The Maximum Modulus Principle & The Mean Value Property

November 18th, 2020 and November 20th, 2020

The open mapping theorem

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Let's prove that $f(U)$ is open. Let $b \in f(U)$ then $b = f(a)$ for some $a \in U$.
Note that a is a zero of $z \mapsto f(z) - f(a)$ but since this function is non-constant, by the isolated zero theorem there exists $r > 0$ such that $\overline{D_r(a)} \subset U$ and $\forall z \in \overline{D_r(a)} \setminus \{a\},\ f(z) \neq f(a)$.
Set $m = \min_{|z-a|=r} |f(z) - f(a)| > 0$. Let's prove that $D_m(b) \subset f(U)$.

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Let $w \in D_m(b)$. Let z be such that $|z - a| = r$ then

$$|(f(z) - w) - (f(z) - b)| = |w - b| < m \leq |f(z) - b|$$

By Rouché's theorem, $f(z) - w$ and $f(z) - b$ have exactly the same number of zeroes on $D_r(a)$ (counted with multiplicities) so $f(z) - w = 0$ for some $z \in D_r(a)$ and $w \in f(D_r(a)) \subset f(U)$. ∎

The maximum modulus principle – 1

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Proof. Assume that z_0 is a local maximum of $|f|$ then there exists $r > 0$ such that $D_r(z_0) \subset U$ and $\forall z \in D_r(z_0),\ |f(z)| \leq |f(z_0)|$.
Assume by contradiction that f is not constant, then, by the open mapping theorem, $f(D_r(z_0))$ is open so $\exists \delta > 0,\ D_\delta(f(z_0)) \subset f(D_r(z_0))$.
Set $w = \left(1 + \frac{\delta}{2|f(z_0)|}\right) f(z_0)$ then $w \in D_\delta(f(z_0)) \subset f(D_r(z_0))$ but $|w| > |f(z_0)|$.
Contradiction. ∎

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Once again this phenomenon is specific to complex calculus: it is possible for a real differentiable function f to be non-constant whereas $|f|$ has local maxima.

The maximum modulus principle – 2

Corollary

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a non-constant holomorphic/analytic function.

1. $\mathfrak{R}(f)$ has no local extremum on U.
2. $\mathfrak{I}(f)$ has no local extremum on U.

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Proof.

1. Define $g = e^f$, then g and $1/g$ are holomorphic and non-constant.

Hence, by the maximum modulus principle, neither $|g| = e^{\mathfrak{R}(f)}$ nor $\left|\frac{1}{g}\right| = e^{-\mathfrak{R}(f)}$ have a local maximum.

Since the real exponential is increasing, we get that neither $\mathfrak{R}(f)$ nor $-\mathfrak{R}(f)$ have a local maximum.

So $\mathfrak{R}(f)$ has no local maximum and no local minimum.

The maximum modulus principle – 2

Corollary

Let $U \subset \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a non-constant holomorphic/analytic function.

1. $\Re(f)$ has no local extremum on U.
2. $\Im(f)$ has no local extremum on U.

Proof.

1. Define $g = e^f$, then g and $1/g$ are holomorphic and non-constant.

Hence, by the maximum modulus principle, neither $|g| = e^{\Re(f)}$ nor $\left|\frac{1}{g}\right| = e^{-\Re(f)}$ have a local maximum.
Since the real exponential is increasing, we get that neither $\Re(f)$ nor $-\Re(f)$ have a local maximum.
So $\Re(f)$ has no local maximum and no local minimum.

2. Apply 1 to $-if$. ∎

Schwarz's lemma

Assume that $f : D_1(0) \to \mathbb{C}$ is holomorphic/analytic.

If
1. $f(0) = 0$
2. $\forall z \in D_1(0),\ |f(z)| \leq 1$

then
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2. $|f'(0)| \leq 1$

Moreover, if there exists $z_0 \neq 0$ s.t. $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = \lambda z$ where $|\lambda| = 1$.

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Remark

Once again, Schwarz's lemma doesn't hold for real differentiable functions.

For instance, define $u(x) = \dfrac{2x}{x^2+1}$ on $[-1, 1]$:

it is C^1, $u(0) = 0$, $|u(x)| \leq 1$ but $|u(x)| > |x|$ on $[-1, 1] \setminus \{0\}$.

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Moreover, if there exists $z_0 \neq 0$ s.t. $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then $f(z) = \lambda z$ where $|\lambda| = 1$.

Proof.

We define $g : D_1(0) \to \mathbb{C}$ by $g(z) = \begin{cases} \frac{f(z)}{z} & \text{if } z \neq 0 \\ f'(0) & \text{otherwise} \end{cases}$. Then g is holomorphic/analytic.

By the maximum modulus principle, for $r \in (0,1)$, $\max_{|z|\leq r} |g| = \max_{|z|=r} |g| \leq \frac{1}{r}$.

As $r \to 1$, we see that $|g(z)| \leq 1$, i.e. $|f(z)| \leq |z|$. It follows that $|f'(0)| \leq 1$.

Assume that there exists $z_0 \neq 0$ such that $|f(z_0)| = |z_0|$, then z_0 is local max of g. Hence, by the maximum modulus principle, g is constant. Besides it is of modulus 1, i.e. $g(z) = \lambda$ where $|\lambda| = 1$. Hence $f(z) = \lambda z$.
Assume that $|f'(0)| = 1$ then $|g(0)| = |f'(0)| = 1|$ and 0 is a local max of g. Hence we may conclude as in the above case. ∎

Maximum modulus and boundary – 1

In the above proof we used the fact that if f is defined on $\overline{U}$ and holomorphic on U, then the local maximum of $|f|$, it there are some, are located on ∂U.

Corollary

Let $U \subset \mathbb{C}$ be a domain and $f : \overline{U} \to \mathbb{C}$ a function.
Assume that f is holomorphic/analytic and non-constant on U.
Then the possible

1. local extrema of $\Re(f)$,
2. local extrema of $\Im(f)$, and,
3. local maxima of $|f|$

are located on ∂U.

Corollary

Let $U \subset \mathbb{C}$ be a bounded domain and $f : \overline{U} \to \mathbb{C}$ a continuous function.
Assume that f is holomorphic/analytic.
If $f_{|\partial U} = 0$ then $f = 0$.

Proof. $|f|$ is continuous on the compact set $\overline{U}$ (closed and bounded), hence it admits a maximum.
By the above corollary, either the function is constant equal to 0 on $\overline{U}$ or the local maxima of $|f|$ are located on ∂U (and so are the global maxima).
In either case the maximum of $|f|$ is reached on the boundary of U so that it has to be 0.
Hence $\forall z \in \overline{U},\ f(z) = 0$. ∎

The mean value property for holomorphic functions

Theorem – The mean value property for holomorphic functions

Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic/analytic.
Let $z_0 \in U$ and $r > 0$ be such that $\overline{D_r(z_0)} \subset U$. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \mathrm{d}t$$

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Proof. Define $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = z_0 + re^{it}$ then, by Cauchy's theorem,

$$f(z_0) = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z_0} \mathrm{d}w = \frac{1}{2i\pi} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}} rie^{it} \mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) \mathrm{d}t$$

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Let $\mathscr{U} \subset \mathbb{R}^2$ be open and simply connected. Let $u : \mathscr{U} \to \mathbb{R}$ be harmonic.
Let $p_0 = (x_0, y_0) \in \mathscr{U}$ and $r > 0$ be such that $\overline{D_r(p_0)} \subset \mathscr{U}$. Then

$$u(p_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(x_0 + r\cos t, y_0 + r\sin t\right) \mathrm{d}t$$

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Proof. Since u is harmonic on $\mathscr{U}$ simply connected, we know that u is the real part of a holomorphic function $f = u + iv$.
By the previous theorem

$$f(x_0 + iy_0) = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 + iy_0 + re^{it}) \mathrm{d}t$$

We conclude by taking the real part of both sides. ∎