Concepts in Abstract Mathematics

Homework questions – Week 3

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Exercise 1.
Let $a, b \in \mathbb{Z}$. Prove that if $a^2 = b^2$ then $|a| = |b|$.

Exercise 2.
Let $n \in \mathbb{N} \setminus \{0\}$. Prove that given n consecutive integers, one is divisible by n.
The above result is very useful and from now on you can use it without proving it again.

Exercise 3.
1. Compute $\gcd(816, 2260)$.
2. Find $(u, v) \in \mathbb{Z}^2$ such that $816u + 2260v = \gcd(816, 2260)$.

Exercise 4.
1. Does the divisibility relation $|$ define an order on $\mathbb{Z}$? If so, is it total?
2. Does the divisibility relation $|$ define an order on $\mathbb{N}$? If so, is it total?

Exercise 5.
Prove that $\forall n \in \mathbb{N},\ 7 | 3^{2n+1} + 2^{4n+2}$

Exercise 6.
Let $a, b, c, d \in \mathbb{Z}$ be such that $ad + bc \neq 0$. Prove that if $ad + bc$ divides a, b, c, d then $|ad + bc| = 1$.

Exercise 7.
Prove that $\forall n \in \mathbb{N},\ \gcd(n^2 + n, 2n + 1) = 1$

Exercise 8.
Let $a, b \in \mathbb{Z}$. Prove that if $\gcd(a, b) = 1$ then $\gcd(a^2, b^2) = 1$.

Exercise 9.
Prove that
1. $\forall a, b \in \mathbb{Z} \setminus \{0\},\ a^2 | b^2 \implies a | b$
2. Prove that $\forall a, b, c \in \mathbb{Z} \setminus \{0\},\ \gcd(a, b) = 1$ and $c | b \implies \gcd(a, c) = 1$

Exercise 10.
For each of the following statements: is it true? If so, prove it. Otherwise, give a counter-example.
1. If $a, b \in \mathbb{Z}$ are coprime then $a + b$ and ab are too.
2. If $a, b \in \mathbb{Z}$ are coprime then $a + b$ and $a^2 + b^2$ are too.

We say that a and b are coprime if $\gcd(a, b) = 1$.

Exercise 11.
1. Prove that among 42 distinct integers, there are always two distinct integers a and b such that $b - a$ is a multiple of 41.
2. Prove that among five integers, there are always three with sum divisible by 3.

Exercise 12.

Seven friends have a dinner in a restaurant. When he brings the bill, the waiter makes the following offer: *"I'll put on each of your foreheads a sticky note with a day of the week*[1] *written on it, so that each of you will see the other six notes but not yours. Then you will have to guess the day written on your note (by secretly writing your guess on your napkin). If at least one of you has the correct answer, then the bill is on me. By the way, there is no rule concerning my choices for the days, for instance I can assign several times the same day."*
While the waiter left to write the sticky notes, one of the friends, who happens to be a mathematician, exclaims: *"I found a way so that we are 100% sure that one of us is correct!"*.
And then he explains his winning strategy to his friends.
Can you guess what it is?

[1] Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday.

Sample solutions to Exercise 1.

Let $a, b \in \mathbb{Z}$ such that $a^2 = b^2$. Then $0 = a^2 - b^2 = (a-b)(a+b)$. Hence either $a = b$ or $a = -b$. In both cases $|a| = |b|$.

Sample solutions to Exercise 2.

Consider n consecutive integers $a, a+1, \dots, a+(n-1)$.
By Euclidean division, there exists $b, q \in \mathbb{Z}$ such that $a+(n-1) = bn + r$ and $0 \leq r < n$.
Then $a+(n-1)-r = bn$ and $0 \leq (n-1)-r \leq n-1$. Thus $a+(n-1)-r$ is an element of the above list which is divisible by n.

Sample solutions to Exercise 3.

1. We use Euclid's algorithm:

$$\begin{array}{ccccccccc} 2260 & = & 816 & \times & 2 & + & 628 \\ 816 & = & 628 & \times & 1 & + & 188 \\ 628 & = & 188 & \times & 3 & + & 64 \\ 188 & = & 64 & \times & 2 & + & 60 \\ 64 & = & 60 & \times & 1 & + & 4 \\ 60 & = & 4 & \times & 15 & + & 0 \end{array}$$

   Thus $\gcd(816, 2260) = 4$.

2. To find a Bézout's relation for 816 and 2260, we follow Euclid's algorithm backward: at each step we plug the previous remainder starting from the last Euclidean division with non-zero remainder.

$$\begin{aligned} 4 &= 64 - 60 \\ &= 64 - (188 - 64 \times 2) \\ &= -188 + 64 \times 3 \\ &= -188 + (628 - 188 \times 3) \times 3 \\ &= 628 \times 3 + 188 \times (-10) \\ &= 628 \times 3 + (816 - 628) \times (-10) \\ &= 816 \times (-10) + 628 \times 13 \\ &= 816 \times (-10) + (2260 - 816 \times 2) \times 13 \\ 4 &= 2260 \times 13 + 816 \times (-36) \end{aligned}$$

Sample solutions to Exercise 4.

1. Divisibility doesn't define an order on $\mathbb{Z}$ since it is not antisymmetric.
   Indeed $1 | -1$ and $-1 | 1$ but $-1 \neq 1$.

2. Divisibility defines an order on $\mathbb{N}$:
   - *Reflexivity.* Let $a \in \mathbb{N}$ then $a = a \times 1$ so that $a|a$.
   - *Transitivity.* Let $a, b, c \in \mathbb{N}$ be such that $a|b$ and $b|c$. Then $b = ka$ and $c = lb$ for some $k, l \in \mathbb{Z}$. Thus $c = lk = lka$. Henre $a|c$.
   - *Antisymmetry.* Let $a, b \in \mathbb{N}$ such that $a|b$ and $b|a$. Then $|a| = |b|$. But since $a, b \in \mathbb{N}$, $|a| = a$ and $|b| = b$. Therefore $a = b$.

   It is not total since $2 \nmid 3$ and $3 \nmid 2$.

Sample solutions to Exercise 5.
Let's prove by induction that $\forall n \in \mathbb{N},\ 7 | 3^{2n+1} + 2^{4n+2}$.
Base case at $n = 0$: $3^{2\times 0+1} + 2^{4\times 0+2} = 7$ and $7|7$.
Induction step: assume that $7|3^{2n+1} + 2^{4n+2}$ for some $n \in \mathbb{N}$.
Then $3^{2n+1} + 2^{4n+2} = 7k$ for some $k \in \mathbb{Z}$ and

$$\begin{aligned}
3^{2(n+1)+1} + 2^{4(n+1)+2} &= 9 \times 3^{2n+1} + 16 \times 2^{4n+2} \\
&= (7+2) \times 3^{2n+1} + (7 \times 2 + 2) \times 2^{4n+2} \\
&= 7 \times (3^{2n+1} + 2 \times 2^{4n+2}) + 2 \times (3^{2n+1} + 2^{4n+2}) \\
&= 7 \times (3^{2n+1} + 2^{4n+3}) + 6 \times 7k \\
&= 7 \times (3^{2n+1} + 2^{4n+3} + 6k)
\end{aligned}$$

Hence $7|3^{2(n+1)+1} + 2^{4(n+1)+2}$ which ends the induction step.

Sample solutions to Exercise 6.
Since $ad + bc$ divides a, b, c, d, there exist α, β, δ such that $a = \alpha(ad + bc)$, $b = \beta(ad + bc)$, $c = \gamma(ad + bc)$ and $d = \delta(ad + bc)$.
Then $ad + bc = \alpha(ad + bc)\delta(ad + bc) + \beta(ad + bc)\gamma(ad + bc) = (\alpha\delta + \beta\gamma)(ad + bc)^2$.
Since $ad + bc \neq 0$, we get that $1 = (\alpha\delta + \beta\gamma)(ad + bc)$.
Therefore $(ad + bc)|1$, and obviously $1|(ad + bc)$, hence $|ad + bc| = |1| = 1$.

Sample solutions to Exercise 7.
Let $n \in \mathbb{N}$. Set $d = \gcd(n^2 + n, 2n + 1)$. Then $d| \left((2n + 1)^2 - 4(n^2 + n)\right) = 1$. Thus $d = 1$.

Sample solutions to Exercise 8.
Let $a, b \in \mathbb{Z}$ be such that $\gcd(a, b) = 1$.
By Bézout's identity, there exist $u, v \in \mathbb{Z}$ such that $au + bv = 1$.
Hence $1 = (au + bv)^3 = a^2(au^3 + 3u^2bv) + b^2(bv^3 + 3auv^2)$.
Thus if $d|a^2$ and $d|b^2$ then $d|a^2(au^3 + 3u^2bv) + b^2(bv^3 + 3auv^2) = 1$.
Therefore $\gcd(a^2, b^2) = 1$.

Sample solutions to Exercise 9.

1. Let $a, b \in \mathbb{Z} \setminus \{0\}$ such that $a^2|b^2$.
   Set $d = \gcd(a, b)$, then $a = \alpha d$ and $b = \beta d$ for some $\alpha, \beta \in \mathbb{Z}$.
   Then $d = \gcd(a, b) = d \gcd(\alpha, \beta)$, thus $\gcd(\alpha, \beta) = 1$.
   And $\gcd(a^2, b^2) = \gcd(d^2\alpha^2, d^2\beta^2) = d^2 \gcd(\alpha^2, \beta^2) = d^2$ from the previous exercise.
   Since $a^2|b^2$, we know that $\gcd(a^2, b^2) = a^2$.
   Hence $a^2 = d^2$ and thus $d = \pm a$.
   Therefore $a = \pm d|b$.

2. Let $a, b, c \in \mathbb{Z} \setminus \{0\}$ be such that $\gcd(a, b) = 1$ and $c|b$.
   From Bézout's identity, there exist $u, v \in \mathbb{Z}$ such that $au + bv = 1$.
   Let $d \in \mathbb{Z}$ such that $d|a$ and $d|c$. Then, $d|b$ since $c|b$.
   Hence $d|au + bv = 1$.
   Therefore $\gcd(a, b) = 1$.

Sample solutions to Exercise 10.

1. Let $a, b \in \mathbb{Z}$ be such that $\gcd(a, b) = 1$. From Bézout's identity, there exist $u, v \in \mathbb{Z}$ such that $au + bv = 1$.
   Squaring both sides, we get $a^2u^2 + b^2v^2 + 2abuv = 1$.
   Let $d = \gcd(a + b, ab)$. Note that $a^2 = a(a + b) - ab$, $b^2 = b(a + b) - ab$ hence $d|a^2$ and $d|b^2$. Besides $d|ab$.
   Therefore $d| \left(a^2u^2 + b^2v^2 + 2abuv\right) = 1$. Hence $d = 1$.

2. Take $a = b = 1$. Then $\gcd(a, b) = 1$ but $\gcd\left(a + b, a^2 + b^2\right) = 2$. So the statement is false.

Sample solutions to Exercise 11.
We are going to use the *pigeonhole principle* also called *Dirichlet's drawer principle*.

1. The remainder of an Euclidean division by 41 satisfies $0 \leq r < 41$. Hence, there are 41 possible remainders. Therefore, among 42 distinct integers, at least two, say a and b, have the same remainder (otherwise the number of remainders will be 42). Then $a = 41q + r$ and $b = 41q' + r$ for $q, q', r \in \mathbb{Z}$ such that $0 \leq r < 41$. And finally $b - a = 41(q' - q)$.

2. Either we can find 3 of these integers which have the same remainder by Euclidean division by 3, i.e. $x_1 = 3q_1 + r$, $x_2 = 3q_2 + r$, $x_3 = 3q_3 + r$. And then $x_1 + x_2 + x_3 = 3(q_1 + q_2 + q_3 + r)$.
Otherwise, we can find one integer for all the possible remainders: $x_1 = 3q_1 + 0$, $x_2 = 3q_2 + 1$ and $x_3 = 3q_3 + 2$. And then $x_1 + x_2 + x_3 = 3(q_1 + q_2 + q_3 + 1)$.

Sample solutions to Exercise 12.
First, he replaces days with numbers as follows:
- Sunday $\leftrightarrow 0$
- Monday $\leftrightarrow 1$
- Tuesday $\leftrightarrow 2$
- Wednesday $\leftrightarrow 3$
- Thursday $\leftrightarrow 4$
- Friday $\leftrightarrow 5$
- Saturday $\leftrightarrow 6$

So we can assume that the sticky notes contain a number between 0 and 6 (included).
Then he numbers the participants (including himself) from 0 to 6 (which is possible since there are seven friends).
And then he explains: *"each of us will add to the sum of the six days he can see, the unique number of* $\{0, 1, \dots, 6\}$ *such that the remainder of the Euclidean division by 7 of the obtained sum corresponds to its assigned number."*

Let me explain why it works.
First, by Euclidean division, the actual sum N of the seven numbers sticked on their forheads can be uniquely written $N = 7 \times q + r$ with $0 \leq r < 7$, i.e. the possible remainders are $r \in \{0, 1, \dots, 6\}$.
I claim that the participant whose assigned number is r gets the correct answer.
Indeed, if the sum of the six numbers he sees is M, then there is a unique $a \in \{0, 1, \dots, 6\}$ such that the Euclidean division of $M + a$ by 7 is r, i.e. $M + a = 7 \times q' + r$.
Since $N - M \in \{0, 1, \dots, 6\}$ and since N and $M + a$ have same remainder by 7, then necessarily $N = M + a$.
So a is exactly the day on the sticky note of the participant whose assigned number is r. And he gets the good answer.