MAT1061 FINAL EXAM

Please hand in this exam to J. Colliander in BA6110. The exam is due by 3:00p.m. on Friday 13 April 2007. Please work out 3 of 4 of the following problems.

1. Suppose U is connected and $u \in W^{1,p}(U)$ satisfies $Du = 0$ *a.e.* in U. Prove u is constant *a.e.* in U.

2. Use the methods of [Evans; §8.4.1] to show the existence of a nontrivial (not identically 0) weak solution $u \in H_0^1(U)$, of the boundary value problem

$$\begin{cases} -\Delta u = |u|^{q-1} u, & \text{in } U \\ \quad u = 0, & \text{on } \partial U \end{cases}$$

for $1 < q < \frac{n+2}{n-2}$ with $n > 2$.

3. Let

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + c(x) u$$

where a^{ij} is a positive definite constant matrix. Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\ ,\ ]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided $c(x) \geq -\mu$ for $x \in U$. (See [Evans; §6.2.1].)

4. Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f, \text{ in } \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \geq 0$. Prove that $u \in H^2(\mathbb{R}^n)$. (Hint: Mimic the proof of Theorem 1 in [Evans; §6.3.1] but without the cutoff function ζ.)