2- Geometry in the complex plane

Norm and dot product

In what follows, we identify $\mathbb{C}$ with $\mathbb{R}^2$: $\begin{array}{ccc} \mathbb{C} & \longleftrightarrow & \mathbb{R}^2 \\ z = x+iy & \longleftrightarrow & (x,y) \end{array}$

Let $z_1 = x_1 + y_1$ and $z_2 = x_2 + iy_2$ then

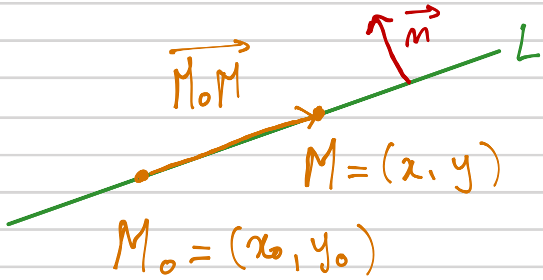
$$\mathrm{Re}(\bar{z}_1 z_2) = \mathrm{Re}(z_1 \bar{z}_2) = x_1 x_2 + y_1 y_2 = (x_1, y_1) \cdot (x_2, y_2)$$

dot product

$$|z_1| = \sqrt{x_1^2 + y_1^2} = \|(x_1, y_1)\|$$

Straight lines

Let $L \subset \mathbb{C}$ be a line, $M_0 = (x_0, y_0) \in L$ and $\vec{m} = (a,b)$ be orthogonal to L.



$$M = (x,y) \in L \iff \overrightarrow{MM_0} \perp \vec{m} \iff \overrightarrow{MM_0} \cdot \vec{m} = 0 \iff (x - x_0, y - y_0) \cdot (a,b) = 0$$

If we set $z_0 = x_0 + iy_0$, $z = x + iy$, $w = a + ib$ then

$$M \in L \iff \overrightarrow{MM_0} \cdot \vec{m} = 0 \iff \mathrm{Re}(\bar{w}(z - z_0)) = 0 \iff \mathrm{Re}(\bar{w} z) = \mathrm{Re}(\bar{w} z_0)$$

Note that $\mathrm{Re}(\bar{w} z_0) \in \mathbb{R}$.

Conversely, we can check that for $w \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{R}$, $\mathrm{Re}(\bar{w} z) = k$ is the equation of a straight line.

Conclusion:

Equation of straight lines in the complex plane:

$$\mathrm{Re}(\bar{w} z) = k \text{ where } w \in \mathbb{C} \setminus \{0\},\ k \in \mathbb{R}$$

Since $\mathrm{Re}(z) = \dfrac{z + \bar{z}}{2}$, we also get the following equivalent equation

$$\bar{w} z + w \bar{z} = r \text{ where } w \in \mathbb{C} \setminus \{0\} \text{ and } r \in \mathbb{R} \quad (r = 2k)$$

⚠ Note that $w = a + ib$ correspond to a vector orthogonal to the line

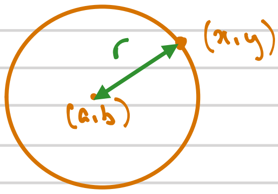
Circles : let $\omega = a+ib$ and $r \in \mathbb{R}_{>0}$

$M=(x,y)$ lies on the circle centered at (a,b) and of radius r

$\iff d((x,y),(a,b)) = r$

$\iff \|(x,y)-(a,b)\| = r$

$\iff |z-\omega| = r$ where $z = x+iy$



Conclusion : the circle centered at $\omega \in \mathbb{C}$ and of radius $r>0$ is described by $|z-\omega| = r$

or, equivalently since $|z-\omega|^2 = (z-\omega)\overline{(z-\omega)} = (z-\omega)(\bar{z}-\bar{\omega}) = z\bar{z} - \bar{\omega}z - \omega\bar{z} + \omega\bar{\omega}$

$$z\bar{z} - \bar{\omega}z - \omega\bar{z} = r^2 - |\omega|^2$$

Circle-line equation

Proposition : The equation $a\bar{z}z - \bar{\eta}z - \eta\bar{z} + k = 0$, $a,k \in \mathbb{R}$, $\eta \in \mathbb{C}$, $|\eta|^2 - ak > 0$ describes :
- a line if $a = 0$
- a circle if $a \neq 0$

And conversely, any circle or line has such an equation

Δ $\underline{a=0}$: $\bar{\eta}z + \eta\bar{z} = k$

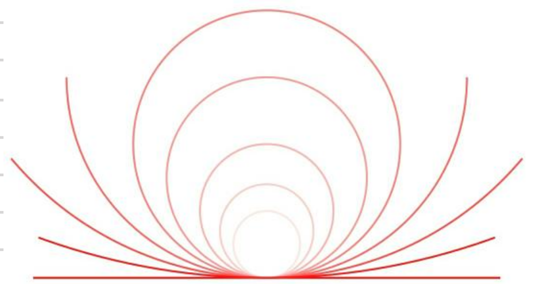
$\underline{a\neq 0}$: $\bar{z}z - \overline{\left(\frac{\eta}{a}\right)}z - \frac{\eta}{a}\bar{z} = -\frac{k}{a}$

$\iff \bar{z}z - \bar{\omega}z - \omega\bar{z} + \bar{\omega}\omega = |\omega|^2 - \frac{k}{a}$ where $\omega = \frac{\eta}{a}$

$\iff |z-\omega|^2 = |\omega|^2 - \frac{k}{a}$ $\square$

Intuitively, a line is a degenerate circle in $\hat{\mathbb{C}} = \mathbb{C} \sqcup \{\infty\}$ (this statement will be made precise later)

Eg : take the circle centered at $(0,r)$ with radius r then, when $r \to +\infty$, we get :



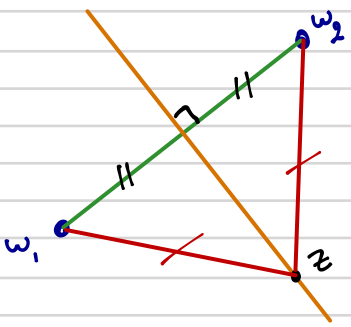
<u>Proposition</u>: take $w_1, w_2 \in \mathbb{C}$, $\rho \in (0, \infty)$

$$|z - w_1| = \rho |z - w_2|$$

describes: • a line if $\rho = 1$

• a circle if $\rho \neq 1$

Δ • $\underline{\rho = 1}$: $|z-w_1| = |z-w_2|$ is the set of points equidistant to w_1 and w_2, it is the bisector to the segment line from w_1 to w_2:



• $0 < \rho < 1$:

$$|z-w_1| = \rho|z-w_2| \iff |\tilde{z} - c|^2 = \rho^2 |\tilde{z}|^2 \quad \text{where } \tilde{z} = z - w_2, \ c = w_1 - w_2$$

$$\iff (\tilde{z} - c)(\bar{\tilde{z}} - \bar{c}) = \rho^2 |\tilde{z}|^2$$

$$\iff (1-\rho^2)|\tilde{z}|^2 - \bar{c}\tilde{z} - c\bar{\tilde{z}} + |c|^2 = 0$$

$$\iff |\tilde{z}|^2 - \left(\frac{\bar{c}}{1-\rho^2}\right)\tilde{z} - \left(\frac{c}{1-\rho^2}\right)\bar{\tilde{z}} + \left|\frac{c}{1-\rho^2}\right|^2 = \frac{\rho^2|c|^2}{(1-\rho^2)^2}$$

$$\iff \left|\tilde{z} - \frac{c}{1-\rho^2}\right|^2 = \left(\frac{\rho|c|}{1-\rho^2}\right)^2$$

$$\iff \left|z - w_2 - \frac{w_1 - w_2}{1-\rho^2}\right| = \frac{\rho|c|}{1-\rho^2}$$

Circle centered at: $w_2 + \dfrac{w_1 - w_2}{1-\rho^2} = \dfrac{w_1 - \rho^2 w_2}{1-\rho^2}$ with radius: $\dfrac{\rho|c|}{1-\rho^2}$

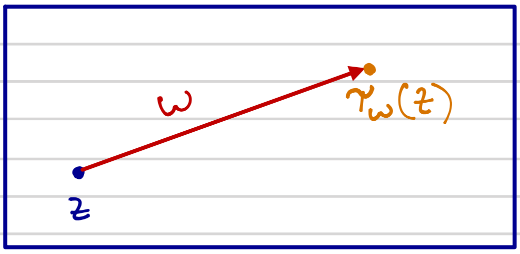
• If $\rho > 1$: $|z-w_1| = \rho|z-w_2| \iff |z-w_2| = \frac{1}{\rho}|z-w_1|$

same as before with $\rho \rightsquigarrow 1/\rho$, $w_1 \rightsquigarrow w_2$, $w_2 \rightsquigarrow w_1$ $\square$

<u>Homework</u>: Read p18-20 (Appolonius circles)

Some transformations:

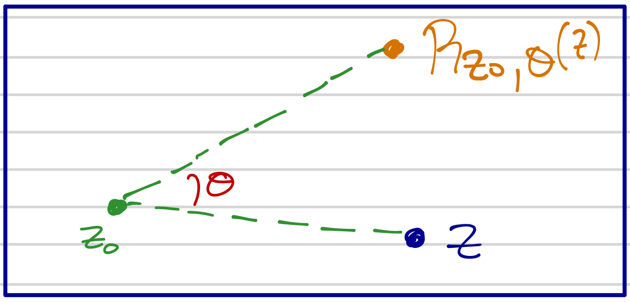
- Translation by $w \in \mathbb{C}$: $\tau_w : \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & z+w \end{array}$



Maps
- lines to lines
- circles to circles

- Rotation centered at z_0 with angle $\theta \in \mathbb{R}$

$$R_{z_0,\theta} : \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & (z-z_0)e^{i\theta} + z_0 \end{array}$$



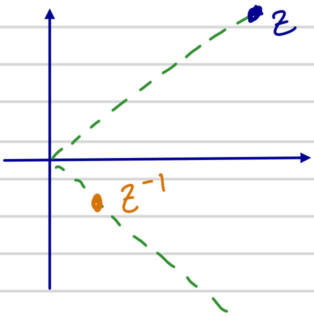
Maps
- lines to lines
- circles to circles

- Scaling by $\lambda \in \mathbb{R}_{>0}$

$$s_\lambda : \begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ z & \longmapsto & \lambda z \end{array}$$

Maps
- lines to lines
- circles to circles

- Inversion: $i : \begin{array}{ccc} \mathbb{C}\setminus\{0\} & \longrightarrow & \mathbb{C}\setminus\{0\} \\ z & \longmapsto & 1/z \end{array}$



$|z^{-1}| = |z|^{-1}$

$\arg(z^{-1}) \equiv -\arg(z) \mod 2\pi$

It maps {circles, lines} to {circles, lines}

⚠ It may maps a circle to a line or vice-versa

⚠ It should not be confused with the geometric inversion : $z \mapsto \overline{z^{-1}}$

Homework: We consider the inversion $i: z \mapsto z^{-1}$

① What's the image of a line passing through the origin?
② What's the image of a circle centered at 0?
③ What's the image of a circle passing through 0?
④ What's the image of a line not passing through 0?

Sample solutions:

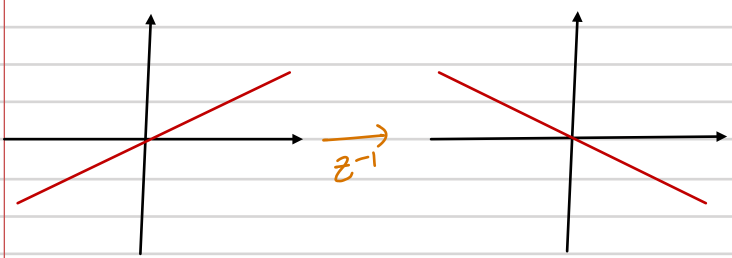
① A line passing through the origin admits an equation of the form: $\bar{\omega} z + \omega \bar{z} = 0$ where $\omega \in \mathbb{C}\setminus\{0\}$ is orthogonal to the line

set $z' = 1/z \iff z = 1/z'$ then $\bar{\omega} z + \omega \bar{z} = 0$

$\iff \bar{\omega}\, \overline{z'} + \omega z' = 0$

which is the equation of a line orthogonal to $\bar{\omega}$

So we get the reflection of the original w.r.t. the real axis



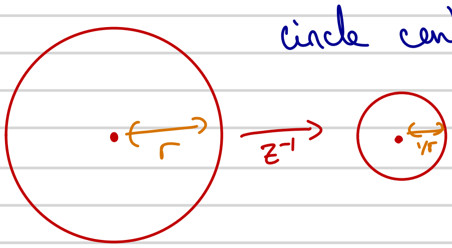
Conclusion: a line passing through the origin, but minus the origin ($z \mapsto z^{-1}$ not defined at 0, yet) is mapped to its reflection w.r.t. the real axis (minus the origin)

② A circle centered at the origin has an eqn of the form

$$\bar{z} z = r^2, \quad r > 0 \text{ radius}$$

$$z' = \frac{1}{z} : \bar{z} z = r^2 \iff \overline{z'} z' = \frac{1}{r^2}$$

circle centered at 0 of radius $1/r$



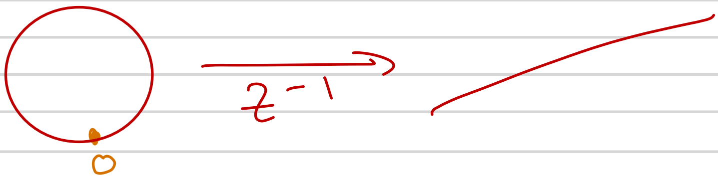
The circle centered at the origin of radius r is mapped to the circle centered at the origin of radius $1/r$

③ A circle passing through the origin has an equation of the form

$$\bar{z}z - \bar{\omega}z - \omega\bar{z} = 0$$

$z' = 1/z$: $\Leftrightarrow 1 - \bar{\omega}\overline{z'} - \omega z' = 0$

which is a line (not passing through 0)



A circle passing through the origin, minus the origin, ($z \mapsto z^{-1}$ not defined at 0) is mapped to a line not passing through the origin

④ A line which doesn't pass through the origin has an eqn of the form $\bar{\omega}z + \omega\bar{z} = k$, $k \in \mathbb{R}\setminus\{0\}$, $\omega \in \mathbb{C}\setminus\{0\}$

$z' = 1/z$ $\Leftrightarrow \bar{\omega}\overline{z'} + \omega z' = k\overline{z'}z$

$$\Leftrightarrow \overline{z'}z - \left(\overline{\frac{\omega}{k}}\right)\overline{z'} - \left(\frac{\omega}{k}\right)z' = \left|\frac{\omega}{k}\right|^2 - \left|\frac{\omega}{k}\right|^2$$

it's the equation of a circle centered at $\frac{\bar{\omega}}{k}$ of radius $\left|\frac{\bar{\omega}}{k}\right|$, so $z \mapsto z^{-1}$ maps a line not passing through the origin to a circle passing throug the origin minus the origin: