6 - Line integrals / Green's theorem

- Curves

Definitions:

- A curve is a continuous function $\gamma : [a,b] \longrightarrow \mathbb{C}$
  then $\gamma(a)$ is the start point of the curve and $\gamma(b)$ is its endpoint.

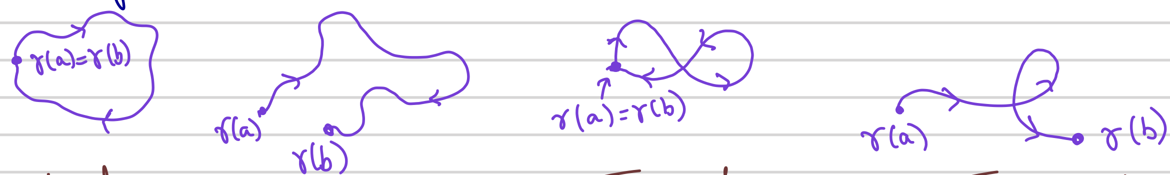
(Be careful: in practice we often use "curve" to talk about either γ or $\mathrm{Range}(\gamma)$)

- We say that a curve $\gamma : [a,b] \to \mathbb{C}$ is simple if γ doesn't admit double points except maybe $\gamma(a)=\gamma(b)$:

$$\left.\begin{array}{l} a \leqslant t_1 < t_2 \leqslant b \\ \gamma(t_1)=\gamma(t_2) \end{array}\right\} \Rightarrow t_1 = a \text{ and } t_2 = b$$

- We say that a curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ is closed if $\gamma(a)=\gamma(b)$

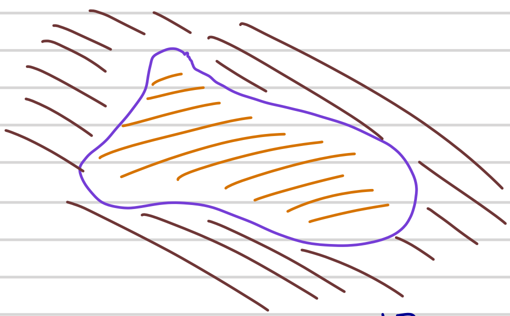
Eg:



simple	simple	NOT simple	NOT simple
closed	NOT closed	closed	NOT closed

The following theorem seems obvious but is difficult to prove.

Jordan curve theorem: If $\gamma : [a,b] \longrightarrow \mathbb{C}$ is a simple closed curve then $\mathbb{C} \setminus \gamma([a,b])$ consists of 2 disjoint open connected sets, one bounded (the inside) and one unbounded (the outside).



i.e.: $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ are C^1; here $t \in [a,b] \subset \mathbb{R}$

- We say that the curve $\gamma : [a,b] \to \mathbb{C}$ is smooth if γ is C^1.
- We say that $\gamma : [a,b] \longrightarrow \mathbb{C}$ is piecewise smooth if there exist $t_0, t_1, \dots, t_n \in [a,b]$ s.t. $a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$ and

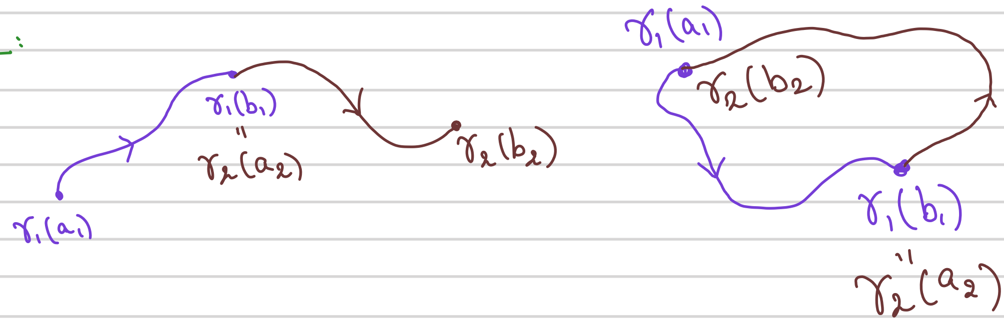
$$\gamma \text{ is } C^1 \text{ on } [t_k, t_{k+1}],\ k = 0, \dots, n-1$$

- Given $\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$ two curves such that $\gamma_1(b_1) = \gamma_2(a_2)$, we define the concatenation (or sum) of γ_1 and γ_2 by

$$(\gamma_1 + \gamma_2) : [a_1, b_1 + b_2 - a_2] \longrightarrow \mathbb{C}$$

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [a_1, b_1] \\ \gamma_2(t + a_2 - b_1) & \text{if } t \in [b_1, b_1 + b_2 - a_2] \end{cases}$$

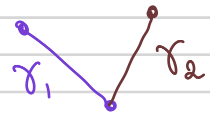
Eg:



Prop: if γ_1 and γ_2 are piecewise-smooth then $(\gamma_1 + \gamma_2)$ is too

⚠ γ_1, γ_2 C^1 $\not\Longrightarrow$ $(\gamma_1 + \gamma_2)$ C^1

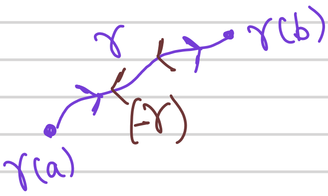
Eg:



Curves are "naturally" oriented by the usual orientation on $[a,b]$ (ie $\gamma : [a,b] \to \mathbb{C}$ goes from $\gamma(a)$ to $\gamma(b)$)

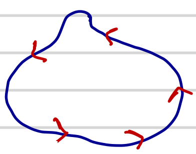
- Given $\gamma : [a,b] \longrightarrow \mathbb{C}$ a curve, we define $(-\gamma) : [a,b] \longrightarrow \mathbb{C}$ by $(-\gamma)(t) := \gamma(a+b-t)$

It is the same curve but traversed from $\gamma(b)$ to $\gamma(a)$

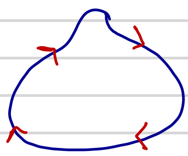


• We say that a simple closed curve is positively oriented if γ keeps its inside on its left:

Eg:



is positively oriented

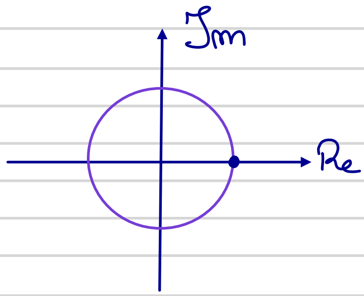


is NOT positively oriented

Some examples:

• $\gamma : [0,1] \longrightarrow \mathbb{C}$

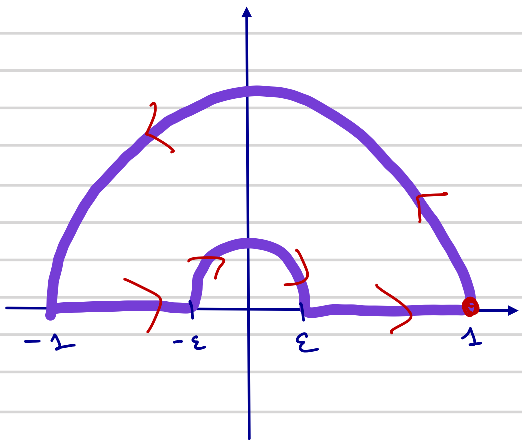
$\gamma(t) = (1-t)z_0 + t z_1$



• $\gamma : [0,1] \longrightarrow \mathbb{C}$

$\gamma(t) = e^{2i\pi t}$

• $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$



$\gamma_1 : \begin{array}{ccc} [0,\pi] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & e^{it} \end{array}$

$\gamma_2 : \begin{array}{ccc} [-1,-\varepsilon] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & t \end{array}$

$\gamma_3 : \begin{array}{ccc} [0,\pi] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & \varepsilon e^{i(\pi - t)} \end{array}$

$\gamma_4 : \begin{array}{ccc} [\varepsilon,1] & \longrightarrow & \mathbb{C} \\ t & \longmapsto & t \end{array}$

- Integrals :

Definition : for $f : [a,b] \longrightarrow \mathbb{C}$ continuous we set

$$\int_a^b f(t)\,dt := \int_a^b \mathrm{Re}(f(t))\,dt + i\int_a^b \mathrm{Im}(f(t))\,dt$$

Prop : $\mathrm{Re}\left(\int_a^b f(t)dt\right) = \int_a^b \mathrm{Re}(f(t))\,dt$

$$\mathrm{Im}\left(\int_a^b f(t)\,dt\right) = \int_a^b \mathrm{Im}\,(f(t))\,dt$$

- Line integrals

Def : $U \subset \mathbb{C}$, $f : U \longrightarrow \mathbb{C}$ continuous, $\gamma : [a,b] \longrightarrow \mathbb{C}$ a smooth curve s.t. $\forall t \in [a,b]$, $\gamma(t) \in U$ then the line integral of f along γ is $\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt$

Def : $U \subset \mathbb{C}$, $f : U \longrightarrow \mathbb{C}$ continuous, $\gamma : [a,b] \longrightarrow \mathbb{C}$ piecewise smooth curve s.t. $\forall t \in [a,b]$, $\gamma(t) \in U$, assume that we have a subdivision $a = t_0 < t_1 < \cdots < t_m = b$ s.t. γ is C^1 on $[t_k, t_{k+1}]$ then the line integral of f along γ is defined by

$$\int_\gamma f(z)\,dz := \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} f(\gamma(t))\,\gamma'(t)\,dt$$

(and it doesn't depend on the choice of the subdivision)

Remark: it is common to use the notation $\oint_\gamma f(z)\,dz$ when γ is closed

Proposition : $\int_{-\gamma} f = -\int_\gamma f$

$$\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

$$\int_\gamma \alpha f + \beta g = \alpha\int_\gamma f + \beta\int_\gamma f \quad , \alpha, \beta \in \mathbb{C}$$

⚠ Homework : check these properties □

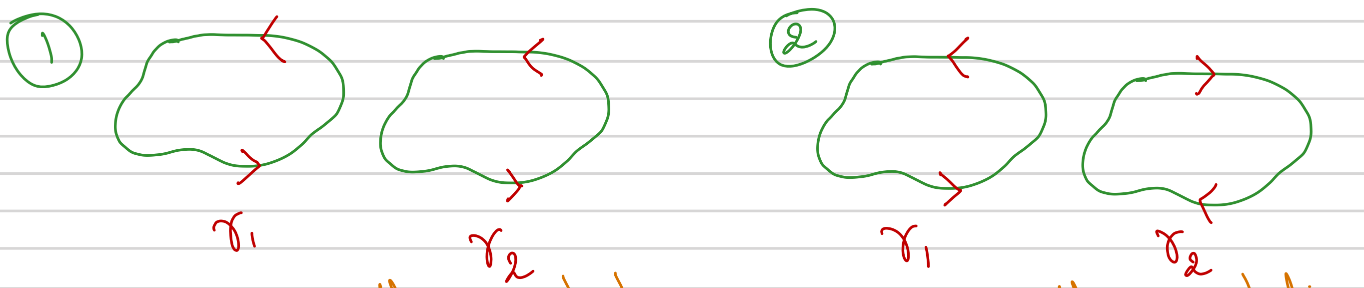
Theorem: (reparametrization)

Let $\gamma_1 : [a_1, b_1] \longrightarrow \mathbb{C}$, $\gamma_2 : [a_2, b_2] \longrightarrow \mathbb{C}$ be two curves

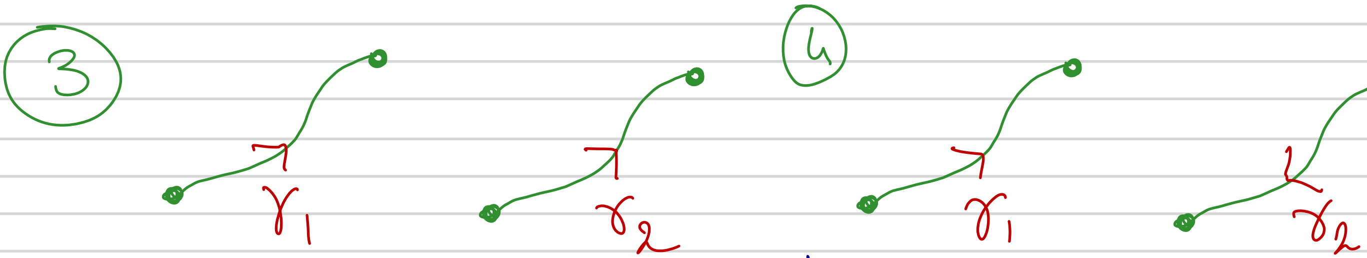
Assume that $\gamma_1 = \gamma_2 \circ \varphi$ where $\varphi : [a_1, b_1] \longrightarrow [a_2, b_2]$ is a C^1-diffeo

If φ is increasing then $\int_{\gamma_1} f = \int_{\gamma_2} f$ (we say that φ preserves the orientation)

If φ is decreasing then $\int_{\gamma_1} f = -\int_{\gamma_2} f$ (we say that φ reverses the orientation)



preserves the orientation reverses the orientation



Δ $\displaystyle\int_{\gamma_1} f(z)\,dz = \int_{a_1}^{b_1} f(\gamma_1(t))\,\gamma_1'(t)\,dt = \int_{a_1}^{b_1} f(\gamma_2(\varphi(t)))\,\gamma_2'(\varphi(t))\,\varphi'(t)\,dt$

φ increasing : $\varphi = |\varphi|$

φ decreasing : $\varphi = -|\varphi|$

$\displaystyle = \pm \int_{a_1}^{b_1} f(\gamma_2(\varphi(t)))\,\gamma_2'(\varphi(t))\,|\varphi'(t)|\,dt$

$\displaystyle = \pm \int_{a_2}^{b_2} f(\gamma_2(s))\,\gamma_2'(s)\,ds$ $\square$

Definition: the length of a smooth curve $\gamma : [a,b] \to \mathbb{C}$ is defined by:

$$\text{length}(\gamma) := \int_a^b |\gamma'(t)|\,dt$$

Proposition: $S \subset \mathbb{C}$, $f : S \longrightarrow \mathbb{C}$ continuous, $\gamma : [a,b] \to \mathbb{C}$ a smooth curve such that $\forall t \in [a,b]$, $\gamma(t) \in S$ then $\left|\int_\gamma f(z)\,dz\right| = \left(\max_{\gamma([a,b])} |f|\right) \cdot \text{length}(\gamma)$

Δ $\displaystyle\left|\int_\gamma f(z)\,dz\right| = \left|\int_a^b f(\gamma(t))\,\gamma'(t)\,dt\right| \leq \int_a^b |f(\gamma(t))\,\gamma'(t)|\,dt$

$\displaystyle \leq \max_{\gamma([a,b])} |f| \int_a^b |\gamma'(t)|\,dt$ $\square$

the max is achieved since $\gamma([a,b])$ is compact and $|f|$ continuous

Green's theorem: (from multivariable calculus)

Let $S \subset \mathbb{R}^2$ be a regular region with piecewise smooth boundaries ∂S assumed to be positively oriented.

Let $F: \mathcal{U} \longrightarrow \mathbb{R}^2$ be a C^1-vector field where $\mathcal{U} \subset \mathbb{R}^2$ open satisfies $S \subset \mathcal{U}$

Then: $\displaystyle\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$ where $F = (P, Q)$

or with the alternative notation you may have seen:

$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \quad \text{where } \vec{F} = (P, Q)$$

Recall: • S regular region means S bounded and $\overline{\mathring{S}} = \overline{S}$

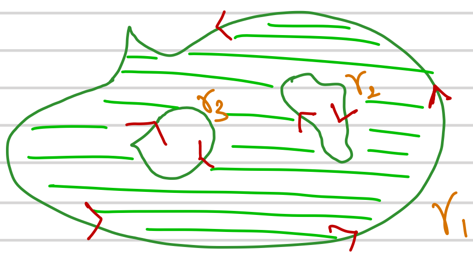
or equivalently: S bounded and $\forall u \in \partial S, \forall r > 0, D_r(u) \cap \mathring{S} \neq \emptyset$

Eg: 

$\overline{D_1(0)}$ is a regular region

$[0,1] \times \{0\}$ is not a regular region

• ∂S positively oriented means that each piece of ∂S is parametrized by a simple closed curve s.t. the interior $\mathring{S}$ is on the left of the parametrization



• If $\partial S = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_p$ then $\displaystyle\int_{\partial S} = \int_{\gamma_1} + \int_{\gamma_2} + \cdots + \int_{\gamma_p}$

Green's theorem: (Complex version)

Let $S \subset \mathbb{C}$ be a regular region with piecewise smooth boundaries ∂S assumed to be positively oriented.

Let $\mathcal{U} \subset \mathbb{C}$ be open s.t. $S \subset \mathcal{U}$ and $f : \mathcal{U} \longrightarrow \mathbb{C}$ C^1 in the real sense: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $(x,y) \mapsto (\mathrm{Re}(f(x+iy)), \mathrm{Im}(f(x+iy)))$ C^1

Then:

Version 1:
$$\int_{\partial S} f(z)\,dz = i \iint_S \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) dx\,dy$$

usual line integral $\int_\gamma f = \int f(\gamma(t))\gamma'(t)$

where:
$$\frac{\partial f}{\partial x} := \frac{\partial \mathrm{Re}(f)}{\partial x} + i \frac{\partial \mathrm{Im}(f)}{\partial x}$$
$$\frac{\partial f}{\partial y} := \frac{\partial \mathrm{Re}(f)}{\partial y} + i \frac{\partial \mathrm{Im}(f)}{\partial y}$$

Version 2:
$$\int_{\partial S} f(z)\,dz = 2i \iint_S \frac{\partial f}{\partial \bar{z}}\, dx\,dy$$

where:
$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$
$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$$

Δ Proof:
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt = \int_a^b \left[P(\gamma(t))\gamma_1'(t) - Q(\gamma(t))\gamma_2'(t) \right] + i \left[P(\gamma(t))\gamma_2'(t) + Q(\gamma(t))\gamma_1'(t) \right] dt$$

where $f = P + iQ$.

hence
$$\int_\gamma f = \int_{\partial S} P\,dx - Q\,dy + i \int P\,dy + Q\,dx \overset{\text{Green's thm}}{=} -\iint_S \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y} \right) dx\,dy + i \iint_S \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} \right)$$
$$= i \iint_S \left[\left(\frac{\partial P}{\partial x} + i \frac{\partial Q}{\partial x} \right) + i \left(\frac{\partial P}{\partial y} + i \frac{\partial Q}{\partial y} \right) \right] dx\,dy$$
$$= i \iint_S \left[\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right] dx\,dy$$
$$= 2i \iint_S \frac{\partial f}{\partial \bar{z}}\, dx\,dy$$