R} f\right| \leq \pi R \sup_{\gamma_R} |f| \xrightarrow[R\to+\infty]{} 0$ since $\displaystyle\lim_{|z|\to\infty} zf(z) = 0$.

Hence $\displaystyle\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\mathrm{d}x = 2i\pi \sum_{z \text{ s.t. } \mathfrak{I}(z)>0} \operatorname{Res}\left(\frac{P}{Q}, z\right)$.

Example: $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} \mathrm{d}x$ (rational function) – 2

Example

Compute $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)}$ où $0 < a < b$.

The poles of $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$ are $-ia$, ia, $-ib$ and ib which are simple. Hence

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} &= 2i\pi \operatorname{Res}(f, ia) + 2i\pi \operatorname{Res}(f, ib) \\ &= \frac{2i\pi}{2i(b^2-a^2)a} + \frac{2i\pi}{2i(a^2-b^2)b} \\ &= \frac{\pi}{ab(a+b)}\end{aligned}$$

Example: $\int_0^{2\pi} R(\cos t, \sin t)\mathrm{d}t$

Set $z = e^{it}$ then $\cos(t) = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin(t) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, and $\frac{\mathrm{d}z}{iz} = \mathrm{d}t$.

Set $f(z) = \frac{1}{iz} R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)$. Then $\int_0^{2\pi} R(\cos t, \sin t)\mathrm{d}t = \int_{S^1} f(z)\mathrm{d}z = 2i\pi \sum_{z \in D_1(0)} \mathrm{Res}(f, z)$.

Example

Compute $\int_0^{2\pi} \frac{a}{a^2 + \sin^2 t}\mathrm{d}t$ where $a > 0$.

Set $f(z) = \frac{1}{iz} \frac{a}{a^2 - \frac{1}{4}\left(z - \frac{1}{z}\right)^2} = -\frac{4iaz}{(z^2 + 2az - 1)(z^2 - 2az - 1)}$.

Note that the singularity at 0 of the LHS is removable since we may extend f through 0 using the RHS, so that $\mathrm{Res}(f, 0) = 0$.

Then, the only poles of f within the unit disk are $z_1 = -a + \sqrt{a^2 + 1}$ and $z_2 = a - \sqrt{a^2 + 1}$ which are simple.

Hence

$$\int_0^{2\pi} \frac{a}{a^2 + \sin^2 t}\mathrm{d}t = 2i\pi\,\mathrm{Res}(f, z_1) + 2i\pi\,\mathrm{Res}(f, z_2) = \frac{2\pi}{\sqrt{a^2 + 1}}$$

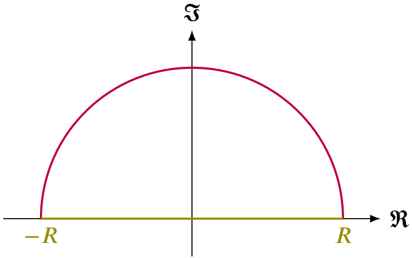
Jordan's lemma

The following trick called, *Jordan's lemma*, can be very useful.
Let $\gamma_R : [0,\pi] \to \mathbb{C}$ be defined by $\gamma_R(t) = Re^{it}$.

$$\begin{aligned}
\left|\int_{\gamma_R} g(z)e^{iz}\mathrm{d}z\right| &= \left|\int_0^\pi g(Re^{it})e^{iR(\cos t + i\sin t)} iRe^{it}\mathrm{d}t\right| \\
&\leq \int_0^\pi \left|g(Re^{it})e^{iR(\cos t + i\sin t)} iRe^{it}\right| \mathrm{d}t \\
&\leq R\sup_{\gamma_R}|g| \int_0^\pi e^{-R\sin t}\mathrm{d}t \\
&= 2R\sup_{\gamma_R}|g| \int_0^{\pi/2} e^{-R\sin t}\mathrm{d}t \\
&\leq 2R\sup_{\gamma_R}|g| \int_0^{\pi/2} e^{-2Rt/\pi}\mathrm{d}t \quad \textit{by Jordan's inequality: } \forall x \in \left[0, \frac{\pi}{2}\right], \ \frac{2}{\pi}x \leq \sin(x) \leq x \\
&\leq \pi\sup_{\gamma_R}|g| \left(1 - e^{-R}\right) \\
&\leq \pi\sup_{\gamma_R}|g|
\end{aligned}$$

Example: $\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{iax} \mathrm{d}x$ – 1

We want to compute $I(a) := \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{iax} \mathrm{d}x$ where P, Q are real polynomials and $a \in \mathbb{R}$. Assume that Q has no real root.
Note that $I(-a) = \overline{I(a)}$, so we may restrict our attention to $a > 0$.
The integral is convergent if and only if $\deg Q \geq \deg P + 1$ (integration by parts), so we assume the latter.



Define $\gamma_R : [0, 1] \to \mathbb{C}$ by $\gamma_R(t) = Re^{i\pi t}$.

For $R > 0$ big enough, all the poles of $f(z) = \frac{P(z)}{Q(z)} e^{iaz}$ whose imaginary part is positive are included within the upper-half disk centered at 0 and of radius R.

Then $\int_{-R}^{R} f(z) \mathrm{d}z + \int_{\gamma_R} f(z) \mathrm{d}z = 2i\pi \sum_{z \text{ s.t. } \mathfrak{I}(z)>0} \operatorname{Res}(f, z) \quad (*)$

But, by Jordan's lemma,

$$\left| \int_{\gamma_R} \frac{P(z)}{Q(z)} e^{iz} \mathrm{d}z \right| \leq \pi \sup_{\gamma_R} |P/Q| \xrightarrow[R \to +\infty]{} 0$$

Hence, by taking $R \to +\infty$ in $(*)$, we get

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{iax} \mathrm{d}x = 2i\pi \sum_{z \text{ s.t. } \mathfrak{I}(z)>0} \operatorname{Res}\left(\frac{P(z)}{Q(z)} e^{iaz}, z \right)$$

Example

Compute $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2+\alpha^2}\mathrm{d}x$ where $\alpha > 0$.

Note that $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2+\alpha^2}\mathrm{d}x = \Re\left(\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+\alpha^2}\mathrm{d}x\right)$.

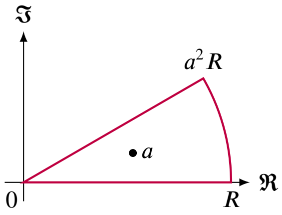
The poles of $f(z) = \dfrac{e^{iz}}{z^2+\alpha^2}$ are $i\alpha$ and $-i\alpha$ which are simple.

By the previous slide, $\displaystyle\int_{-\infty}^{+\infty} \frac{e^{iz}}{z^2+\alpha^2}\mathrm{d}z = 2i\pi \operatorname{Res}(f, i\alpha) = 2i\pi\frac{e^{-\alpha}}{2i\alpha} = \pi\frac{e^{-\alpha}}{\alpha}$.

Hence $\displaystyle\int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2+\alpha^2}\mathrm{d}x = \pi\frac{e^{-\alpha}}{\alpha}$.

Example: $\int_0^{+\infty} \frac{x^p}{1+x^n}\mathrm{d}x,\ n,p\in\mathbb{N}$

We know that the integral $\displaystyle\int_0^{+\infty}\frac{x^p}{1+x^n}\mathrm{d}x$ is convergent if and only if $n \geq p+2$. Set $f(z)=\dfrac{z^p}{1+z^n}$ and $a=e^{i\frac{\pi}{n}}$.
We consider the following sector of the circle centered at 0 and of radius R, such that the only pole of f enclosed in its inside is a.



Let $\gamma : \left[0, \frac{2\pi}{n}\right] \to \mathbb{C}$ be defined by $\gamma(t) = Re^{it}$.

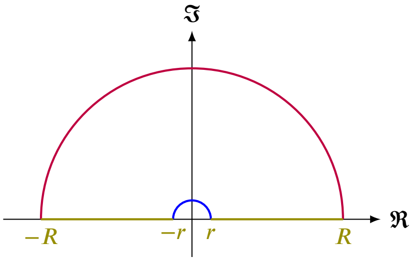
By the residue theorem, $2i\pi\,\mathrm{Res}(f,a) = \displaystyle\int_{[0,R]} f + \int_\gamma f + \int_{[a^2R,0]} f.$

- $\mathrm{Res}(f,a) = \frac{a^p}{na^{n-1}} = -\frac{a^{p+1}}{n}.$
- $\displaystyle\int_{[a^2R,0]} f(z)\mathrm{d}z = -a^2\int_0^R \frac{a^{2p}t^p}{1+a^{2n}t^n}\mathrm{d}t = -a^{2(p+1)}\int_0^R\frac{t^p}{1+t^n}\mathrm{d}t$
- $\displaystyle\left|\int_\gamma f(z)\mathrm{d}z\right| \leq \frac{2\pi}{n}R\sup_\gamma|f| \xrightarrow[R\to+\infty]{} 0$ since $\displaystyle\lim_{z\to\infty} zf(z) = 0.$

Hence, by taking the limit as $R\to+\infty$, we get $\displaystyle -2i\pi\frac{a^{p+1}}{n} = \int_0^{+\infty}\frac{x^p}{1+x^n}\mathrm{d}x - a^{2(p+1)}\int_0^{+\infty}\frac{x^p}{1+x^n}\mathrm{d}x.$

Finally $\displaystyle\int_0^{+\infty}\frac{x^p}{1+x^n}\mathrm{d}x = \frac{2i\pi}{n}\frac{a^{p+1}}{a^{2(p+1)}-1} = \frac{\pi}{n}\frac{2i}{a^{p+1}-a^{-(p+1)}} = \frac{\pi}{n\sin\frac{(p+1)\pi}{n}}.$

Example: $\int_0^{+\infty} \frac{\sin t}{t} dt$



Define $\gamma_R : [0,1] \to \mathbb{C}$ by $\gamma_R(t) = Re^{i\pi t}$ and $\gamma_r : [0,1] \to \mathbb{C}$ by $\gamma_r(t) = re^{i\pi t}$.

By Cauchy's integral theorem

$$\int_{\gamma_R} \frac{e^{iz}}{z} dz - \int_{\gamma_r} \frac{e^{iz}}{z} dz + \int_{-R}^{-r} \frac{e^{it}}{t} dt + \int_r^R \frac{e^{it}}{t} dt = 0.$$

- By Jordan's lemma: $\left| \int_{\gamma_R} \frac{e^{iz}}{z} dz \right| \le \pi \sup_{\gamma_R} |1/z| \xrightarrow[R \to +\infty]{} 0$
- Since 0 is a simple pole of $f(z) = \frac{e^{iz}}{z}$, we have that $f(z) = \operatorname{Res}(f,0) z^{-1} + g(z)$ where g is holomorphic.

Then $\int_{\gamma_r} f(z) dz = \int_{\gamma_r} \operatorname{Res}(f,0) z^{-1} dz + \int_{\gamma_r} g(z) dz$ but $\int_{\gamma_r} \operatorname{Res}(f,0) z^{-1} dz = \operatorname{Res}(f,0) i\pi$ and $\left| \int_{\gamma_r} g(z) dz \right| \le \pi r \sup_{\gamma_r} |g| \xrightarrow[r \to 0]{} 0.$

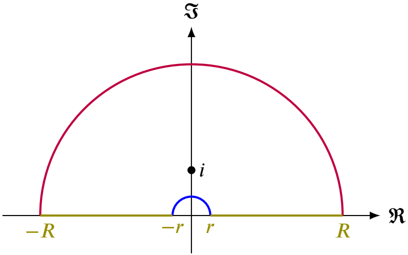
Hence $\int_{\gamma_r} f(z) dz \xrightarrow[r \to 0]{} \operatorname{Res}(f,0) i\pi = i\pi.$

$$\int_r^R \frac{\sin t}{t} dt = \frac{1}{2i} \int_r^R \frac{e^{it} - e^{-it}}{t} dt = \frac{1}{2i} \int_r^R \frac{e^{it}}{t} dt - \frac{1}{2i} \int_r^R \frac{e^{-it}}{t} dt = \frac{1}{2i} \int_r^R \frac{e^{it}}{t} dt + \frac{1}{2i} \int_{-R}^{-r} \frac{e^{it}}{t} dt = \frac{1}{2i} \left(\int_{\gamma_r} \frac{e^{iz}}{z} dz - \int_{\gamma_R} \frac{e^{iz}}{z} dz \right)$$

Hence, taking $r \to 0$ and $R \to +\infty$ we get that $\int_0^{+\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$

Example: $\int_0^{+\infty} \frac{(\log t)^2}{1+t^2} \mathrm{d}t$

We set $f(z) = \frac{(\log z)^2}{1+z^2}$ but for that we need to fix a branch of the logarithm. Let's fix $\log : \mathbb{C} \setminus \{iy \ : \ y \leq 0\} \to \mathbb{C}$ defined by $\log z = \log|z| + i\,\mathrm{Arg}(z)$ where $\mathrm{Arg}(z) \in \left(-\frac{\pi}{2}, 3\frac{\pi}{2}\right)$.



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By Cauchy's residue theorem

$$\int_{\gamma_R} f(z)\mathrm{d}z - \int_{\gamma_r} f(z)\mathrm{d}z + \int_{-R}^{-r} f(z)\mathrm{d}z + \int_r^R f(z)\mathrm{d}z = 2i\pi\,\mathrm{Res}(f,i) = -\frac{\pi^3}{4}.$$

- Since $|\log z| \leq |\ln r| + \pi$ on γ_r, $\int_{\gamma_r} f(z)\mathrm{d}z \leq \pi r \frac{(|\ln r| + \pi)^2}{1+r^2} \xrightarrow[r\to+\infty \text{ or } 0]{} 0$.
- Since $z = te^{i\pi}$ on $[-R,-r]$, we have

$$\int_{-R}^{-r} f(z)\mathrm{d}z = \int_r^R \frac{(\ln t + i\pi)^2}{1+t^2}\mathrm{d}t = \int_r^R \frac{(\ln t)^2}{1+t^2}\mathrm{d}t + 2i\pi\int_r^R \frac{\ln t}{1+t^2}\mathrm{d}t - \int_r^R \frac{\pi^2}{1+t^2}\mathrm{d}t$$

By taking the limits $r \to 0$ and $R \to +\infty$ we get $\int_0^{+\infty} \frac{(\ln t)^2}{1+t^2}\mathrm{d}t + 2i\pi\int_0^{+\infty} \frac{\ln t}{1+t^2}\mathrm{d}t - \frac{\pi^3}{2} + \int_0^{+\infty}\frac{(\ln t)^2}{1+t^2}\mathrm{d}t = -\frac{\pi^3}{4}$.

By taking the real part, we get $\int_0^{+\infty} \frac{(\ln t)^2}{1+t^2}\mathrm{d}t = \frac{\pi^3}{8}$.