

NONLINEAR SCHRÖDINGER EQUATIONS

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1. BACKGROUND

fixedpointexercise

Exercise 1.1. State and prove Banach's fixed point theorem. Use this theorem to prove the fundamental existence theorem for ODEs (under an appropriate Lipschitz hypothesis).

1.1. **Fourier transform and oscillatory integrals.** Definition of F.T.; Derivatives and F.T., smoothness/decay; Plancherel's Theorem; Riemann-Lebesgue Lemma; Hausdorff Young; Products/Convolutions; Stationary Phase

We closely follow [33], [42].

Fourier transform on L^1

Definition 1.1. $\forall f \in L^1(\mathbb{R}^n)$ we define the Fourier transform of f by

ft_defined

$$(1.1) \quad \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi x \cdot \xi} f(x) dx \quad \forall \xi \in \mathbb{R}^n.$$

L1FT

Theorem 1.2. Let $f \in L^1(\mathbb{R}^n)$. Then

(1) $f \mapsto \widehat{f}$ is a bounded linear map $L^1(\mathbb{R}^n) \mapsto L^\infty(\mathbb{R}^n)$ and

FTL1Linfty

$$(1.2) \quad \|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}.$$

(2) \widehat{f} is continuous.

(3) (Riemann-Lebesgue) $\widehat{f}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$.

(4) $\forall h \in \mathbb{R}^n$ define the translation operator $\tau_h f(x) = f(x - h)$. Then

FTtranslation

$$(1.3) \quad \widehat{\tau_h f}(\xi) = e^{-2\pi i \xi \cdot h} \widehat{f}(\xi)$$

and

$$(e^{-2\pi i x \cdot h} f)(\xi) = \tau_{-h} \widehat{f}(\xi).$$

(5) $\forall a > 0$ define the dilation operator $\delta_a f(x) = f(ax)$. Then

FTdilation

$$(1.4) \quad \widehat{\delta_a f}(\xi) = a^{-n} \widehat{f}(a^{-1} \xi).$$

(6) Let $g \in L^1(\mathbb{R}^n)$ and define the convolution $f * g(x) = \int f(x - y)g(y)dy$. Then

FTConvolution

$$(1.5) \quad \widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

(7) Let $g \in L^1(\mathbb{R}^n)$. Then

FTadjoint

$$(1.6) \quad \int \widehat{f}(y)g(y)dy = \int f(y)\widehat{g}(y)dy.$$

Exercise 1.2. Prove Theorem ^{L1FT}1.2.

Exercise 1.3. Calculate the Fourier transform of $f(x) = \chi_{(a,b)}(x)$ and $g(x) = e^{-a|x|^2}$. **Q:** If $f \in L^1(\mathbb{R}^n)$ is $\widehat{f} \in L^1(\mathbb{R}^n)$?

Fourier transform and differentiation

FTdiffthm

Theorem 1.3. Suppose $f \in L^1(\mathbb{R}^n)$ and $x_k f \in L^1(\mathbb{R}^n)$. Then \widehat{f} is differentiable w.r.t. ξ_k and

FTdiffFormula

$$(1.7) \quad \partial_{\xi_k} \widehat{f}(\xi) = (-2\pi i x_k \widehat{f})(\xi).$$

The proof is direct.

Definition 1.4. $f \in L^p(\mathbb{R}^n)$ is differentiable in $L^p(\mathbb{R}^n)$ w.r.t. k th variable if $\exists g \in L^p(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \left| \frac{f(x + he_k) - f(x)}{h} - g(x) \right|^p dx \rightarrow 0 \text{ as } h \rightarrow 0$$

where e_k is the k th coordinate unit vector. If g exists, it is unique, and is called the partial derivative of f w.r.t. k th variable in L^p .

FTDerivConv

Theorem 1.5. Let $f \in L^1(\mathbb{R}^n)$ and g be its partial derivative w.r.t. k th variable in L^1 . Then

$$\widehat{g}(\xi) = 2\pi i \xi_k \widehat{f}(\xi).$$

Exercise 1.4. Prove Theorem ^{FTDerivConv} 1.5.

Remark 1.6. If P is a polynomial in n variables and $P(D)$ denotes the associated operator then

$$(\widehat{P(D)f})(\xi) = P(2\pi i \xi) \widehat{f}(\xi).$$

Recovery of a function from its Fourier transform

Suppose we are given \widehat{f} . Can we find f ? The natural guess is to use the formula

$$f(x) = \int \mathbb{R}^n \widehat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi.$$

But, examples show that $\widehat{f} \notin L^1$ for certain $f \in L^1$ so the integral is not absolutely convergent. What to do? Reinterpret the integral...

FTInversionTh

Theorem 1.7 (Fourier Inversion). Let $f \in L^1(\mathbb{R}^n)$. Then we have the inversion formula

FTInversion

$$(1.8) \quad f(x) = \lim_{\theta \rightarrow 0} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} e^{-4\pi^2 \theta |\xi|^2} \widehat{f}(\xi) d\xi$$

where the limit is taken in the L^1 -norm. Moreover, if f is cts. at x_0 then

FTpw

$$(1.9) \quad f(x_0) = \lim_{\theta \rightarrow 0} \int_{\mathbb{R}^n} e^{2\pi i x_0 \cdot \xi} e^{-4\pi^2 \theta |\xi|^2} \widehat{f}(\xi) d\xi.$$

Fourier transform on L^2

We have a theory of the Fourier transform on $L^1(\mathbb{R}^n)$ which we now extend to $L^2(\mathbb{R}^n)$.

Plancherel

Theorem 1.8 (Plancherel). Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\widehat{f} \in L^2(\mathbb{R}^n)$ and $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$.

Proof. ¹ Let's define $g(x) = \overline{\widehat{f}(-x)}$. We have

$$\begin{aligned} \|\widehat{f}\|_{L^2}^2 &= \int \widehat{f}(\xi) \overline{\widehat{f}(\xi)} d\xi = \int \widehat{f}(\xi) \widehat{g}(\xi) d\xi \\ &= \int (\widehat{f * g})(\xi) d\xi = (f * g)(0) \\ &= \int f(x) g(0 - x) dx = \int f(x) \overline{\widehat{f}(x)} dx = \|f\|_{L^2}^2. \end{aligned}$$

□

¹The equalities in the second line of the proof require some justification based upon Young's inequality for Convolutions (1.11) and (1.9).

We have shown that the Fourier transform defines a bounded linear map $L^1 \cap L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ and is an isometry in the L^2 norm. Thus, \exists a unique bounded extension defined on all of $L^2(\mathbb{R}^n)$ which we define by \mathcal{F} . $\forall f \in L^2(\mathbb{R}^n)$ we shall write $\mathcal{F}[f] = \widehat{f}$. The Fourier transform defines a unitary operator on $L^2(\mathbb{R}^n)$.

Oscillatory Integrals/Stationary Phase in one dimension

Q: What is the asymptotic behavior of $I(\lambda)$ when $\lambda \rightarrow \infty$ where

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)} f(x) dx$$

and ϕ is a smooth \mathbb{R} -valued “phase” function and f is a smooth \mathbb{C} -valued function? Locations x_0 where $\phi'(x_0) = 0$ dominate the large λ asymptotics for $I(\lambda)$. Where $\phi' \neq 0$, the phase is “nonstationary” and the oscillating exponential shreds f .

vdc

Lemma 1.9 (Van der Corput). *Let $k \in \mathbb{Z}^+$ and $|\phi^{(k)}(x)| \geq 1 \forall x \in [a, b]$ with $\phi'(x)$ monotone in case $k = 1$. Then*

vandercorput

$$(1.10) \quad \left| \int_a^b e^{i\lambda\phi(x)} f(x) dx \right| \leq c_k \lambda^{-1/k} (\|f\|_{L^\infty} + \|f\|_{L^1})$$

with c_k independent of a, b .

Exercise 1.5. *Prove Lemma 1.9. Hint: $\frac{1}{i\lambda\phi'(x)} \left(\frac{d}{dx} e^{i\lambda\phi(x)} \right) = e^{i\lambda\phi(x)}$ and integrate by parts.*

1.2. Some interpolation theory. Riesz Interpolation Theorem; Marcinkiewicz Interpolation Theorem; Stein’s Interpolation Theorem; Young’s inequality

threelines

Lemma 1.10 (Three lines theorem). *Let F be a continuous bounded function defined on $S = \{z = x + iy : 0 \leq x \leq 1\}$ which is also analytic in the interior of S . If $\forall y \in \mathbb{R} |F(iy)| \leq M_0$ and $|F(1 + iy)| \leq M_1$ then $\forall z = x + iy \in S$*

$$|F(x + iy)| \leq M_0^{1-x} M_1^x.$$

Proof. (chalkboard...) □

Notation: If $T : L^p(X) \mapsto L^q(Y)$ is a linear operator which is continuous or bounded then the number

$$\sup_{f \neq 0} \frac{\|Tf\|_{L^q(Y)}}{\|f\|_{L^p(X)}} < \infty$$

is called the norm of the operator T .

RieszThorinThm

Theorem 1.11 (Riesz-Thorin interpolation). *Let $p_0 \neq p_1$, $q_0 \neq q_1$. Let T be a linear operator such that*

$$T : L^{p_0}(X, \mu) \mapsto L^{q_0}(Y, \nu) \text{ bounded w. norm } M_0$$

$$T : L^{p_1}(X, \mu) \mapsto L^{q_1}(Y, \nu) \text{ bounded w. norm } M_1.$$

Then

$$T : L^{p_\theta}(X, \mu) \mapsto L^{q_\theta}(Y, \nu) \text{ bounded w. norm } M_\theta$$

where

$$M_\theta \leq M_0^{1-\theta} M_1^\theta$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}; \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

Proof. (...chalkboard) □

Having shaken the tree, we pick up the fruit.

Theorem 1.12 (Young's inequality for convolutions). *Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$; $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} \geq 1$. Then $f * g \in L^r(\mathbb{R}^n)$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Moreover,*

Young (1.11)
$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof. For given g , we define the operator $Tf(x) = \int_{\mathbb{R}^n} g(x-y)f(y)dy$. By Minkowski's integral inequality

$$\|Tf\|_{L^q} \leq \|g\|_{L^q} \|f\|_{L^1}.$$

By Hölder's inequality,

$$\|Tf\|_{L^\infty} \leq \|g\|_{L^q} \|f\|_{L^{q'}}.$$

Now use the Riesz-Thorin interpolation inequality to finish the proof. □

Theorem 1.13 (Hausdorff-Young inequality). *Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. Then $\widehat{f} \in L^{p'}(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and*

HY (1.12)
$$\|\widehat{f}\|_{L^{p'}} \leq \|f\|_{L^p}.$$

Proof. By the definition ^(ft. defined) (1.1) and Hölder's inequality $\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$. (In fact we know that $\widehat{f} \in C_0(\mathbb{R}^n)$ when $f \in L^1$ so the Fourier image of L^1 is actually a very small subspace of L^∞ .) By Plancherel ^(Plancherel) (1.8), $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$. Now interpolate using Riesz-Thorin. □

Marcinkiewicz Interpolation Theorem (Diagonal Case)

Let (X, μ) be a measure space.

Definition 1.14. \forall measurable $f : X \mapsto \mathbb{C}$ we define the distribution function

$$m[f](\lambda) = \mu\{x \in X : |f(x)| > \lambda\} = \mu(E_f^\lambda).$$

As a function of $\lambda \in [0, \infty]$, $m[f](\lambda)$ is

- well-defined.
- takes values in $[0, \infty]$.
- nonincreasing.
- continuous from the right.

Lemma 1.15. $\forall f : X \mapsto \mathbb{C}$ and $\forall \lambda \geq 0$

(1) (Tscheychev)

$$m[f](\lambda) \leq \lambda^{-p} \int_{E_f^\lambda} |f(x)|^p dx \leq \lambda^{-p} \|f\|_{L^p}^p.$$

(2) If $1 \leq p < \infty$

$$\|f\|_{L^p}^p = - \int_0^\infty \lambda^p d\{m[f](\lambda)\} = p \int_0^\infty \lambda^{p-1} m[f](\lambda) d\lambda.$$

If $p = \infty$

$$\|f\|_{L^\infty} = \inf\{\lambda : m[f](\lambda) = 0\}.$$

(3)

$$m[f+g](\lambda) \leq m[f](\lambda/2) + m[g](\lambda/2).$$

Definition 1.16. $\forall 1 \leq p < \infty$ we define

$$L^{p^*}(X, \mu) = \{f : X \mapsto \mathbb{C} \mid \|f\|_{L^{p^*}}^* = \sup_{\lambda > 0} \lambda(m[f](\lambda))^{1/p} < \infty\}.$$

Lemma 1.17. (1) $L^{p^*} \subset L^p$; $L^{p^*} \neq L^p$.

$$(2) \|f + g\|_{L^{p^*}}^* \leq 2(\|f\|_{L^{p^*}}^* + \|g\|_{L^{p^*}}^*).$$

Definition 1.18. Let $M(X)$ be the space of \mathbb{C} -valued measurable functions on X . A linear or sublinear operator $T : L^p(X) \mapsto M(X)$ (with $1 \leq p < \infty$) is weak type (p, q) if $\exists c > 0$ such that $\forall f \in L^p(X)$

$$\|Tf\|_{L^q}^* \leq C\|f\|_{L^p}.$$

T is strong type (p, q) if $\exists c > 0$ such that $\forall f \in L^p$

$$\|Tf\|_{L^q} \leq c\|f\|_{L^p}.$$

MarcinkiewiczThm

Theorem 1.19 (Marcinkiewicz interpolation theorem). Let $1 < r \leq \infty$ and $T : L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n) \mapsto M(\mathbb{R}^n)$ be a quasilinear operator (i.e. $|T(f+g)| \leq |T(f)| + |T(g)|$). If T is weak type $(1, 1)$ and weak type (r, r) then $\forall p \in (1, r)$, T is strong type (p, p) .

Thus, if we can prove two “easier” weak type inequalities we obtain a whole collection of strong type inequalities!

Proof. (...chalkboard) □

1.3. Fractional integrals. Hardy-Littlewood maximal function, Hardy-Littlewood-Sobolev Inequality

Hardy-Littlewood maximal function

Notation:

$$L^1_{loc}(\mathbb{R}^n) = \{f : \mathbb{R}^n \mapsto \mathbb{C} \mid \int_K |f(x)| dx < \infty \forall K \subset\subset \mathbb{R}^n\}.$$

$$B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\|_{\mathbb{R}^n} < r\}$$

$$\omega_n = |B_1(0)|.$$

Definition 1.20. $\forall f \in L^1_{loc}(\mathbb{R}^n)$ we define the Hardy-Littlewood Maximal function associated to f as

$$\begin{aligned} \mathcal{M}[f](x) &= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy \\ &= \sup_{r>0} \frac{1}{\omega_n} \int_{B_1(0)} |f(x - ry)| dy \\ &= \sup_{r>0} (f * \frac{1}{|B_r(0)|} \chi_{B_r(0)})(x) \end{aligned}$$

Exercise 1.6. Prove the following statements.

- (1) \mathcal{M} is a quasilinear operator: $|\mathcal{M}(f+g)(x)| \leq |\mathcal{M}(f)(x)| + |\mathcal{M}(g)(x)|$.
- (2) If $f \in L^\infty(\mathbb{R}^n)$ then $\|\mathcal{M}f\|_{L^\infty} \leq \|f\|_{L^\infty}$. (Thus, \mathcal{M} is strong type (∞, ∞) . We will soon show that \mathcal{M} is weak type $(1, 1)$.)

Lemma 1.21 (Vitali covering lemma). *Let $E \subset \mathbb{R}^n$ be a measurable set such that $E \subset \cup_{\alpha} B_{r_{\alpha}}(x_{\alpha})$ with the family of open balls $\{B_{r_{\alpha}}(x_{\alpha})\}_{\alpha}$ satisfying $\sup_{\alpha} r_{\alpha} = c_0 < \infty$. Then \exists a countable subfamily $\{B_{r_j}(x_j)\}_j$ of disjoint balls such that*

$$|E| \leq 5^n \sum_{j=1}^{\infty} |B_{r_j}(x_j)|.$$

Proof. (...chalkboard) □

Theorem 1.22 (Hardy-Littlewood maximal function estimates). *Let $1 < p \leq \infty$. \mathcal{M} is a quasilinear strong type (p, p) operator:*

HLeSt

$$(1.13) \quad \|\mathcal{M}f\|_{L^p} \leq c_p \|f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^n).$$

Proof. It suffices to show that \mathcal{M} is weak type $(1, 1)$. So, we want to show \exists constant c_1 such that $\forall f \in L^1(\mathbb{R}^n)$

$$\sup_{\lambda > 0} \lambda m[\mathcal{M}f](\lambda) \leq c_1 \|f\|_{L^1}.$$

Define $\forall \lambda > 0$

$$E_f^{\lambda} = \{x \in \mathbb{R}^n \mid \mathcal{M}f(x) > \lambda\}.$$

If $x \in E_f^{\lambda} \exists B_{r_x}(x)$ such that

$$\int_{B_{r_x}(x)} |f(y)| dy > \lambda |B_{r_x}(x)|.$$

Clearly, $E_f^{\lambda} \subset \cup_{x \in E_f^{\lambda}} B_{r_x}(x)$. By the Vitali covering lemma \exists countable disjoint subfamily $\{B_{r_{x_j}}(x_j)\}_{j \in \mathbb{Z}^+}$ such that

$$\begin{aligned} |E_f^{\lambda}| &\leq 5^n \sum_{j=1}^{\infty} |B_{r_{x_j}}(x_j)| \\ &\leq 5^n \lambda^{-1} \int_{B_{r_j}(x_j)} |f(y)| dy \\ &\leq 5^n \lambda^{-1} \|f\|_{L^1} \end{aligned}$$

which was to be shown. □

Lemma 1.23. *Let $\varphi \in L^1(\mathbb{R}^n)$ be radial, positive, nonincreasing function of $r = |x| \in [0, \infty)$. Then*

radialLoneKernel

$$(1.14) \quad \sup_{t > 0} |\phi_t * f(x)| = \sup_{t > 0} \left| \int_{\mathbb{R}^n} \frac{\varphi(t^{-1}(x-y))}{t^n} f(y) dy \right| \leq \|\varphi\|_{L^1} \mathcal{M}f(x).$$

Proof. (...chalkboard) □

Riesz Potentials/Fractional Integrals

Recall that a solution of $\Delta u = f$ may be constructed by convolving f with the Newtonian potential: On \mathbb{R}^n , $n \geq 3$,

$$u(x) = c_n \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} f(y) dy.$$

This formula gives some meaning to the notation $u = \Delta^{-1}f$. The Riesz potentials generalize this expression.

Definition 1.24. Let $0 < \alpha < n$. The Riesz potential of order α , denoted by I_α , is defined as

$$I_\alpha f(x) = c_\alpha \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = c_\alpha k_\alpha * f(x)$$

where $c_\alpha = \pi^{n/2} 2^\alpha \Gamma(\alpha/2) / \Gamma(n/2 - \alpha/2)$ is a normalizing constant.

HLSTheorem

Theorem 1.25 (Hardy-Littlewood-Sobolev fractional integration theorem). Let $0 < \alpha < n$, $1 \leq p < q < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

- (1) If $f \in L^p(\mathbb{R}^n)$ then the integral defining $I_\alpha f$ is absolutely convergent for a.e. $x \in \mathbb{R}^n$.
- (2) If $p > 1$, I_α is strong type (p, q) :

HLS

$$(1.15) \quad \|I_\alpha f\|_{L^q} \leq C_{p,\alpha,n} \|f\|_{L^p}.$$

Remark 1.26. This would follow from Young's inequality for convolutions ([Young \(I.11\)](#)) if $k_\alpha(x) = c_\alpha |x|^{\alpha-n} \in L^{n/(n-\alpha)}(\mathbb{R}^n)$. But this (barely) fails to be the case. The Hardy-Littlewood-Sobolev theorem may thus be viewed as an endpoint refinement of Young's inequality when one of the convolution factors has a specific form.

Remark 1.27. group property...

Proof of Theorem [1.25](#). ([...chalkboard](#)) □

1.4. Littlewood-Paley theory. frequency localized intuition and tricks

Littlewood-Paley theory is a toolbox for quantifying smoothness and L^p -integrability properties of functions. A key idea is the decomposition of \hat{f} w.r.t. centered dyadic annuli $\{\xi \in \mathbb{R}^n : |\xi| \sim 2^k\}$.

We closely follow [\[47\]](#).

Annular Decomposition of $\mathbb{R}^n \setminus \{0\}$.

Let ϕ be a smooth radial bump function supported on $\{|x| < 2\}$ which satisfies $\phi(\xi) = 1 \forall |\xi| \leq 1$. Let $\psi(\xi) = \phi(\xi) - \phi(2\xi)$. Then $\text{spt } \psi \subset \{\frac{1}{2} < |\xi| < 1\} := A_1 = \{|\xi| \sim 1\}$.

Exercise 1.7. Prove that

lpdecomp

$$(1.16) \quad \sum_{k \in \mathbb{Z}} \psi(\xi/2^k) = 1 \quad \forall \xi \neq 0.$$

We have a partition of unity w.r.t. $\{A_k\}_{k \in \mathbb{Z}}$ where $A_k = \{|\xi| \sim 2^k\} = \{2^{k-1} < |\xi| < 2^{k+1}\}$.

Littlewood-Paley projection operators

We define the projection operators

Pkdef

$$(1.17) \quad \widehat{P}_k f(\xi) = \psi(\xi/2^k) \widehat{f}(\xi),$$

Pleqkdef

$$(1.18) \quad \widehat{P}_{\leq k} f(\xi) = \phi(\xi/2^k) \widehat{f}(\xi).$$

We have that $P_k = P_{\leq k} - P_{< k}$. If $f \in L^2$ then $P_{\leq k} f \xrightarrow{L^2} 0$ as $k \rightarrow -\infty$ and $P_{\leq k} f \xrightarrow{L^2} f$ as $k \rightarrow \infty$.

The Littlewood-Paley decomposition of f is

$$f = \sum_{k \in \mathbb{Z}} P_k f$$

in the, say, L^2 limit sense. The function f is thus represented as a countable sum of essentially constant frequency pieces.

Properties of $P_k f, P_{\leq k} f$

$P_k f, P_{\leq k} f$ are presented as Fourier multiplier operators with multipliers given as a 2^k -related-dilate of ψ, ϕ , respectively. There are corresponding convolution representations, e.g.

$$\boxed{\text{Pleqkconv}} \quad (1.19) \quad P_{\leq k} f(x) = (f * 2^{nk} \widehat{\phi}(2^k \cdot))(x) = \int f(x + 2^{-k} y) \widehat{\phi}(y) dy.$$

Note that $\int \widehat{\phi}(y) dy = \phi(0) = 1$ so $P_{\leq k}$ is an average of f on scales $\lesssim 2^{-k}$. The convolution representation makes $P_{\leq k} f$ estimable using Young's inequality ^{Young} (I.II).

Since $P_k f = P_{\leq k+2} P_k f$ we have the reproducing formula

$$\boxed{\text{reproduce}} \quad (1.20) \quad P_k f(x) = \int P_k f(x + 2^{-k-2} y) \widehat{\phi}(y) dy$$

$\implies P_k f$ is essentially constant at physical scales $\ll 2^{-k}$. Also, $P_{\leq k-2} P_k f = 0$ so

$$\int P_k f(x + 2^{-k+2} y) \widehat{\phi}(y) dy = 0 \quad \forall x \in \mathbb{R}^n$$

$\implies P_k f$ has zero mean at scales $\lesssim 2^{-k+2}$. So, the oscillatory behavior of $P_k f$ is constrained: on each ball of radius 2^{-k} , $P_k f$ is smooth at scales $\ll 2^{-k}$ and has ~ 1 oscillation.

bernsteinlemma

Lemma 1.28 (Bernstein's inequality). *If $1 \leq p \leq q \leq \infty$ and f has frequency support in an interval I (spt $f \subset I$) then*

$$\boxed{\text{bernstein}} \quad (1.21) \quad \|f\|_{L^q} \lesssim |I|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^p}.$$

Proof. The estimate is scaling consistent. Let $\phi_I \in C_0^\infty$ with $\phi_I(\xi) = 1 \quad \forall \xi \in I$, spt $\phi_I \subset \tilde{I}$ with \tilde{I} a double of I . Then

$$f(x) = \mathcal{F}^{-1}(\phi_I(\cdot) \widehat{f}(\cdot))(\xi) = (f * \mathcal{F}^{-1}(\phi_I))(x).$$

Young's inequality for convolutions ^{Young} (I.II) gives

$$\|f\|_{L^q} \leq \|f\|_{L^p} \|\mathcal{F}^{-1}(\phi_I)\|_{L^r} \quad \text{if } \frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1.$$

Finally, note that

$$\|\mathcal{F}^{-1}(\phi_I)\|_{L^r} \sim |I|^{\frac{1}{p} - \frac{1}{q}}$$

which proves the claim. □

Specializing I in ^{bernstein} (I.21) to the A_k gives for $1 \leq p \leq q \leq \infty$

$$\boxed{\text{Pkbernstein}} \quad (1.22) \quad \|P_k f\|_{L^q} \lesssim 2^{nk(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p},$$

Pleqkbernstein

$$(1.23) \quad \|P_{\leq k} f\|_{L^q} \lesssim 2^{nk(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p}.$$

Thus, higher L^q norms of $P_k f$ are controlled by lower L^p norms with k -dependent factors.

Remark 1.29. Intuitively, $P_k f$ is a function “essentially constant” on scales $\sim 2^{-k}$. Thus, the L^∞ norm can't be wildly large without be wildly large all across a set with diameter of scale 2^{-k} . So, we have the estimate $\|P_k\|_{L^\infty} \leq 2^{kn} \|P_k\|_{L^1}$. With this intuition, we also see that the worst case of Bernstein's inequality involves a function as localized-in- x as allowed by P_k . If $P_k f$ is spread out on a set much larger than scale 2^{-k} then, since the L^1 norm feels the size of the support, the right side grows and Bernstein's estimate is not sharp.

Derivatives and Littlewood-Paley pieces

Roughly, $\nabla P_k f \sim 2^k P_k f$.

Lemma 1.30. Let $k \in \mathbb{Z}$, f such that $\text{spt } \widehat{f} \subset \{2^{k-1} < |\xi| < 2^{k+1}\}$ then

$$\boxed{\text{PkderivLp}} \quad (1.24) \quad \|\nabla f\|_{L^p} \sim 2^p \|f\|_{L^p} \quad \forall 1 \leq p \leq \infty.$$

Proof. (...chalk...) □

Estimates relating $P_k f$ and f

We have

$$\|P_{\leq k} f\|_{L_x^p} = \left\| \int f(x + 2^{-k} y) \widehat{\phi}(y) dy \right\|_{L_x^p} \\ \lesssim \|f\|_{L^p}.$$

Similarly,

$$\|P_k f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Since $f = \sum_k P_k f$, we have the cheap Littlewood-Paley inequality

$$\boxed{\text{cheaplp}} \quad (1.25) \quad \sup_k \|P_k f\|_{L^p} \lesssim \|f\|_{L^p} \lesssim \sum_k \|P_k f\|_{L^p}.$$

There is also a more refined statement....

Definition 1.31. The Littlewood-Paley square function, denoted $|Sf|$, is defined

$$\boxed{\text{Sf}} \quad (1.26) \quad |Sf(x)| = \left(\sum_k |P_k f(x)|^2 \right)^{1/2}$$

Theorem 1.32 (Littlewood-Paley square function estimate). $\forall 1 < p < \infty$

$$\boxed{\text{lpsf}} \quad \boxed{\text{lpsfest}} \quad (1.27) \quad \| |Sf| \|_{L_x^p} \sim \|f\|_{L_x^p}.$$

We will omit the proof.

1.5. Sobolev spaces and embeddings. We turn our attention to developing the two parameter scale of Sobolev spaces $W^{s,p}$.

“The” Sobolev inequality

Nonendpoint Sobolev embedding

Endpoint Sobolev embedding

2. CONSERVATION AND SYMMETRY PROPERTIES

We closely follow parts of [SulemSulem \[45\]](#) and consulted [EvanEvansNLW \[?\], \[?\]](#).

Lagrangian formulation of NLS; Hamiltonian formulation of NLS; symplectic structure on L^2 for NLS

Noether's theorem; conservation of mass/energy/momentum; pseudoconformal transformation; Mass-Virial identity; Energy-Virial identity

2.1. Classical Mechanics. Suppose we are given $L : \mathbb{R}_y^n \times \mathbb{R}_y^n \times I \mapsto \mathbb{R}$ where I is an interval in \mathbb{R} . Assume that L is smooth and convex with respect to the first slot \dot{y} . The function L is called the *Lagrangian*. Form the *Action Functional*

$$I[w] = \int_{t_0}^{t_1} L(\dot{w}(t), w(t), t) dt$$

for $w \in \mathcal{A} = \{\text{competitors subject to some boundary conditions}\}$.

Hamilton's Principle: *Motions of a mechanical system governed by the Lagrangian L coincide with extremals of the Action Functional $I[\cdot]$.*

By the calculus of variations, extremizers of $I[\cdot]$ solve the *Euler-Lagrange equations*

EulerLagrangeEquations

$$(2.1) \quad -\frac{d}{dt}(L_{\dot{y}}(\dot{y}(t), y(t), t) + L_y(\dot{y}(t), y(t), t)) = 0.$$

This is a system of n second order equations in $y(t)$.

Define $p(t) = L_{\dot{y}}(\dot{y}(t), y(t), t)$ and $H(p(t), y(t), t) = \sup_{\dot{y}} [p(t) \cdot \dot{y} - L(\dot{y}, y(t), t)]$.

Notice that H is the *Legendre transform* of L . Then, the Euler-Lagrange equations (2.1) can be rewritten

HamiltonsEquations

$$(2.2) \quad \begin{cases} \dot{p} = -\frac{\partial H}{\partial y}, \\ \dot{y} = \frac{\partial H}{\partial p} \end{cases}$$

This is a system of $2n$ first order equations in $(p(t), y(t))$. The equations (2.2) are called *Hamilton's Equations*.

Remarks

- Mechanical motions are *canonically* described as a consequence of a variational principle.
- Hamilton's equations are special among a general class of equations.

$$\begin{cases} \dot{x} = F(x, y), \\ \dot{y} = G(x, y). \end{cases}$$

In particular, the form of Hamilton's equations implies phase space volume is conserved under the dynamics. Under certain conditions, this property implies the mechanical motion is recurrent (Poincaré recurrence).

Suppose we write $w = (p, y)$, $w \in \mathbb{R}^{2n}$. We rewrite Hamilton's equations by writing $H_z = (H_p, H_y)$ and defining the matrix

Jmatrix

$$(2.3) \quad \mathbb{J} = \begin{bmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{bmatrix}.$$

With these notations, Hamilton's equations (2.2) may be written

complexHamiltonsEqns

$$(2.4) \quad \dot{w} = \mathbb{J}H_w.$$

Note that J is essentially a rotation matrix and in this notation we can interpret the dynamics of Hamilton's equations as "move perpindicular to the gradient of H ". Notice also that $\mathbb{J}^2 = -\mathbb{I}$ and Hamilton's equations are defined on an *even* dimensional space. This suggests there may be a nice connection with complex numbers.

Complex Notation

For $(x, y) \in \mathbb{R}^2$, define $z = x + iy$, $\bar{z} = x - iy$, $i^2 = -1$. Notice that z, \bar{z} give x, y and vice-versa. Define $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Observe that $\partial_{z\bar{z}} = (\partial_x + i\partial_y)(x + iy) = 0$. If $f(x, y)$ and $g(x, y)$ are two \mathbb{R} -valued functions,

we can form $u(x, y) = f(x, y) + ig(x, y)$. Then, u is *complex-analytic* if, upon reexpressing x and y via z, \bar{z} , we see that $\partial_{\bar{z}}u = 0$. Analytic functions depend only upon z and not \bar{z} (or vice versa).

Now, suppose we write $z_1 = y_1 + ip_1, z_2 = y_2 + ip_2, \dots$. Reexpress $H(p, y) = H(z, \bar{z})$. Consider the following system of n equations in the complex variables $z_j(t)$ given by

$$(2.5) \quad \dot{z}_j = iH_{\bar{z}_j}.$$

Calculating \dot{z}_j and taking the $\partial_{\bar{z}_j}$ derivative shows this system is equivalent with [\(2.2\)](#).

Example 2.1. *Let*

$$H = H(p, y) = \frac{1}{2}|p|^2 + \frac{1}{2}|y|^2 = \frac{1}{2} \sum_j p_j^2 + y_j^2.$$

We can rewrite this using the notation above as

$$H(z, \bar{z}) = |z|^2 = \sum_j |z_j|^2 = \sum_j z_j \bar{z}_j.$$

Hamilton's Equations then read

$$\begin{cases} \dot{p}_j = -y_j = -\frac{\partial H}{\partial y_j}, \\ \dot{y}_j = p_j = \frac{\partial H}{\partial p_j} \end{cases}$$

and can be rewritten

$$\dot{z}_j = iH_{\bar{z}_j} = iz_j.$$

This says the dynamics moves the complex components z_j of the complex vector z “perpendicular to z_j with speed $|z_j|$.”

Example 2.2. *(slightly fancier)*

Change the Hamiltonian in the previous example by writing

$$H(z, \bar{z}) = \sum_j j^2 |z_j|^2 = \sum_j j^2 z_j \bar{z}_j.$$

Hamilton's equations are then

$$(2.6) \quad \dot{z}_j = ij^2 z_j$$

which says the complex number z_j moves perpendicular to itself with speed $j^2|z_j|$. The motion of each of the components of z moving independently of the other components; the motion of the components are uncoupled. In this example, the speed of motion also depends on the index j .

Example 2.3. *(NLS)*

Let $u : \mathbb{T} \times I \mapsto \mathbb{C}$ where \mathbb{T} is the 1d torus and I is a time interval. Form the Hamiltonian Functional

$$H = H[u, \bar{u}] = \int_{\mathbb{T}} |\nabla u|^2 = \int_{\mathbb{T}} \nabla u \nabla \bar{u} dx.$$

(Recall that this is just the classical Dirichlet energy.) The usual calculus of variations argument shows

$$\lim_{\tau \rightarrow 0} \frac{H[u, \bar{u} + \tau \bar{v}] - H[u, \bar{u}]}{\tau} = \int_{\mathbb{T}} (-\Delta u) \bar{v} dx = \langle -\Delta u, v \rangle.$$

Therefore, $H_{\bar{u}}[u, \bar{u}] = -\Delta u$ and Hamilton's equations in this case read

$$\dot{u} = iH_{\bar{u}} = -i\Delta u$$

which is precisely the linear Schrödinger equation!

The initial value problem

periodiclinearschro

$$(2.7) \quad \begin{cases} i\partial_t u + \Delta u = 0, & \mathbb{T} \times \{t > 0\} \\ u(x, 0) = \phi(x), & \mathbb{T} \times \{t = 0\}. \end{cases}$$

has solution

$$u(x, t) = \sum_k e^{ikx} e^{-i|k|^2 t} \widehat{\phi}(k).$$

We can reexpress the Hamiltonian in terms of the Fourier transform by writing

$$\begin{aligned} H[u, \bar{u}] &= \int_{\mathbb{T}} \nabla u \nabla \bar{u} dx \\ &= \int_{\mathbb{T}} \left(\sum_k e^{ikx} (ik) e^{-i|k|^2 t} \widehat{\phi}(k) \right) \overline{\left(\sum_{k'} e^{ik'x} (ik') e^{-i|k'|^2 t} \widehat{\phi}(k') \right)} dx \\ &= \sum_k |k|^2 |\widehat{\phi}|^2. \end{aligned}$$

So, Schrödinger's equation is an infinite dimensional Hamiltonian system in which each Fourier coefficient behaves like the solution of a Harmonic oscillator. Recall that the dispersion relation calculation showed that

$$\frac{d}{dt} \widehat{u}(t)(k) = i|k|^2 \widehat{u}(t)(k)$$

which is essentially the same as appeared in [\(2.6\)](#). see it again

2.2. Action symmetries and conservation laws: Noether's theorem. KdV and NLS have been derived as approximate models for various physical phenomena. From physical principles, we expect the KdV and NLS evolutions to take place leaving certain integral quantities, associated with mass, energy, momentum, etc. to be time invariant. Time-invariant quantities are said to be *conserved*. For example

$$\partial_t u + \partial_x^3 u + \frac{1}{2} \partial_x u^2 = 0,$$

can be multiplied by u and rewritten

$$\frac{1}{2} \partial_t u^2 + \partial_x (u u_{xx} - \frac{1}{2} u_x^2 + [\frac{1}{3} u^3]) = 0.$$

Upon integrating over all of $x \in \mathbb{R}$ (assuming that u and its derivatives decay as $|x| \rightarrow +\infty$) we learn

$$\partial_t \int u^2 dx = 0.$$

Therefore, the KdV evolution satisfies *conservation of the L^2 norm*

$$\|u(t)\|_{L_x^2} = \|u(0)\|_{L_x^2}.$$

This places a basic constraint on the dynamics: all the motion takes place on a sphere in L_x^2 . We are interested in finding other conserved quantities to constrain (and therefore better understand) the dynamics. However, the method used above to discover the L^2 conservation law for KdV requires a bit of cleverness that may not

be easily available for discovering more complicated conservation laws. A systematic approach to discovering some conservation laws is contained in: **E. Noether's Principle:** *If a variational principle is invariant under a family of transformations, then solutions of its Euler-Lagrange equation satisfy a conservation law.*

Recall that mechanical motions are characterized as extremizers of the action which is a variational principle. The present discussion borrows from [?], [?], [?]. Evans, Willem, CBMS

Let $L : \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}$, $L = L(p, w)$ and $u : \mathbb{R}^m \mapsto \mathbb{R}$. Let $\mathbb{U} \subset \mathbb{R}^m$. Form $I[u] = \int_{\mathbb{U}} L(\nabla u, u) dx$. The calculus of variations shows that u is a smooth critical point of $I[\cdot]$ if and only if the Euler-Lagrange equation

$$-\sum_{i=1}^m \partial_{x_i} (L_{p_i}(\nabla u, u)) + \partial_w L(\nabla u, u) = 0$$

is satisfied.

Noether's principle allows us to look for invariances of the variational principle instead of for the conservation laws. The former turn out to be easier to find than the latter.

Notation

The following notation is introduced to flexibly describe a one parameter family of transformations. Let $\mathbf{g} : \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}^m$, $w : \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}$, $\phi : \mathbb{R}^m \mapsto \mathbb{R}^m$. We define a *domain variation* $x' = \mathbf{g}(x, \tau) = x + \tau\phi(x) + o(\tau)$, $\phi = \frac{\partial g}{\partial \tau}|_{\tau=0}$. We define a *function or target variation* $u'(x') = w(x(x'), \tau) = u(x) + \tau v(x) + o(\tau)$, $v = \frac{\partial w}{\partial \tau}|_{\tau=0}$. Naturally $\mathbb{U}' = \mathbf{g}(\mathbb{U}, \tau)$.

The value of the domain and target varied action at parameter value τ is

$$i(\tau) = \int_{\mathbb{U}'} L(\nabla u', u') dx'.$$

We are interested in characterizing situations where $i(\cdot)$ is independent of its argument or when $i(\cdot)$ is infinitesimally invariant: $i'(0) = 0$.

Lemma 2.4.

noetherilemma (2.8)
$$i'(0) = \int_{\mathbb{U}} L_{p_i} [v_{x_i} - u_{x_j} \phi_{x_j}^i] + (\partial_w L)v + L\phi_{x_j}^j dx.$$

Proof. follow your nose and change variables. Expand out the jacobian determinant using divergence. □

Theorem 2.5. (Noether) *Given $L : \mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}$, $\mathbb{U} \subset \mathbb{R}^m$, $u : \mathbb{R}^m \mapsto \mathbb{R}$, we form the action functional $I[u] = \int_{\mathbb{U}} L(\nabla u, u) dx$. Suppose that u is a smooth critical point of the action functional $I[\cdot]$. Assume that $I[\cdot]$ is invariant under a family of transformations $u \mapsto u'$, $x \mapsto x'$ (so $\mathbb{U} \mapsto \mathbb{U}'$), for all regions \mathbb{U} ,*

invariantaction (2.9)
$$\int_{\mathbb{U}} L(\nabla u, u) dx = \int_{\mathbb{U}'} L(\nabla u', u') dx'.$$

Then,

realnoether (2.10)
$$\partial_{x_i} \left\{ \partial_{p_i} L(\nabla u, u) [v - u_{x_j} \phi_{x_j}^j] + L(\nabla u, u) \phi^i \right\} = 0.$$

Proof. It suffices to show that the integrand in the lemma may be rewritten as the divergence appearing in realnoether (2.10). This may be verified using the fact that u is a solution to the Euler-Lagrange equation. □

Remark 2.6. In case $u : \mathbb{R}^m \mapsto \mathbb{C}$, the conclusion changes to the statement

$$\boxed{\text{complexnoether}} \quad (2.11) \quad \partial_{x_i} \left\{ \frac{\partial L}{\partial(\partial_{x_i} u)} [v - u_{x_j} \phi^j] + \frac{\partial L}{\partial(\partial_{x_i} \bar{u})} [v - \bar{u}_{x_j} \phi^j] + L \phi^i \right\} = 0$$

2.3. Lagrangian Structure of NLS. The nonlinear Schrödinger dynamics may be described using a variational principle. To show this, we introduce a Lagrangian, form the associated action functional and then calculate NLS as the Euler-Lagrange equation characterizing smooth critical points of the action functional.

Consider the nonlinear Schrödinger equation

$$\boxed{\text{GNLS}} \quad (2.12) \quad i\psi_t + \Delta\psi = F'(|\psi|^2)\psi,$$

posed for $x \in \mathbb{R}^d$. We assume ψ and its derivatives are smooth and vanish as $|x| \rightarrow \infty$. The nonlinearity F' is a smooth function of its argument and we define

$$F(\lambda) = \int_0^\lambda F'(s) ds.$$

Define

$$\boxed{\text{gnslagrangian}} \quad (2.13) \quad L = \frac{i}{2} (\bar{\psi} \psi_t - \psi \bar{\psi}_t) - [|\nabla\psi|^2 - F(|\psi|^2)].$$

Evidently, $L = L(\psi, \bar{\psi}, \psi_t, \bar{\psi}_t, \nabla\psi, \nabla\bar{\psi})$. Form the action functional

$$\boxed{\text{gnlsaction}} \quad (2.14) \quad I[\psi] = \int_{t_0}^{t_1} \int_{\mathbb{R}^d} L \, dx dt,$$

defined for $\psi \in A$, some appropriate class of admissible functions.

The usual calculus of variations argument shows that if ψ is a smooth critical point of $I[\cdot]$ then ψ satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial\psi} = \sum_{i=1}^n \frac{\partial}{\partial x_i} \frac{\partial L}{\partial(\partial_{x_i} \psi)} + \frac{\partial}{\partial t} \frac{\partial L}{\partial(\partial_t \psi)}.$$

We calculate the various terms using L to find (the complex conjugate of) NLS

$$-i\bar{\psi}_t + \Delta\bar{\psi} = f(|\psi|^2)\bar{\psi}.$$

Noether's theorem applied to NLS

We write $\mathbb{R}_y^m = \mathbb{R}_t^1 \times \mathbb{R}_x^d$, distinguishing the time variable t from the spatial variables x . A one parameter family of transformations is defined when we specify ϕ^0 , ϕ and v

$$\boxed{\text{transf}} \quad (2.15) \quad \begin{cases} t \mapsto t' = t + \tau\phi^0(x, t, \psi), \\ x \mapsto x' = x + \tau\phi(x, t, \psi), \\ \psi(x, t) \mapsto \psi'(x', t') = \psi(x, t) + \tau v(x, t). \end{cases}$$

Suppose that the $\boxed{\text{transf}}$ action in (infinitesimally) invariant under the family of transformations (2.15). Noether's theorem implies that (2.11) holds. Recalling the distinguished time variable, if we integrate over the spatial domain, we obtain the conservation law

$$\partial_t \int_{\mathbb{R}^d} \left\{ \frac{\partial L}{\partial(\partial_t \psi)} [v - \psi_t \phi^0 - \nabla_x \psi \cdot \phi] + \frac{\partial L}{\partial(\partial_t \bar{\psi})} [v - \bar{\psi}_t \phi^0 - \nabla_x \bar{\psi} \cdot \phi] + L \phi^0 \right\} dx = 0.$$

We apply this formalism to identify certain invariances of the NLS action functional and thereby infer certain conservation laws for NLS.

Invariance by phase shift/ $U(1)$ gauge invariance

Consider the transformation $\psi \mapsto \tilde{\psi} = e^{i\tau}\psi$. For tiny τ , this is equivalent to

$$(2.16) \quad \begin{cases} t \mapsto \tilde{t} = t, \\ x \mapsto \tilde{x} = x, \\ \psi \mapsto \tilde{\psi} = \psi + i\tau\psi. \end{cases}$$

This transformation leaves the Lagrangian invariant and therefore the action functional is also invariant so Noether's theorem applies. Comparing with (2.15)^{transf}, we see that $\phi^0 = 0$, $\phi = 0$, $v = i\psi$. Recalling the Lagrangian and calculating

$$\frac{\partial L}{\partial(\partial_t\psi)} = \frac{i}{2}\bar{\psi}, \quad \frac{\partial L}{\partial(\partial_t\bar{\psi})} = \frac{i}{2}\psi,$$

we find that

$$\int_{\mathbb{R}^d} \left(\frac{i}{2}\bar{\psi} \right) (i\psi) + \left(-\frac{i}{2}\psi \right) (-i\bar{\psi}) dx = - \int_{\mathbb{R}^d} |\psi|^2 dx,$$

is conserved. Define

$$\boxed{\text{LTwoMass}} \quad (2.17) \quad N = \int_{\mathbb{R}^d} |\psi|^2 dx.$$

The conserved quantity N represents the mass, wave action, plasmon number, or wave power in various applications of NLS as a model equation. Note that the conservation law prior to spatial integration related to this invariance is

$$\boxed{\text{LtwoTransport}} \quad (2.18) \quad \partial_t |\psi|^2 + \nabla_x \cdot \{i(\psi \nabla_x \bar{\psi} - \bar{\psi} \nabla_x \psi)\} = 0.$$

The quantity $\{i(\psi \nabla_x \bar{\psi} - \bar{\psi} \nabla_x \psi)\}$ may then be interpreted as a current.

The invariance by phase shift is sometimes referred to as *gauge invariance*.

Invariance by time translation

We define a transformation

$$(2.19) \quad \begin{cases} t \mapsto t' = t + \tau\phi^0, \\ x \mapsto x' = x, \\ \psi \mapsto \psi' = \psi. \end{cases}$$

This transformation leaves the Lagrangian and, hence, the action functional invariant. In the notation of (2.15)^{transf}, we have $\phi = 0$, $v = 0$, $\phi^0 \neq 0$. Noether's theorem implies the time invariance of

$$\int_{\mathbb{R}^d} \left\{ \frac{i}{2}\bar{\psi} [-\psi_t\phi^0] + \left(-\frac{i}{2}\right) [-\bar{\psi}_t\phi^0] + L\phi^0 \right\} dx.$$

Substituting L from (2.13)^{gnlsLagrangian} reveals that

$$\boxed{\text{gnlsHamiltonian}} \quad (2.20) \quad H = \int_{\mathbb{R}^d} |\nabla\psi|^2 + F(|\psi|^2) dx$$

is conserved. This quantity is the *Hamiltonian* for the NLS equation. It represents the energy in various applications.

Invariance by space translation

We define a transformation

$$(2.21) \quad \begin{cases} t \mapsto t' = t, \\ x \mapsto x' = x + \tau b, \\ \psi \mapsto \psi' = \psi. \end{cases}$$

This transformation leaves L invariant so the associated action functional is also invariant. Here $\phi = b$, $\phi^0 = 0$, $v = 0$. Noether's theorem implies

$$\int_{\mathbb{R}^d} \frac{i}{2} \bar{\psi} [-\nabla_x \psi \cdot b] + \left(-\frac{i}{2} \psi\right) [-\nabla_x \bar{\psi} \cdot b] dx$$

is conserved. Therefore, $\frac{i}{2} \int_{\mathbb{R}^d} [\psi \nabla_x \bar{\psi} - \bar{\psi} \nabla_x \psi] dx \cdot b$ is conserved. Since b was arbitrary, we have that the *linear momentum* of solutions of NLS

linearmomentum

$$(2.22) \quad \mathbf{P} = i \int_{\mathbb{R}^d} [\psi \nabla_x \bar{\psi} - \bar{\psi} \nabla_x \psi] dx$$

is conserved.

Exercise 2.1. (Galilean Invariance) Consider the transformation

$$(2.23) \quad \begin{cases} t \mapsto t' = t, \\ x \mapsto x' = x - \mathbf{c}t, \\ \psi(x, t) \mapsto \psi'(x', t') = e^{-i[\frac{1}{2}\mathbf{c} \cdot x' + \frac{1}{4}|\mathbf{c}|^2 t']} \psi(x' + \mathbf{c}t', t'). \end{cases}$$

Verify that this transformation leaves the action associated to NLS invariant. Use this fact to find a conserved quantity.

2.4. Local conservation laws. This discussion is lifted directly from Section 2 of [CKSTT:Gopher](#) [15].

In this subsection we record some standard facts about the (non)conservation of mass, momentum and energy densities for general nonlinear Schrödinger equations of the form

forced

$$(2.24) \quad i\partial_t \phi + \Delta \phi = \mathcal{N}$$

on the spacetime slab $I_0 \times \mathbb{R}^d$ with I_0 a compact interval.

We begin by introducing some notation which will be used to describe the mass and momentum (non)conservation properties of [\(2.24\)](#).

emtensor

Definition 2.7. Given a (Schwartz) solution ϕ of [\(2.24\)](#) we define the mass density

$$T_{00}(t, x) := |\phi(t, x)|^2,$$

the momentum density

$$T_{0j}(t, x) := T_{j0}(t, x) := 2\text{Im}(\bar{\phi} \phi_j),$$

and the (linear part of the) momentum current

$$L_{jk}(t, x) = L_{kj}(t, x) := -\partial_j \partial_k |\phi(t, x)|^2 + 4\text{Re}(\bar{\phi}_j \phi_k).$$

mmbrackets

Definition 2.8. Given any two (Schwartz) functions $f, g : \mathbb{R}^d \rightarrow \mathbb{C}$, we define the mass bracket

mass-flux-def

$$(2.25) \quad \{f, g\}_m := \text{Im}(f \bar{g})$$

and the momentum bracket

momentum-flux-def

$$(2.26) \quad \{f, g\}_p := \text{Re}(f \nabla \bar{g} - g \nabla \bar{f}).$$

Thus $\{f, g\}_m$ is a scalar valued function, while $\{f, g\}_p$ defines a vector field on \mathbb{R}^d . We will denote the j th component of $\{f, g\}_p$ by $\{f, g\}_p^j$.

With these notions we can now express the mass and momentum (non)conservation laws for [\(2.24\)](#), which can be validated with straightforward computations.

local-conserv

Lemma 2.9 (Local conservation of mass and momentum). *If ϕ is a (Schwartz) solution to ^{forced}(2.24) then we have the local mass conservation identity*

local-mass-conserv

$$(2.27) \quad \partial_t T_{00} + \partial_j T_{0j} = 2\{\mathcal{N}, \phi\}_m$$

and the local momentum conservation identity

local-momentum-conserv

$$(2.28) \quad \partial_t T_{0j} + \partial_k L_{kj} = 2\{\mathcal{N}, \phi\}_p^j.$$

Here we adopt the usual² summation conventions for the indices j, k .

We now specialize to the gauge invariant Hamiltonian case, when $\mathcal{N} = F'(|\phi|^2)\phi$. Observe that

mass-cancel

$$(2.29) \quad \{F'(|\phi|^2)\phi, \phi\}_m = 0$$

and

momentum-cancel-general

$$(2.30) \quad \{F'(|\phi|^2)\phi, \phi\}_p = -\nabla G(|\phi|^2)$$

where $G(z) := zF'(z) - F(z)$. Thus, in the gauge invariant case we can reexpress ^{momentum-flux-def}(2.26) as

momentum-nl-conserv

$$(2.31) \quad \partial_t T_{0j} + \partial_k T_{jk} = 0$$

where

Tjk

$$(2.32) \quad T_{jk} := L_{jk} + \delta_{jk}G(|\phi|^2)$$

is the (linear and nonlinear) *momentum current*. Integrating ^{local-mass-conserv}(2.27) and ^{momentum-nl-conserv}(2.31) in space we see that the total mass

$$\int_{\mathbb{R}^d} T_{00} \, dx = \int_{\mathbb{R}^d} |\phi(t, x)|^2 \, dx$$

and the total momentum

$$\int_{\mathbb{R}^d} T_{0j} \, dx = 2 \int_{\mathbb{R}^d} \text{Im}(\overline{\phi(t, x)} \partial_j \phi(t, x)) \, dx$$

are both conserved quantities. In this Hamiltonian setting one can also verify the local energy conservation law

energy-conserv

$$(2.33) \quad \partial_t \left[\frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} F(|\phi|^2) \right] + \partial_j \left[\text{Im}(\overline{\phi_k} \phi_{kj}) - F'(|\phi|^2) \text{Im}(\overline{\phi} \phi_j) \right] = 0$$

which implies conservation of total energy

$$\int_{\mathbb{R}^d} \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} F(|\phi|^2) \, dx.$$

²Repeated Euclidean coordinate indices are summed. As the metric is Euclidean, we will not systematically match subscripts and superscripts.

2.5. **Generalized virial identity and some applications.** This subsection stems from [15] and [?].

We introduce two related quantities which average the mass and momentum densities (see Definition 2.7) against a weight function $a(x)$.

Definition 2.10. Let $a(x)$ be a function³ defined on the spacetime slab $I_0 \times \mathbb{R}^3$. We define the associated virial potential

$$\text{Vsuba} \quad (2.34) \quad V_0^a(t) = \int_{\mathbb{R}^3} a(x)|\phi(t, x)|^2 dx$$

and the associated Morawetz action

$$\text{Msuba} \quad (2.35) \quad M_0^a(t) = \int_{\mathbb{R}^3} a_j 2\text{Im}(\bar{\phi}\phi_j) dx.$$

A calculation using Lemma 2.9 shows that

$$\text{Vsubadot} \quad (2.36) \quad \partial_t V_0^a = M_0^a + 2 \int_{\mathbb{R}^3} a \{ \mathcal{N}, \phi \}_m dx,$$

so $M_0^a = \partial_t V_0^a$ when $\mathcal{N} = F'(|\phi|^2)\phi$. Using Lemma 2.9, a longer but similar calculation establishes,

Lemma 2.11 (Generalized virial identity). Let ϕ be a (Schwartz) solution of (2.24). Then

$$\text{Madot} \quad (2.37) \quad \partial_t M_0^a = \int_{\mathbb{R}^3} (-\Delta\Delta a)|\phi|^2 + 4a_{jk} \text{Re}(\bar{\phi}_j\phi_k) + 2a_j \{ \mathcal{N}, \phi \}_p^j dx.$$

In the gauge invariant setting of (2.12), this identity specializes to read

$$\text{Madotgauge} \quad (2.38) \quad \partial_t M_0^a = \int_{\mathbb{R}^3} (-\Delta\Delta a)|\phi|^2 + 4a_{jk} \text{Re}(\bar{\phi}_j\phi_k) + 2\Delta a G(|u|^2) dx$$

where $G(z) = zF'(z) - F(z)$.

Example 2.12 (Variance Identity [?, ?]). Set $a(x) = |x|^2$. Then $a_{jk} = 2\delta_{jk}$, $\Delta a = 2d$, $\Delta\Delta a = 0$. We insert these calculations into (2.38):

$$\partial_t^2 V_0^a = 4 \int 2\delta_{jk} \text{Re}(u_k \bar{u}_j) + 4dG(|u|^2) dx.$$

The first term simplifies to an expression with $|\nabla u|^2$. In the focusing case where $F \sim G \leq 0$, the second term is nonpositive. The right side may thus be reexpressed as

$$8H[u] + (\leq 0)$$

where (≤ 0) denotes a nonpositive term. For initial data satisfying $H[u] < 0$ we obtain the variance identity

$$\text{varianceidentity} \quad (2.39) \quad \partial_t^2 \int |x|^2 |u|^2 \leq 8H < 0.$$

Thus, the solution evolving from negative energy data has variance going to zero in finite time. Since

$$\int |u|^2 dx = \int \frac{1}{d} [\nabla \cdot x] u \bar{u} dx$$

³In other contexts it's useful to consider also time dependent weights functions $a(t, x)$

an integration by parts shows that

$$\boxed{\text{uncertainty}} \quad (2.40) \quad \|u\|_{L_x^2} \leq \sqrt{\frac{2}{d}} \|xu\|_{L_x^2} \|\nabla u\|_{L_x^2}.$$

By L^2 conservation, we thus see that $\|\nabla u(t)\|_{L_x^2}$ goes to $+\infty$ in finite time for solutions whose variance goes to zero.

Example 2.13 (Lin-Strauss Morawetz-type identity ^{linstrauss}[32]). Set $a(x) = |x|$. Then ^{Madot gauge} $a_j = \frac{x_j}{|x|}$, $a_{jk} = [\delta_{jk} - \frac{x_j x_k}{|x|^2}] \frac{1}{|x|}$, $\Delta a = \frac{d-1}{|x|}$. We insert these calculations into (2.38):

$$\boxed{\text{linstraussidentity}} \quad (2.41) \quad \partial_t M_0^a(t) = - (d-1) \int_{\mathbb{R}^d} \Delta \left(\frac{1}{|x|} \right) |u|^2(t, x) dx$$

$$\boxed{\text{ls2}} \quad (2.42) \quad + 4 \int_{\mathbb{R}^d} \left[\delta_{jk} - \frac{x_j x_k}{|x|^2} \right] \frac{1}{|x|} \text{Re}(u_j \bar{u}_k) dx$$

$$\boxed{\text{ls3}} \quad (2.43) \quad + 2(d-1) \int_{\mathbb{R}^d} \frac{G(|u|^2)}{|x|} dx.$$

A nice miracle occurs on \mathbb{R}^3 since $-\Delta(\frac{1}{|x|}) = 4\pi\delta_0$ so we get

$$\boxed{\text{lsR3}} \quad (2.44) \quad \partial_t M_0^a(t) = 8\pi |u(t, 0)|^2 + 4 \int_{\mathbb{R}^3} \frac{|\nabla_0 u|^2 + G(|u|^2)}{|x|} dx$$

where ∇_0 denotes the angular part of the gradient. (Observe that the integrand in (2.42) may be reexpressed ^{ls2} $[|\nabla u|^2 - |\partial_r u|^2] \frac{1}{|x|} \geq 0$.) Let's assume we are in the defocusing case so $G \geq 0$. Then, the right side of (2.44) is positive and we observe that M_0^a is an increasing function of t . Recall that ^{lsR3}

$$M_0^a(t) = - \int_{\mathbb{R}^d} \frac{x_j}{|x|} 2\text{Im}(u \bar{u}_j) dx$$

which may be interpreted as the average of the outbound radial mass current. So, for the defocusing evolution, the mass is “repelled” from the spatial origin. We also have that⁴

$$|M_0^a(t)| \leq \|u(t)\|_{L^2} \|\nabla u(t)\|_{L^2}.$$

By conservation of L^2 mass and energy, the right side of this expression is uniformly bounded for finite energy solutions of defocusing NLS. Upon integrating (2.44) w.r.t time, we obtain the Lin-Strauss Morawetz-type estimate ^{lsR3}

$$\boxed{\text{lsMorawetz}} \quad (2.45) \quad \int_0^T |u(t, 0)|^2 dx + 4 \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla_0 u|^2 + G(|u|^2)}{|x|} dx dt \leq 2 \|u(t)\|_{L^2} \|\nabla u\|_{L^2}.$$

Since the right side is bounded independently of T , the $G(|u|^2)$ term in this estimate implies the solution decays with time.

2.6. Globalizing estimates. A Priori H^1 boundedness–(de)focusing distinctions; basic dynamical effects in $NLS_p(\mathbb{R}^d)$; scaling/criticality; almost conservation laws.

2.7. Appendix: symplectic properties. nonsqueezing/capacity; phase space “volume” preservation, invariant Gibbs measures.

3. LINEAR SCHRÖDINGER ESTIMATES

^{GV85} ^{KT}
^[20], ^[30]

⁴In fact, $|M_0^a(t)| \leq \|u(t)\|_{H^{1/2}}^2$.

- 3.1. Representation of $e^{it\Delta}$.
- 3.2. Strichartz estimates via Fourier restriction.
- 3.3. Strichartz estimates via Hardy-Littlewood-Sobolev.
- 3.4. Refinements of Strichartz estimates.
- 3.5. Local smoothing estimates.
- 3.6. Maximal function estimates.

4. STRICHARTZ-BASED LWP THEORY

$\frac{CW}{[10]}, \frac{CWO}{[11]}$

5. $X_{s,b}$ -BASED LWP THEORY

$\frac{BXsb}{[3]}$

Periodic Strichartz estimates; periodic initial value problems; “Denominator games”, “calculus methods” for nonlinear smoothing.

6. ILL-POSEDNESS

7. GWP BELOW ENERGY

7.1. Hi-low truncation method. $\frac{BRefBAMS}{[4], [6]}$

7.2. I-method. $\frac{CKSTCKBTTCKBAM:DNLS1}{[?], [14], [12]}$

8. GLOBAL-IN-TIME BEHAVIOR

8.1. Linear scattering. $\frac{gv:scatterCKSTCKBTTCKBAM}{[19], [36], [14]}$

8.2. Nonlinear scattering.

9. BLOWUP

10. H^1 -CRITICAL QUINTIC NLS ON \mathbb{R}^3

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