Numerical Simulations of Radial Supercritical Defocusing Waves

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Edinburgh, 2011-01-17

This talk reports on [CSS09] with **G. Simpson**, C. Sulem (U. Toronto) and forthcoming work [CPSS10] also with P. Pang.

1. Energy Supercritical NLS

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Consider the defocusing monomial NLS initial value problem:

$$\begin{cases} (i\partial_t + \Delta)u = |u|^{p-1}u \\ u(0,x) = u_0(x). \end{cases} \qquad (NLS_p^+(^d))$$

<u>Dilation Invariance</u>:

$$u : [0,T]\times{}^d \longmapsto \text{ solves } NLS_p^+(^d)$$

R^d

$$\Updownarrow$$

$$\forall\ \lambda > 0,\ u_\lambda : [0, \lambda^2 T]\times{}^d] \longmapsto \text{ solves } NLS_p^+(^d)$$

where

$$u_\lambda(\tau, y) = (\frac{1}{\lambda})^{\frac{2}{p-1}} u(\frac{\tau}{\lambda^2}, \frac{y}{\lambda}).$$

Critical Sobolev Regularity

A simple calculation shows that

$$\|D^s u_\lambda(\tau,\cdot)\|_{L^2} = \left(\frac{1}{\lambda}\right)^{\frac{2}{p-1}+s-\frac{d}{2}} \|D^s u(\tau)\|_{L^2}.$$

We encounter a dilation invariant Sobolev space norm when

$$s = s_c = \frac{d}{2} - \frac{2}{p-1}.$$

The space $\dot{H}^{s_c}({}^d)$ plays a basic role in theory for $NLS_p({}^d)$.

Conservation Laws for $NLS_p^+(\mathbb{R}^d)$

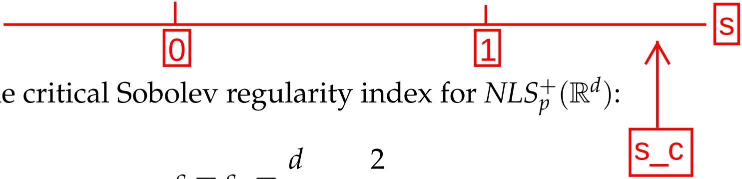
Time invariant quantitites:

$$\begin{aligned} \text{Mass} &= \int |u(t,x)|^2 dx. \\ \text{Momentum} &= 2\Im \int \overline{u}(t)\partial_x u(t) dx. \\ \text{Energy} = H[u(t)] &= \int_T \frac{1}{2} |\partial_x u(t)|^2 dx + \frac{2}{p+1} |u(t)|^{p+1} dx. \end{aligned}$$

Sobolev Controls:

- Mass controls the L^2 (a.k.a. H^0) norm.
- Momentum is at the $H^{1/2}$ regularity level.
- Energy controls H^1 norm.

Criticality Regimes $NLS_p^+(\mathbb{R}^d)$



Recall the critical Sobolev regularity index for $NLS_p^+(\mathbb{R}^d)$:

$$s = s_c = \frac{d}{2} - \frac{2}{p-1}.$$

Conservation laws identify 5 critical regimes:

- $s_c < 0$, Mass Subcritical
- $s_c = 0$, Mass Critical
- $0 < s_c < 1$, Mass Supercritical/Energy Subcrtical
- $s_c = 1$, Energy Critical
- $s_c < 0$, Energy Supercritical.

Global Wellposedness for Energy Subcritical Regime

- H^1-Local Wellposedness Lifetime depends on H^1 norm:

$$T_{lwp} \sim \|u(0)\|_{H^1}^{-\gamma}.$$

- Energy Conservation implies H^1 *a priori* control:

$$\|u(t)\|_{H^1}^2 \leq \text{Energy}[u(0)], \ \forall\, t.$$

- Iterate the local theory with **uniform** time steps $\implies$ GWP.

Remarks:

- Maximal-in-time behavior of solutions? Not clear...
- Low regularity relaxations (high/low frequency, I-method)
- Polynomial in time bounds on $\|u(t)\|_H^s, \ s \gg 1$.

GWP for Energy Critical Regime

LWP iteration FAILS:

- Energy implies H^1 *a priori* control: $\|u(t)\|_{H^1}^2 \leq \text{Energy}[u(0)]$.
- H^1-LWP lifetime not controlled by H^1 norm.
- H^1-LWP iteration does NOT imply GWP.

Energy Critical Scattering via Critical Elements Method:

- Induction on Energy Approach [Bou99]
  - $NLS_5^+(\mathbb{R}^3)$, $NLS_3^+(\mathbb{R}^4)$ radial.
  - $NLS_5^+(\mathbb{R}^3)$ GWP and Scatters [CKS+08].
  - 4 and higher dimensions [RV05], [Vis07].
- Extract and kill critical elements. [KM08a]
  - Progress on focusing problems beneath soliton threshold.
  - Mass Critical Case [KVZ08], [Dod10]
  - Other equations, including Navier-Stokes [KK09]

2. Energy Supercritical Maximal-in-time Theory?

2. Energy Supercritical Maximal-in-time Theory?

- This theory is **wide open**.
- We have no *a priori* control on Sobolev norms with $s > 1$.
- Same issue obstructs Navier-Stokes initial value problem?
- Idea? First resolve $NLS_3^+(\mathbb{R}^3)$? Use the momentum?

Critical Norm Bounded $\implies$ Scattering

Bounded H^{s_c} norm **Assumption**: Suppose $H^{s_c} \ni u(0) \longmapsto u$ solves NLS or NLW with $s_c > 1$ and $\exists\, K < 0$ such that

$$\sup_t \|u(t)\|_{H^{s_c}} < K.$$

- $NLW_p(^d)$:
  Radial + Bounded H^{s_c} norm $\implies$ scattering [KM08b].
- $NLS_p^+(^d)$: Bounded H^{s_c} norm $\implies$ scattering [KV10].

Question: What is the behavior of $\|u(t)\|_{H^{s_c}}$?

3. Numerical Simulations of Supercritical Waves

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- Supercritical NLW behaves nicely [SV78].
- $NLS_5^+(^5)$ has bounded H^2 norm [CSS09].
- Energy supercritical NLW behaves similarly [CPSS10].

Strauss-Vazquez 1978 simulations

JOURNAL OF COMPUTATIONAL PHYSICS 28, 271–278 (1978)

Note

Numerical Solution of a Nonlinear Klein–Gordon Equation

We compute the solutions of the equation $u_{tt} - \Delta u + m^2 u + g u^p = 0$ for p odd and $m, g > 0$. Our computations show that (i) the solutions remain bounded as $t \to \infty$, (ii) the amplitude decreases as p increases, and (iii) the number of oscillations increases as p increases. Because of (i), theoretical results imply that the amplitude goes to zero like $O(t^{-3/2})$ as $t \to \infty$.

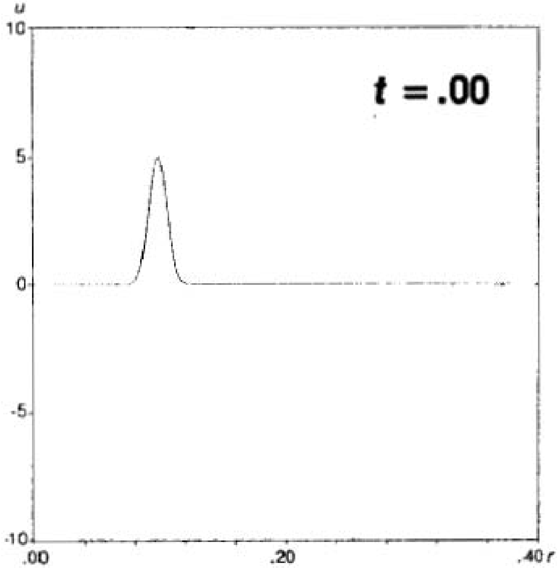
1. Analysis

The nonlinear Klein–Gordon equation (NLKG)

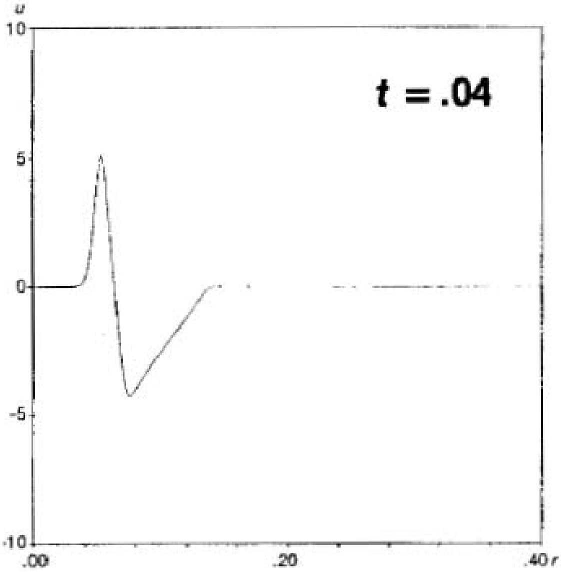
$$u_{tt} - \Delta u + m^2 u + G'(u) = 0, \qquad (1)$$

$$u(x, 0) = \phi(x), \qquad u_t(x, 0) = \Psi(x) \qquad (2)$$

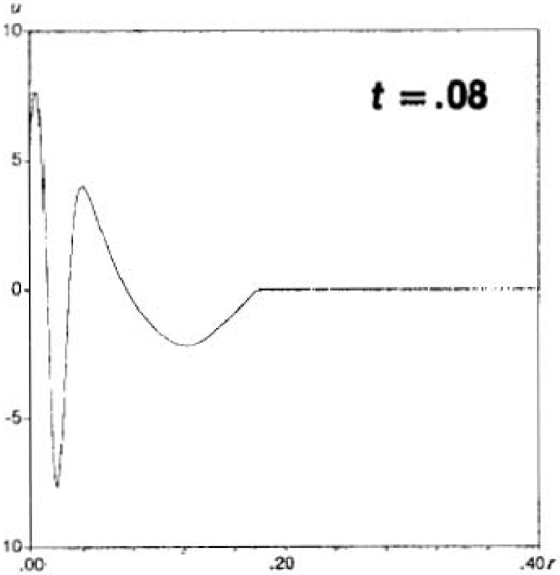
Strauss-Vazquez 1978 simulations



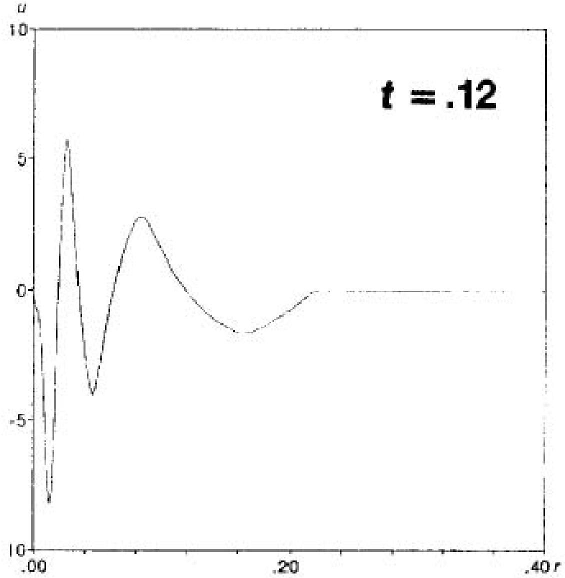
Strauss-Vazquez 1978 simulations



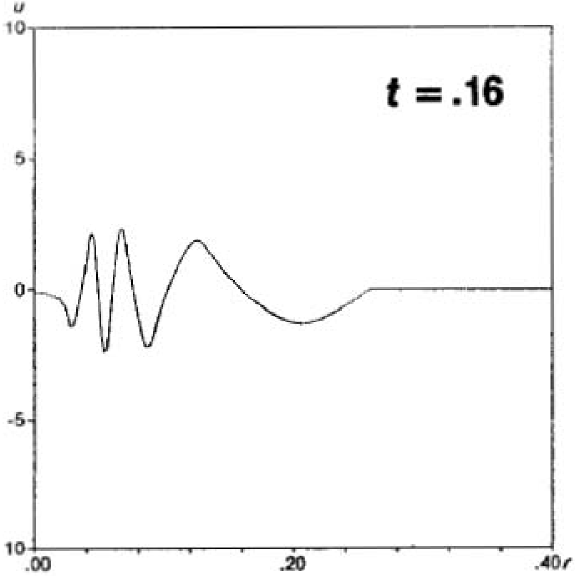
Strauss-Vazquez 1978 simulations



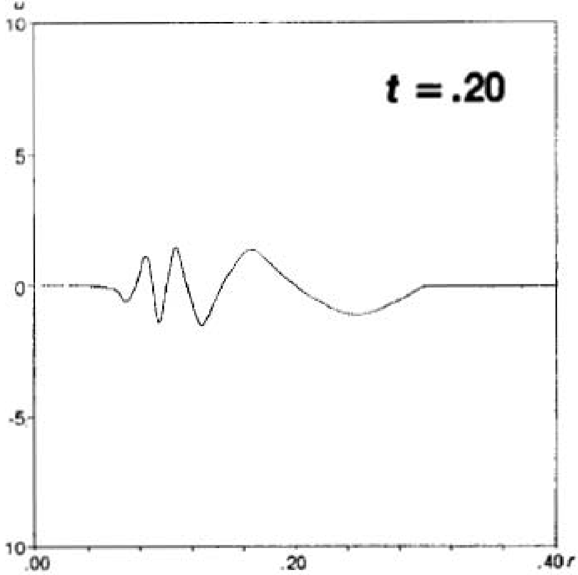
Strauss-Vazquez 1978 simulations



Strauss-Vazquez 1978 simulations



Strauss-Vazquez 1978 Simulations



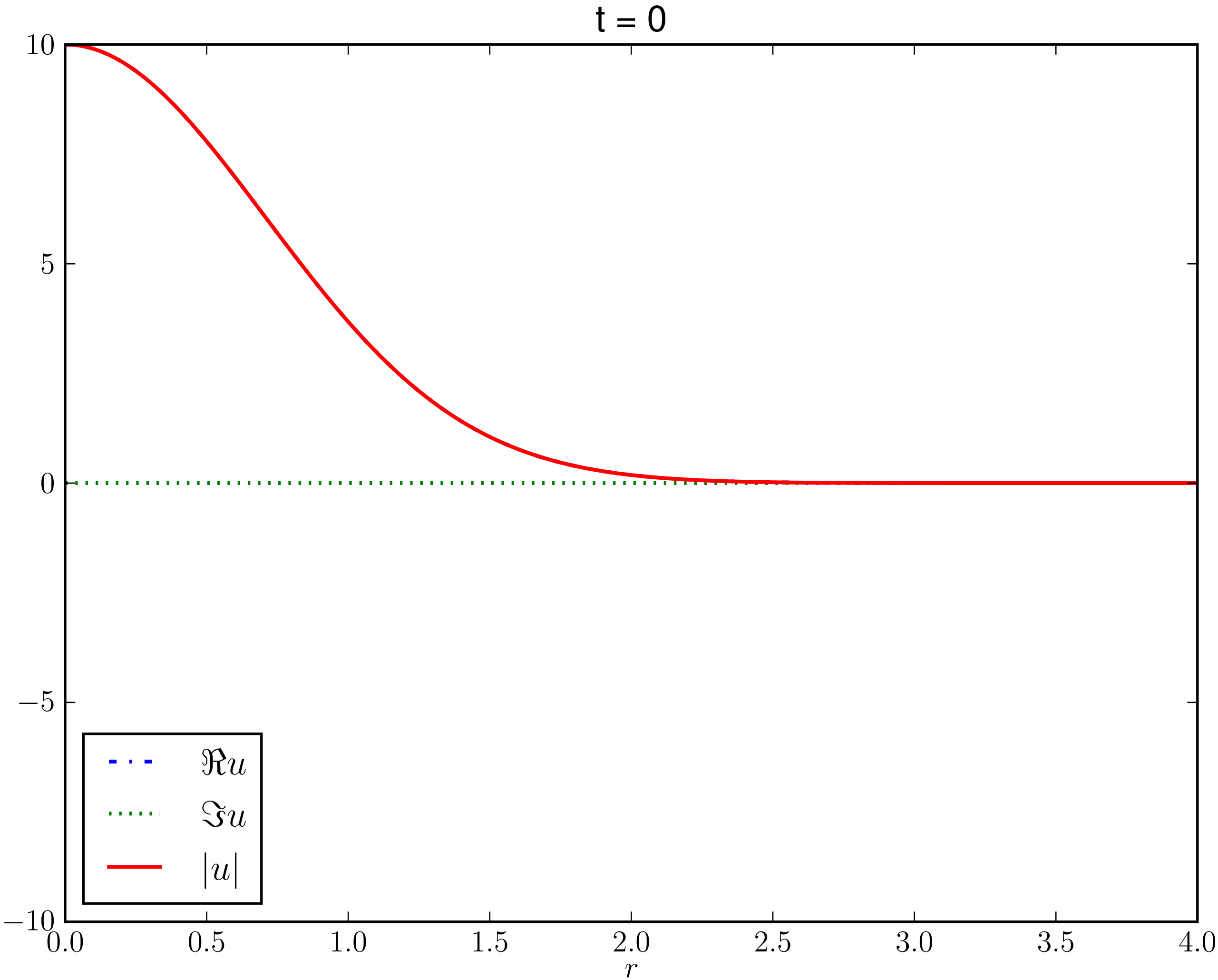
C-Simpson-Sulem simulations

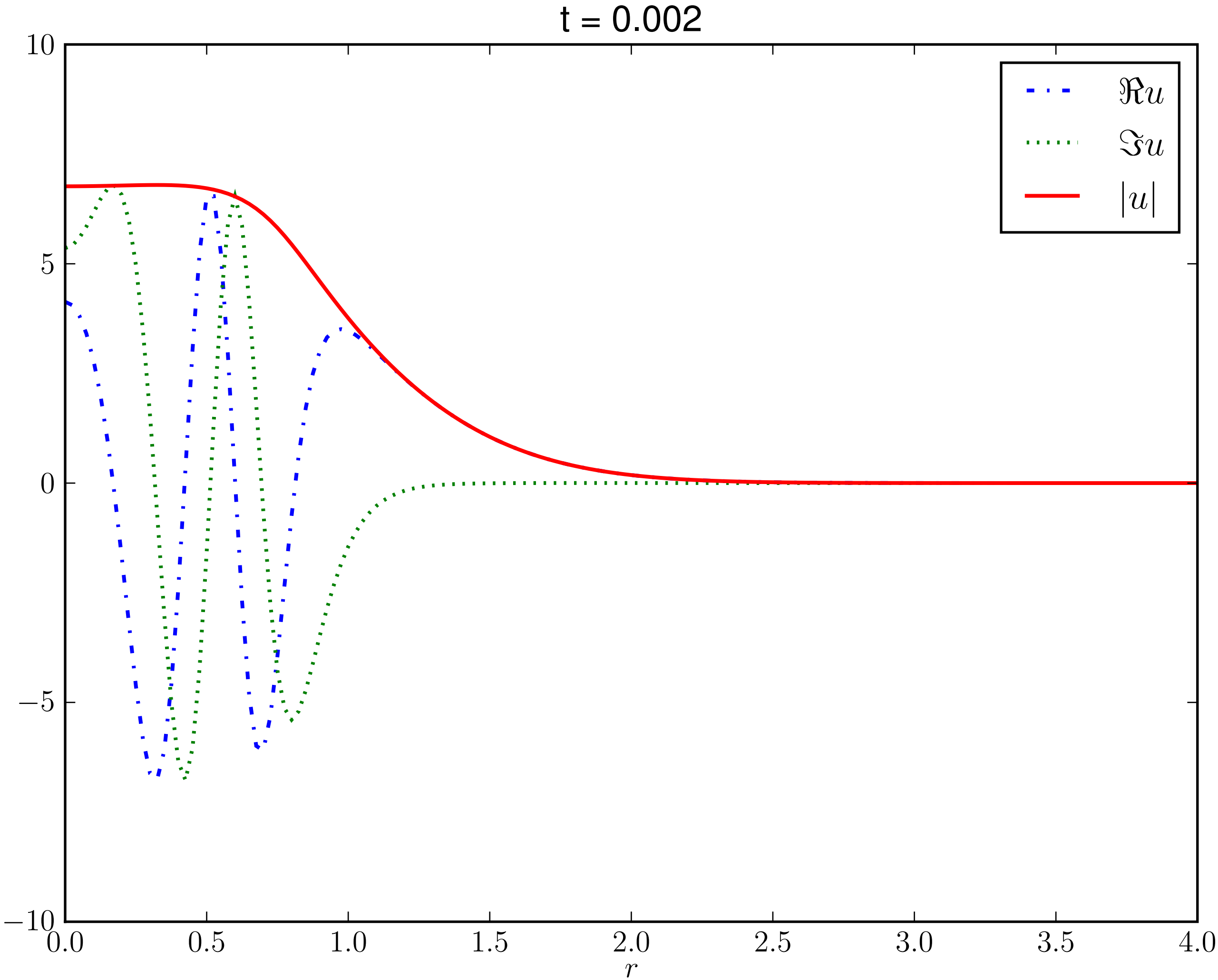
C-Simpson-Sulem simulations

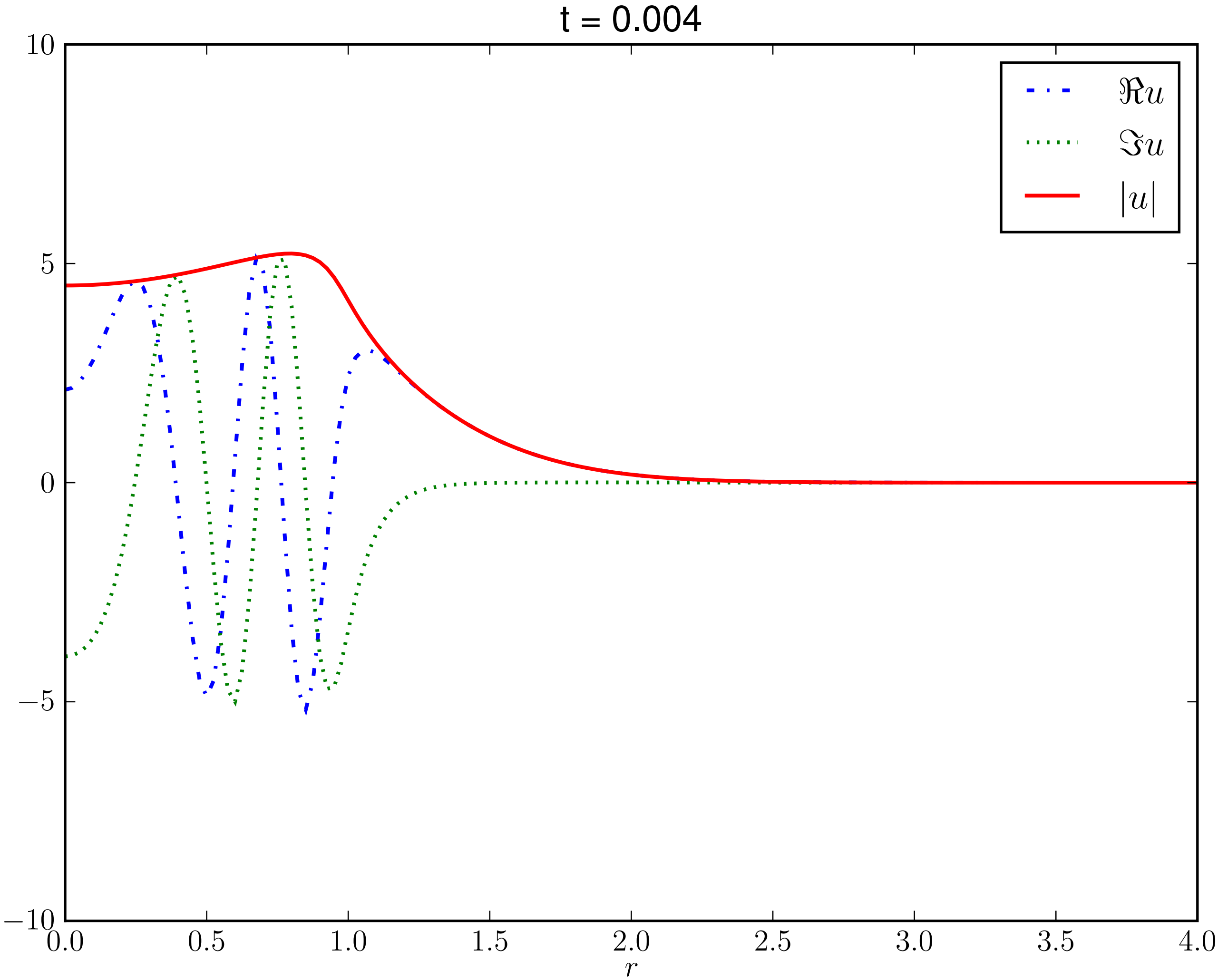
Four simulations of radial data:

- Centered Gaussian
- Phased Centered Gaussian
- Phased Centered Gaussian (Linear flow diagnostic)
- Spherical Ring

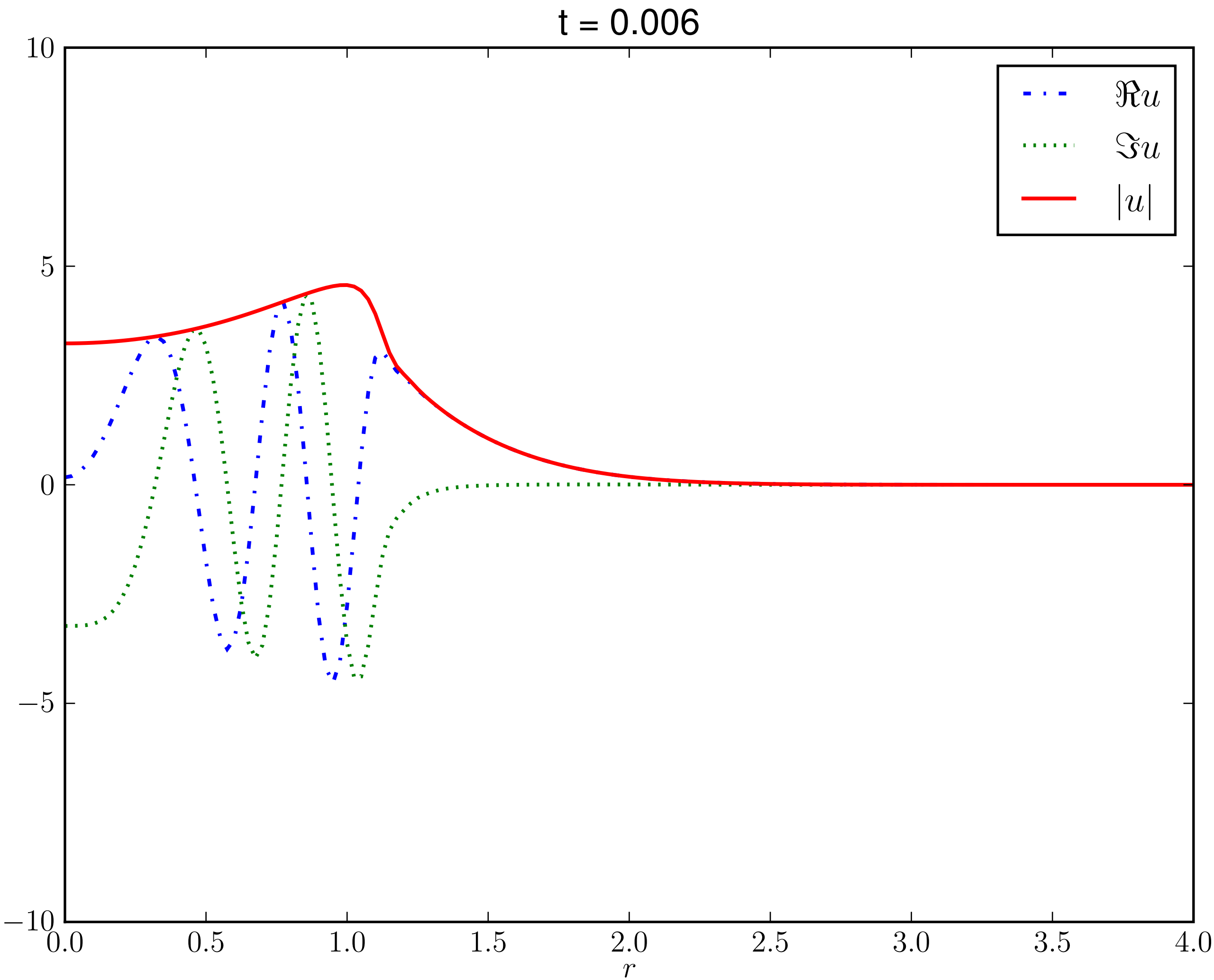
Centered Gaussian Initial Data

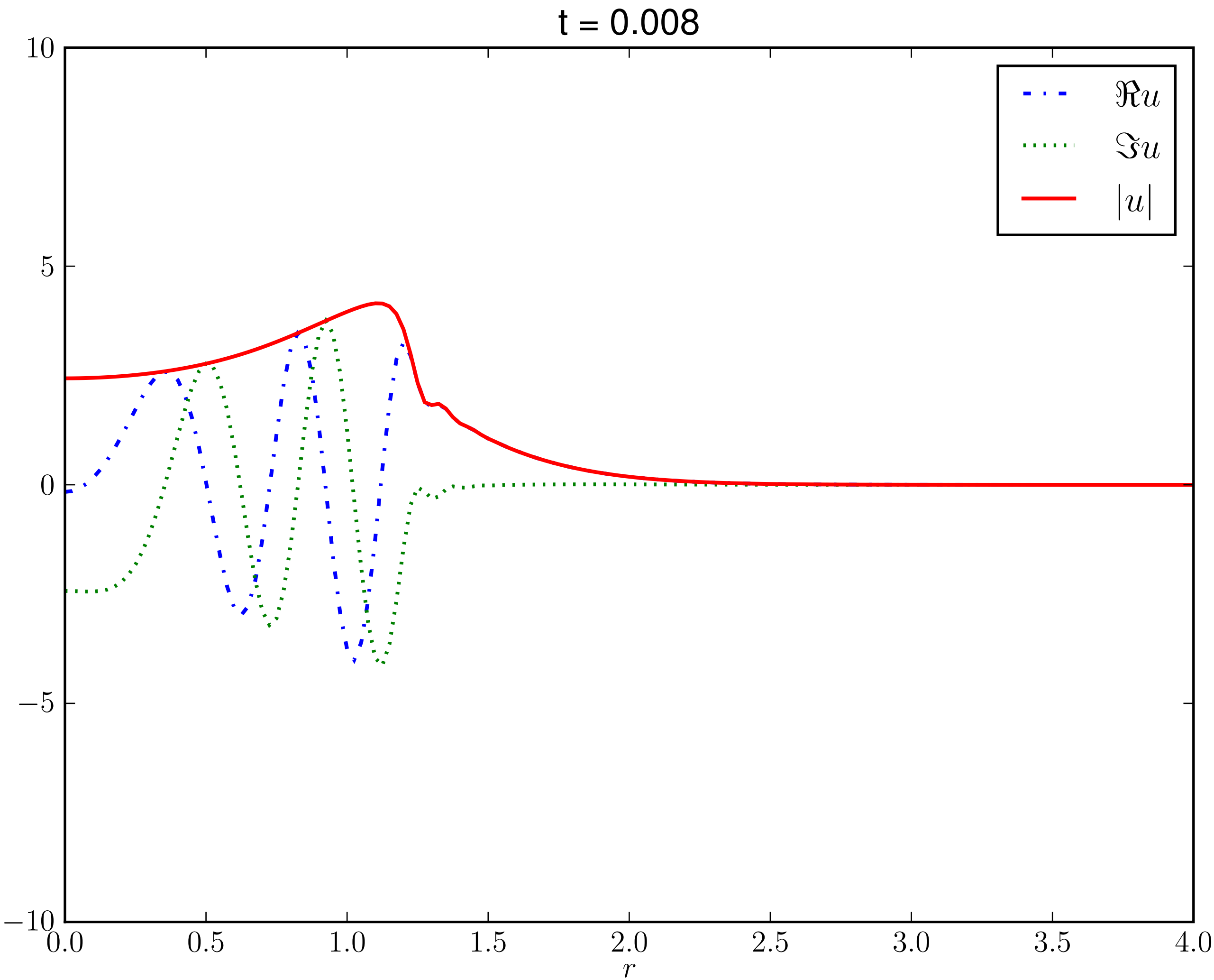



t = 0.002
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

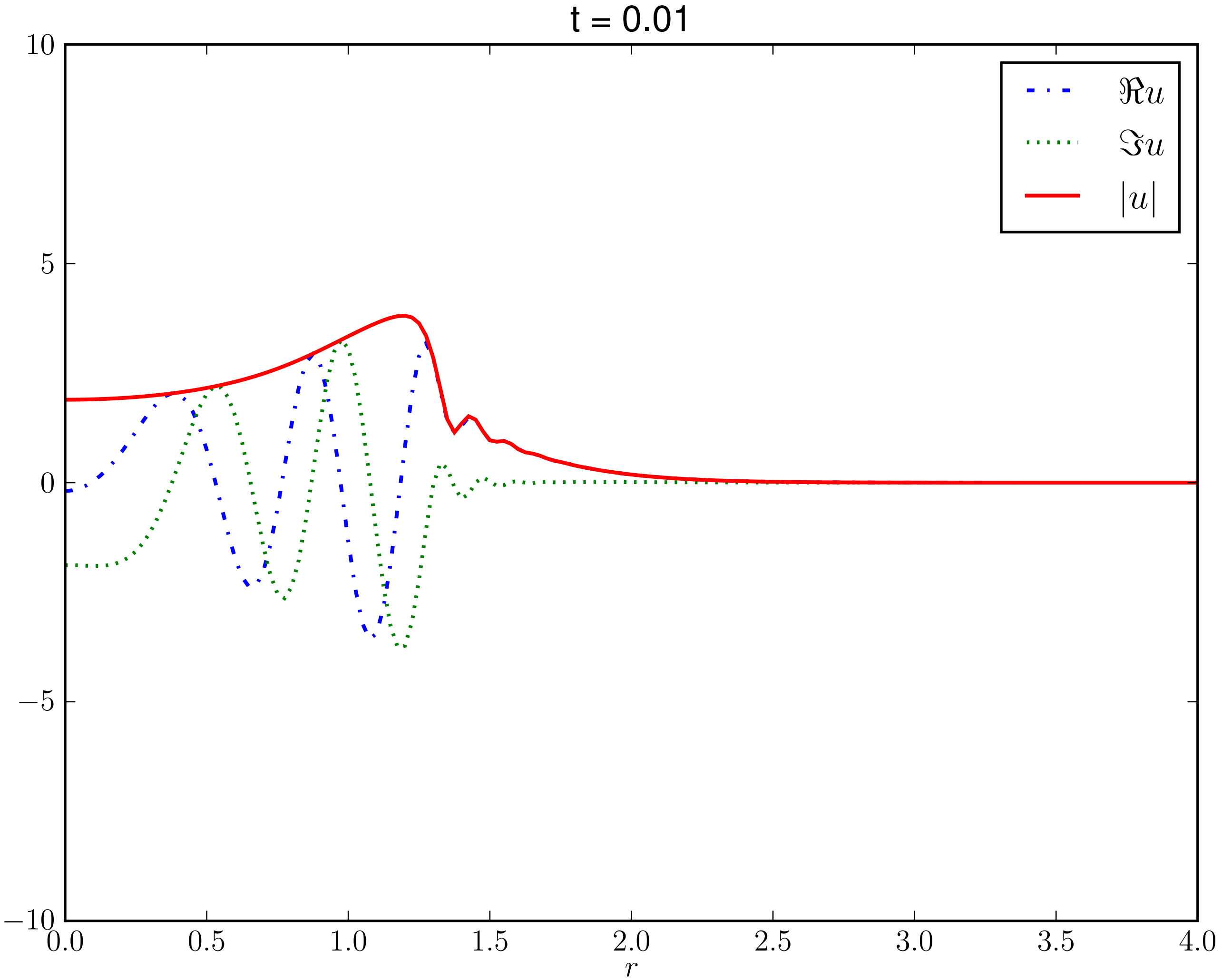


t = 0.004
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

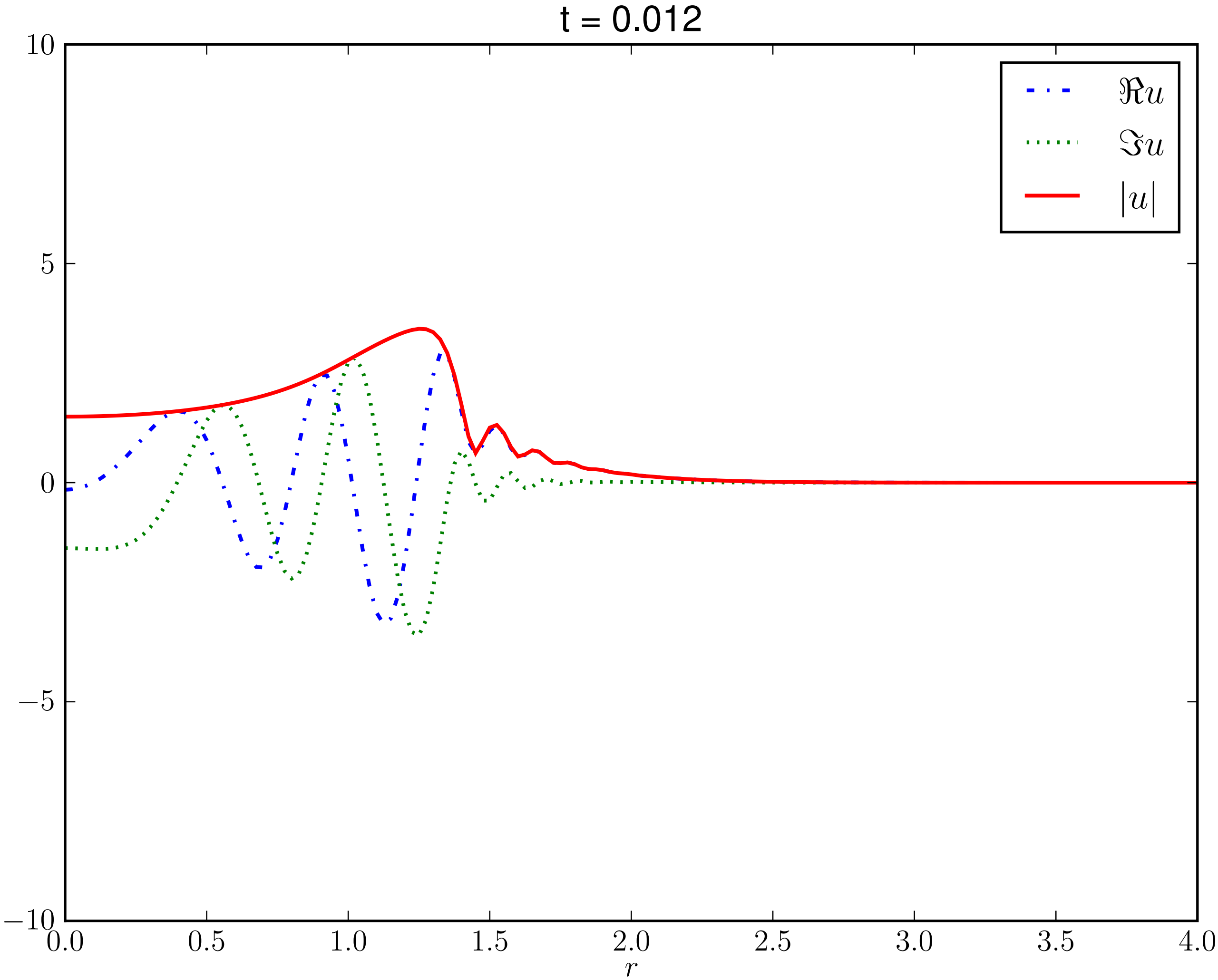

t = 0.006
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



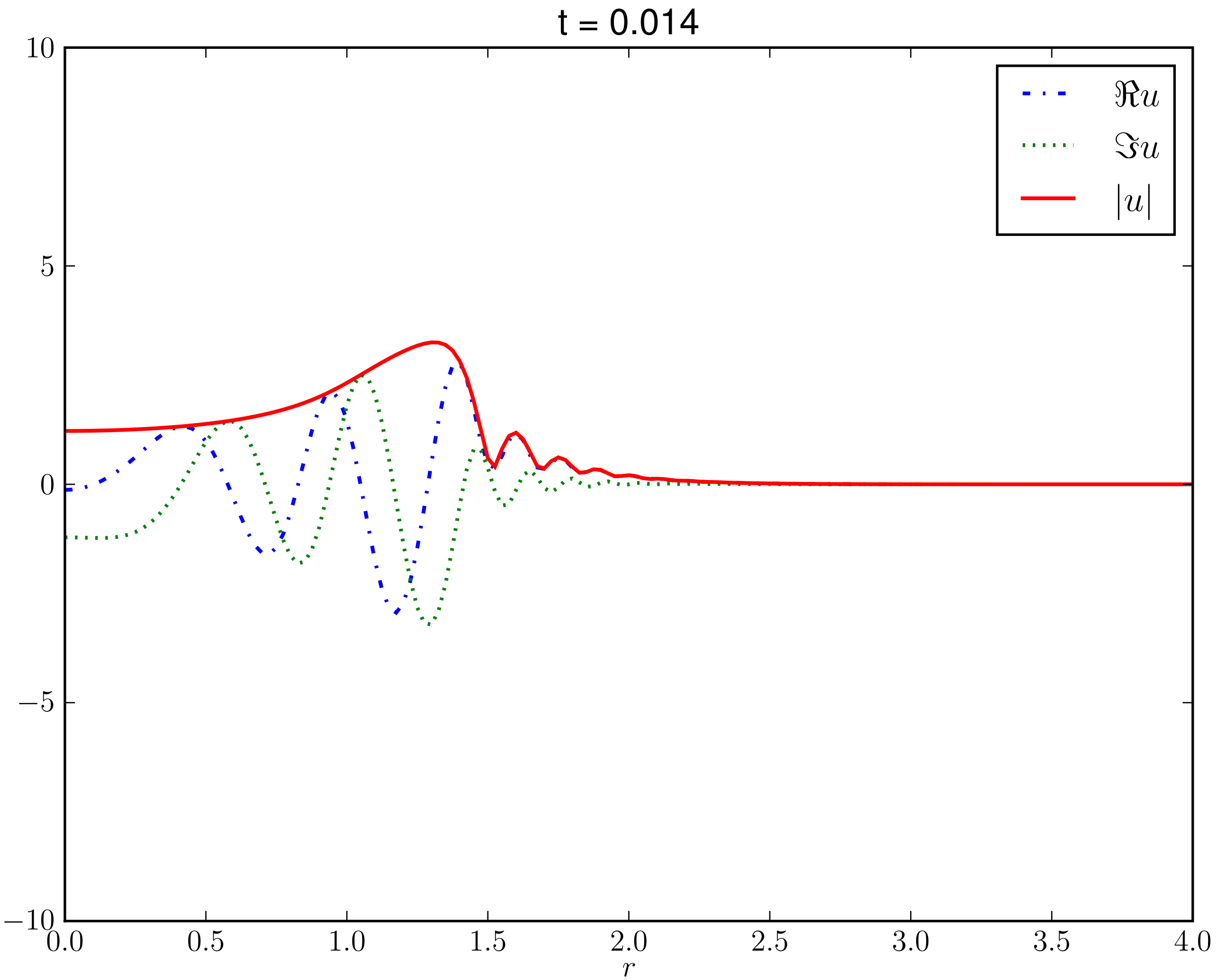
t = 0.008
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



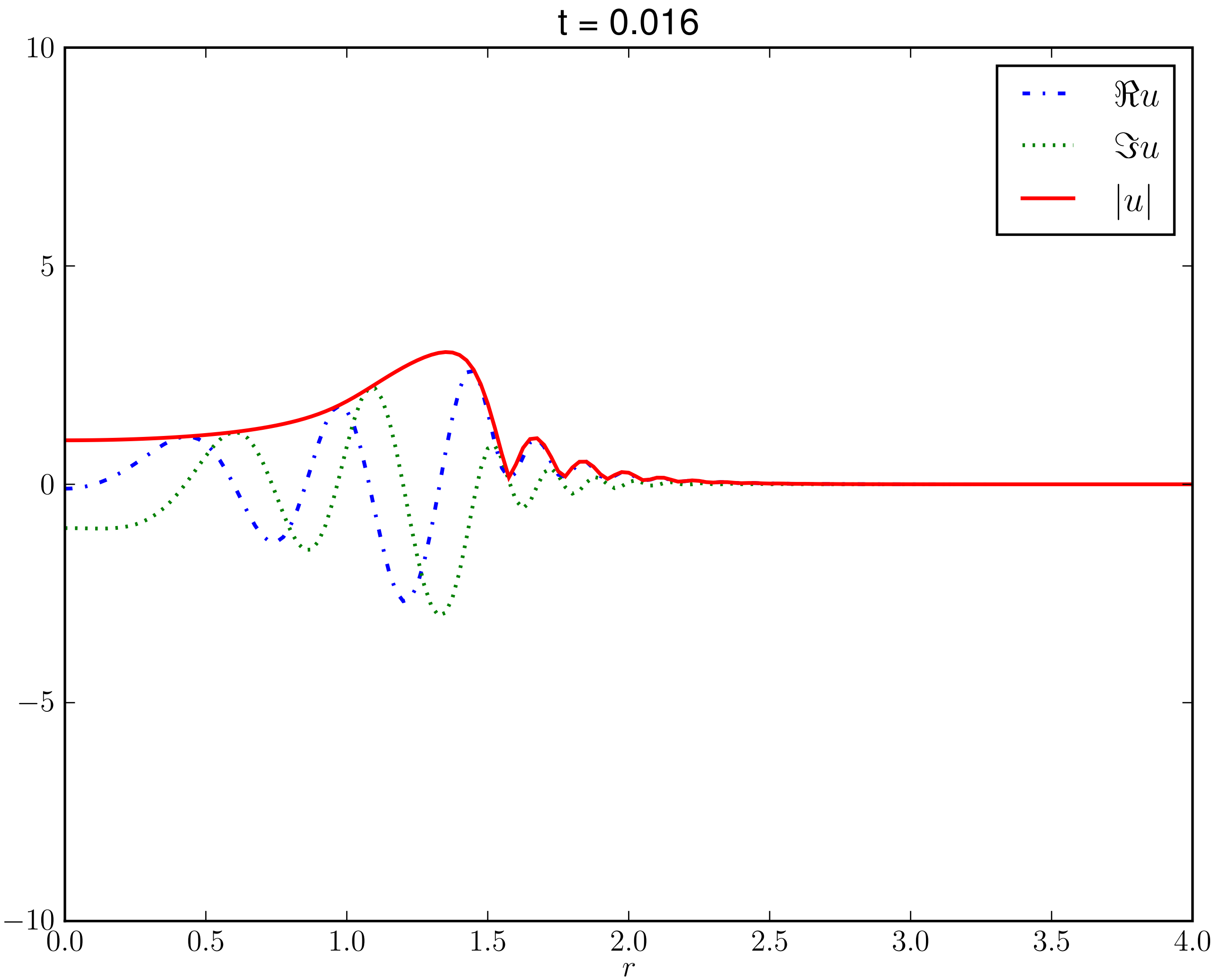
t = 0.01
$\Re u$
$\Im u$
$|u|$
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r



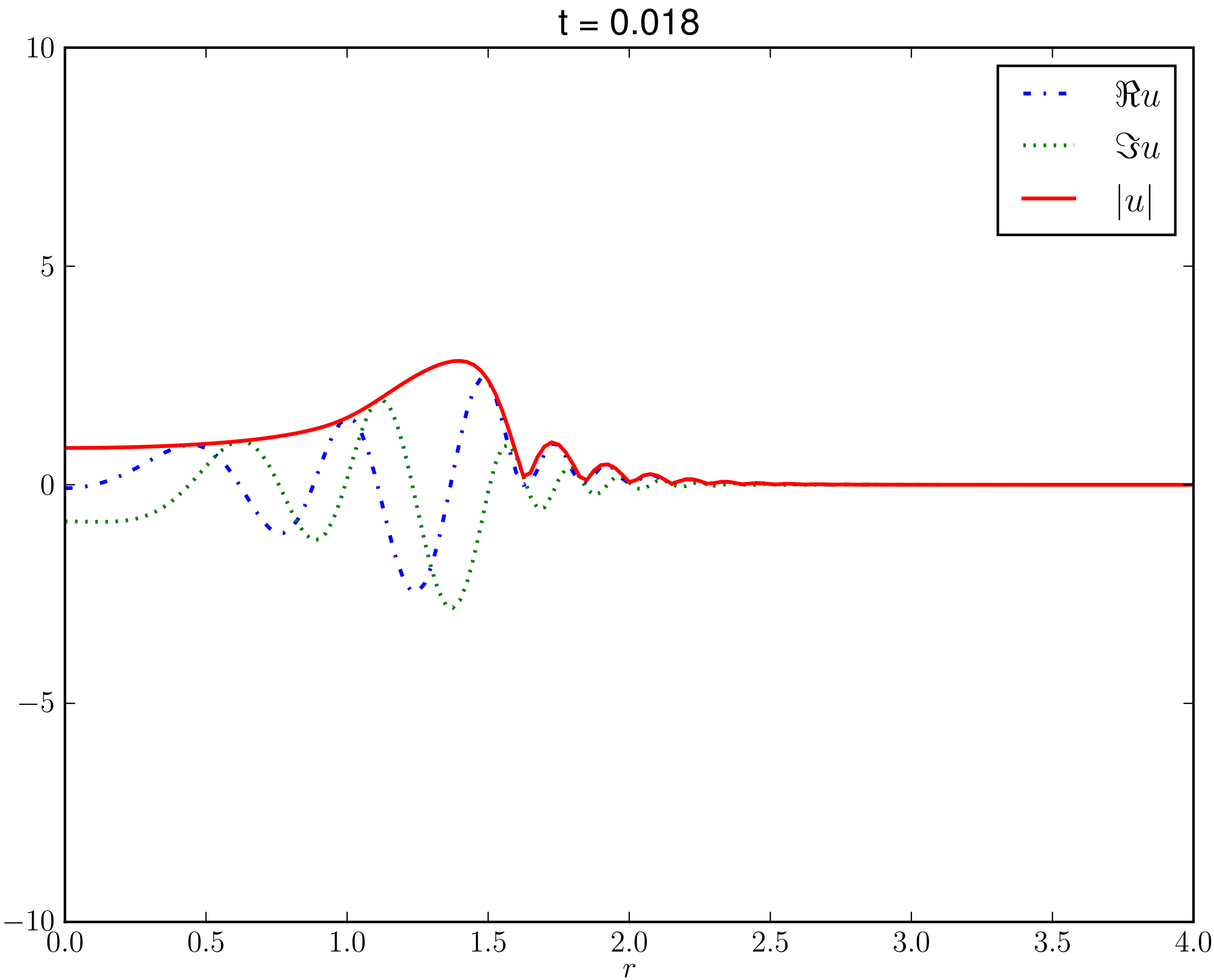
t = 0.012
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

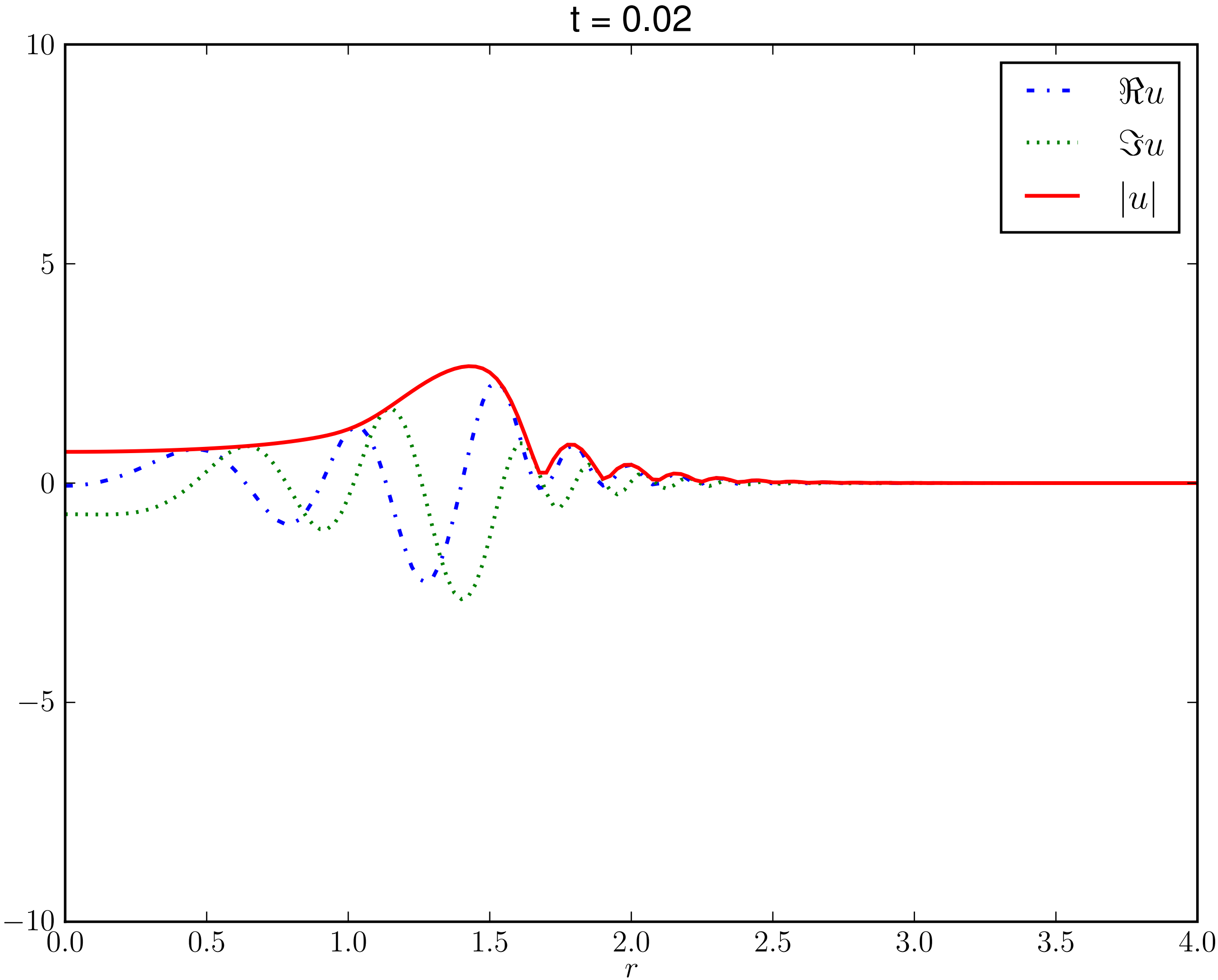


t = 0.014
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

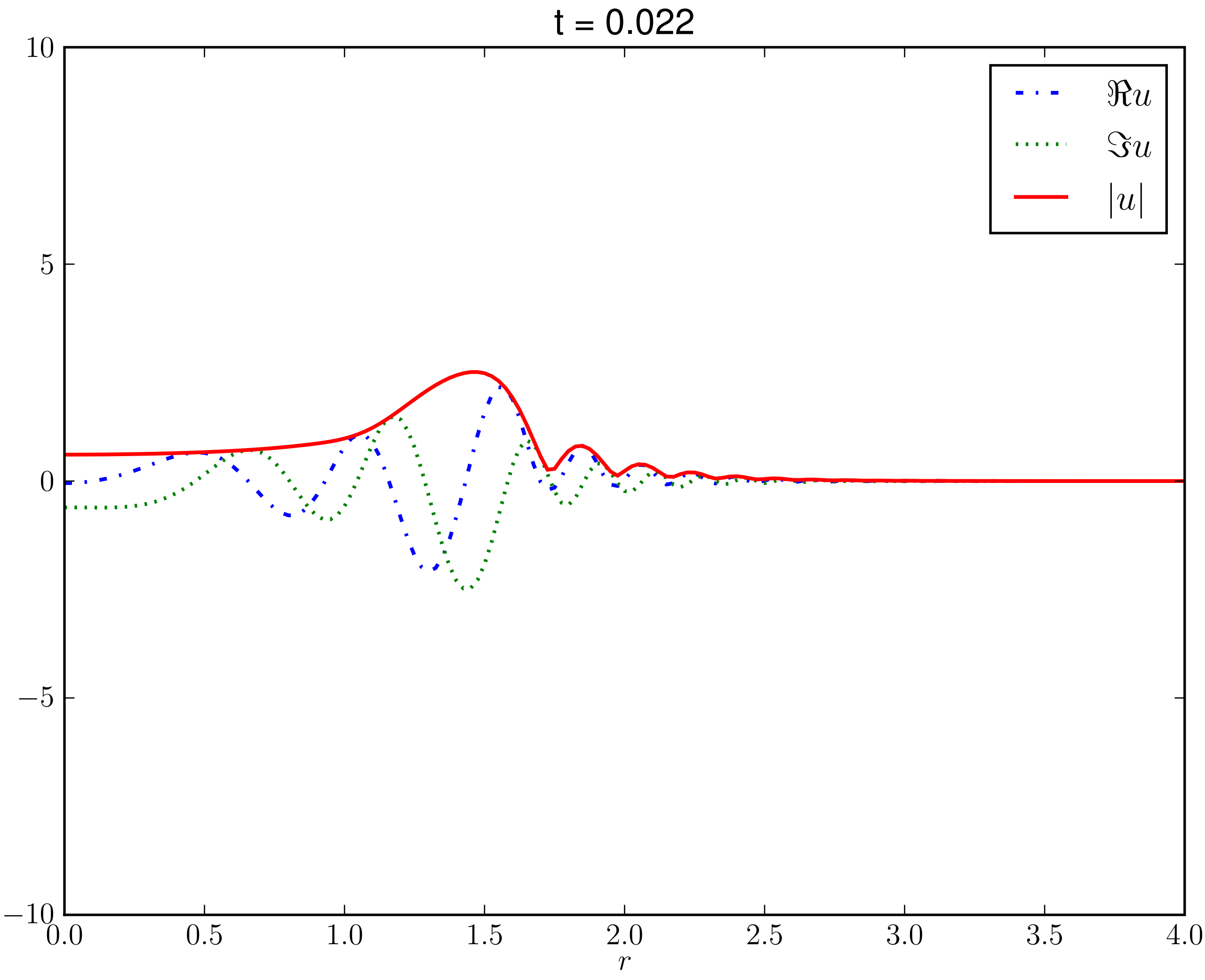


t = 0.016
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

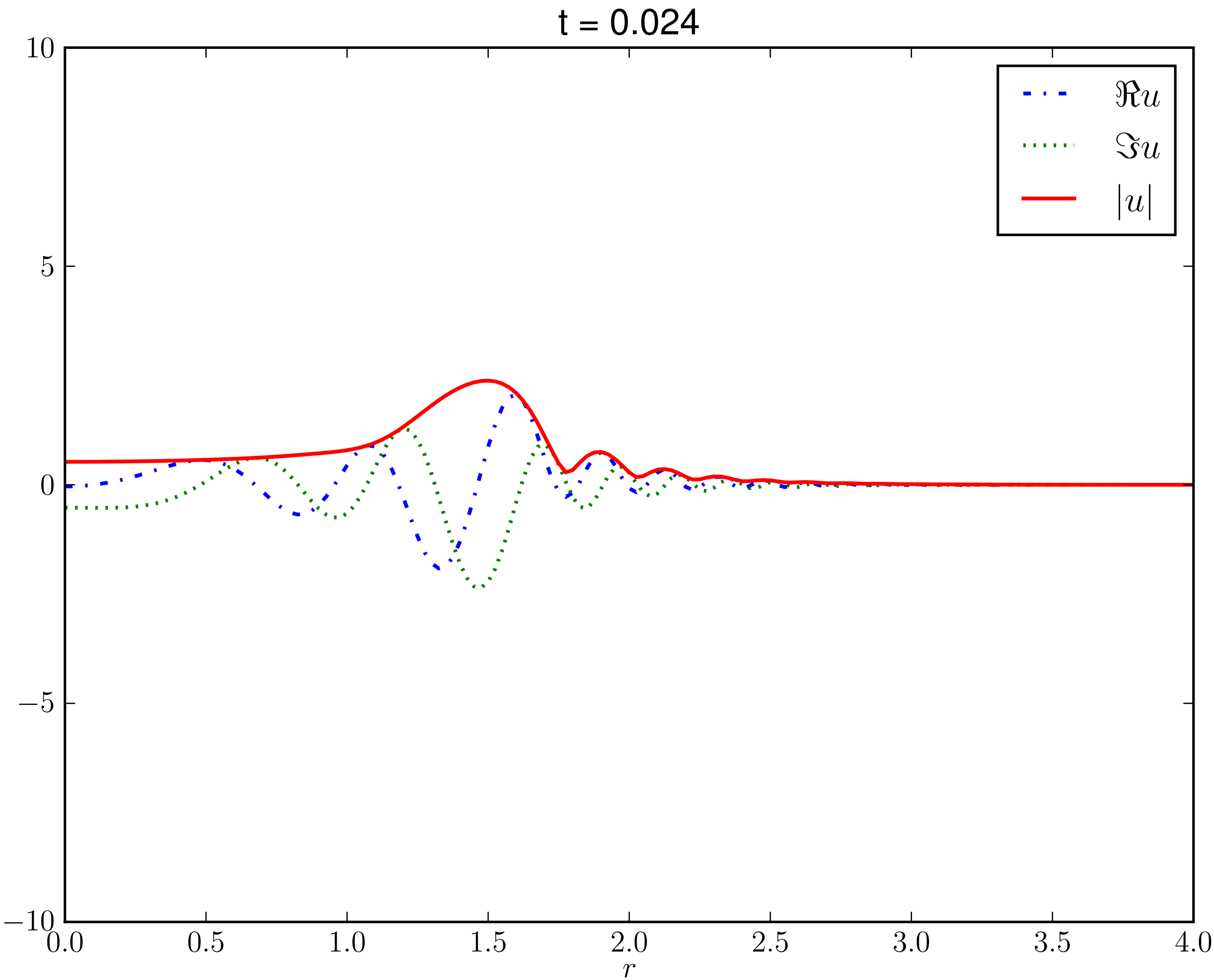

t = 0.018
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

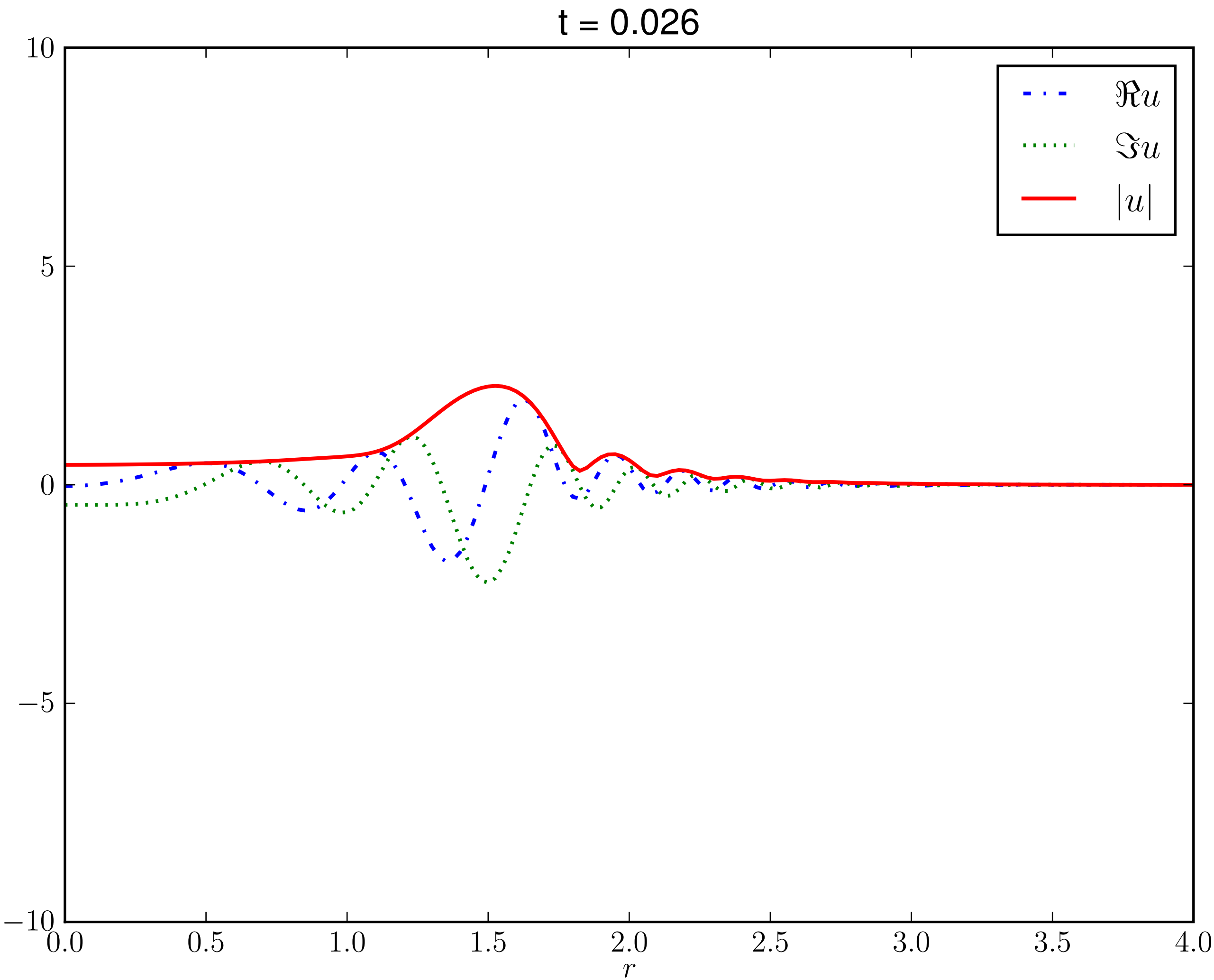


t = 0.02
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

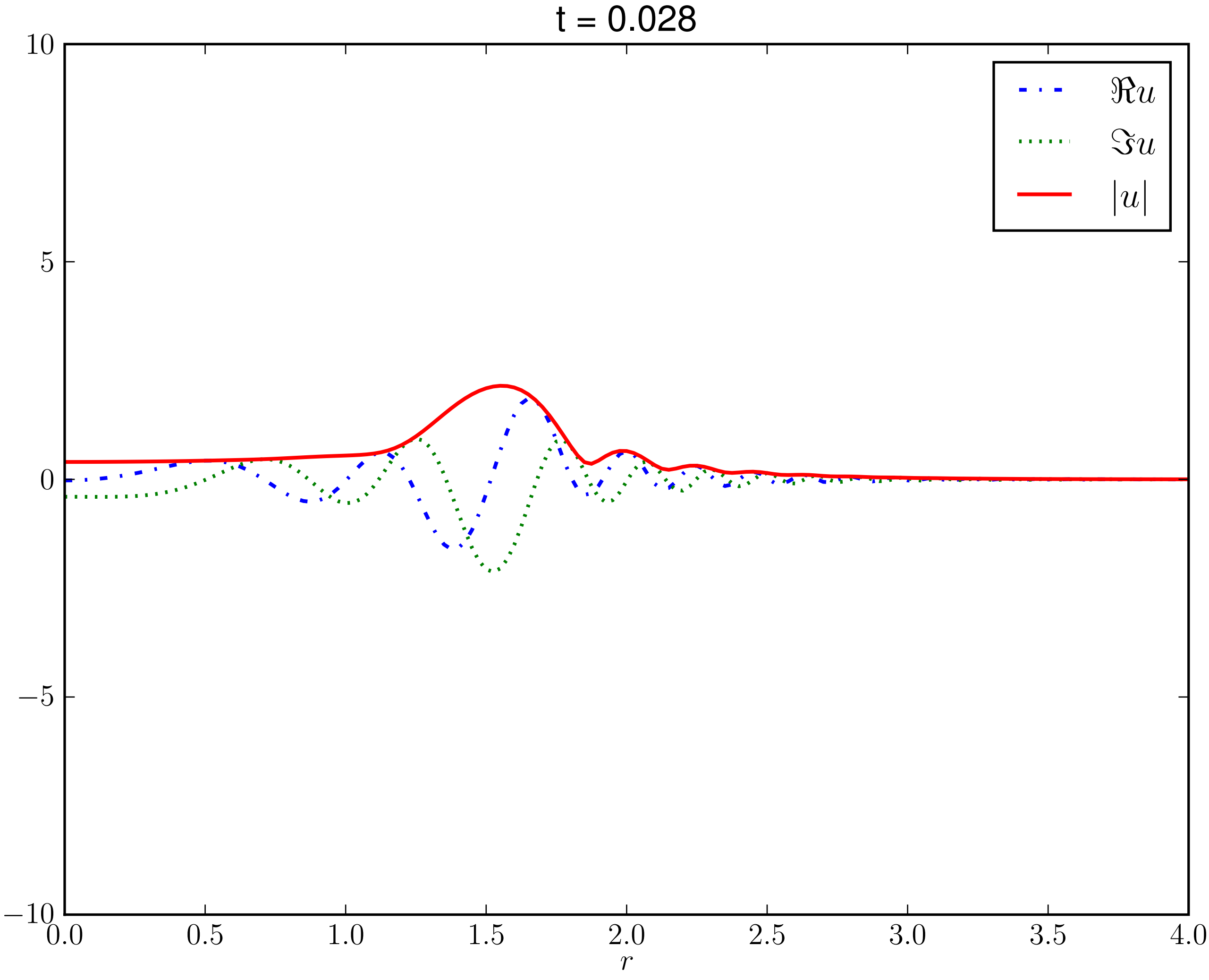


t = 0.022
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

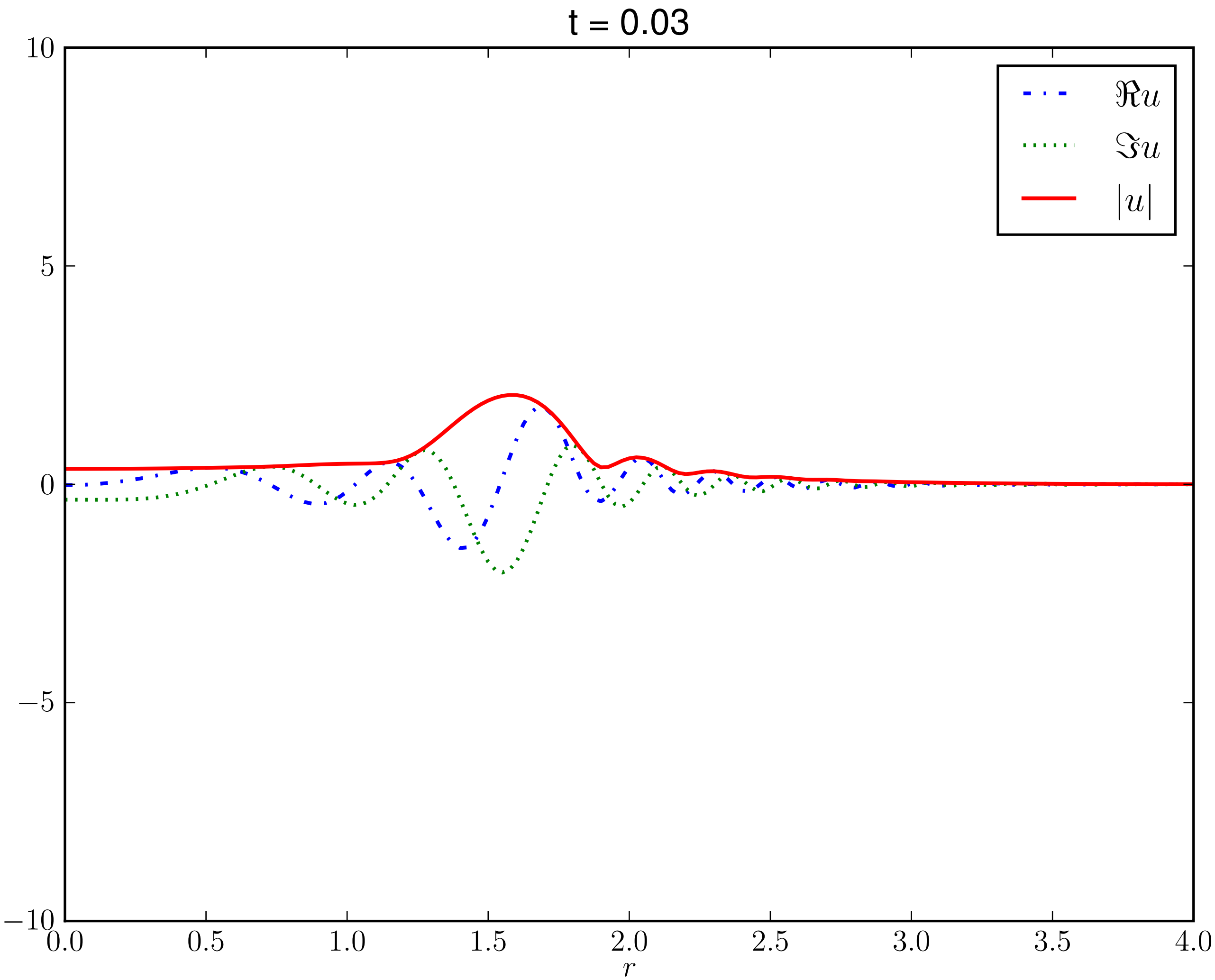

t = 0.024
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



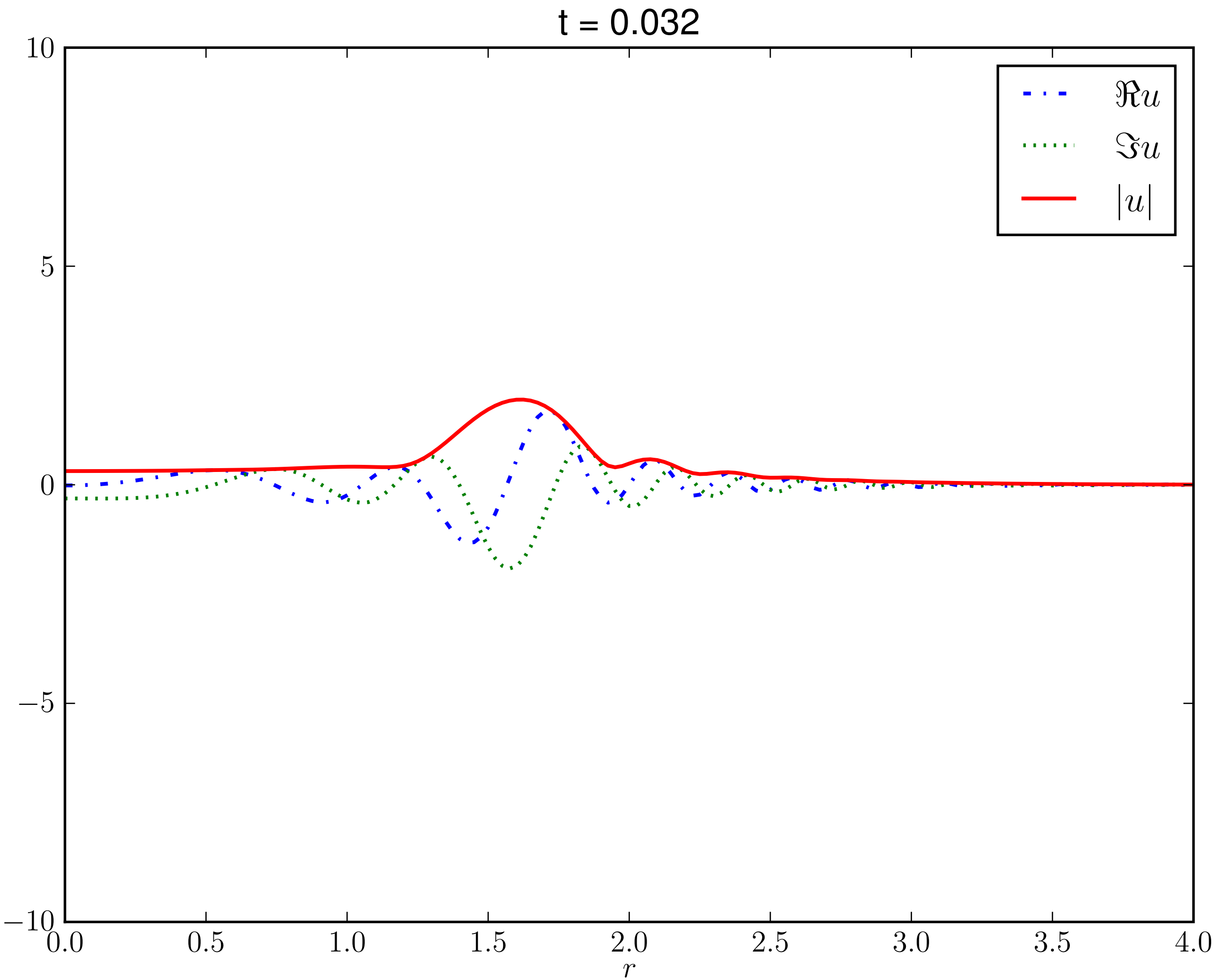
t = 0.026
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



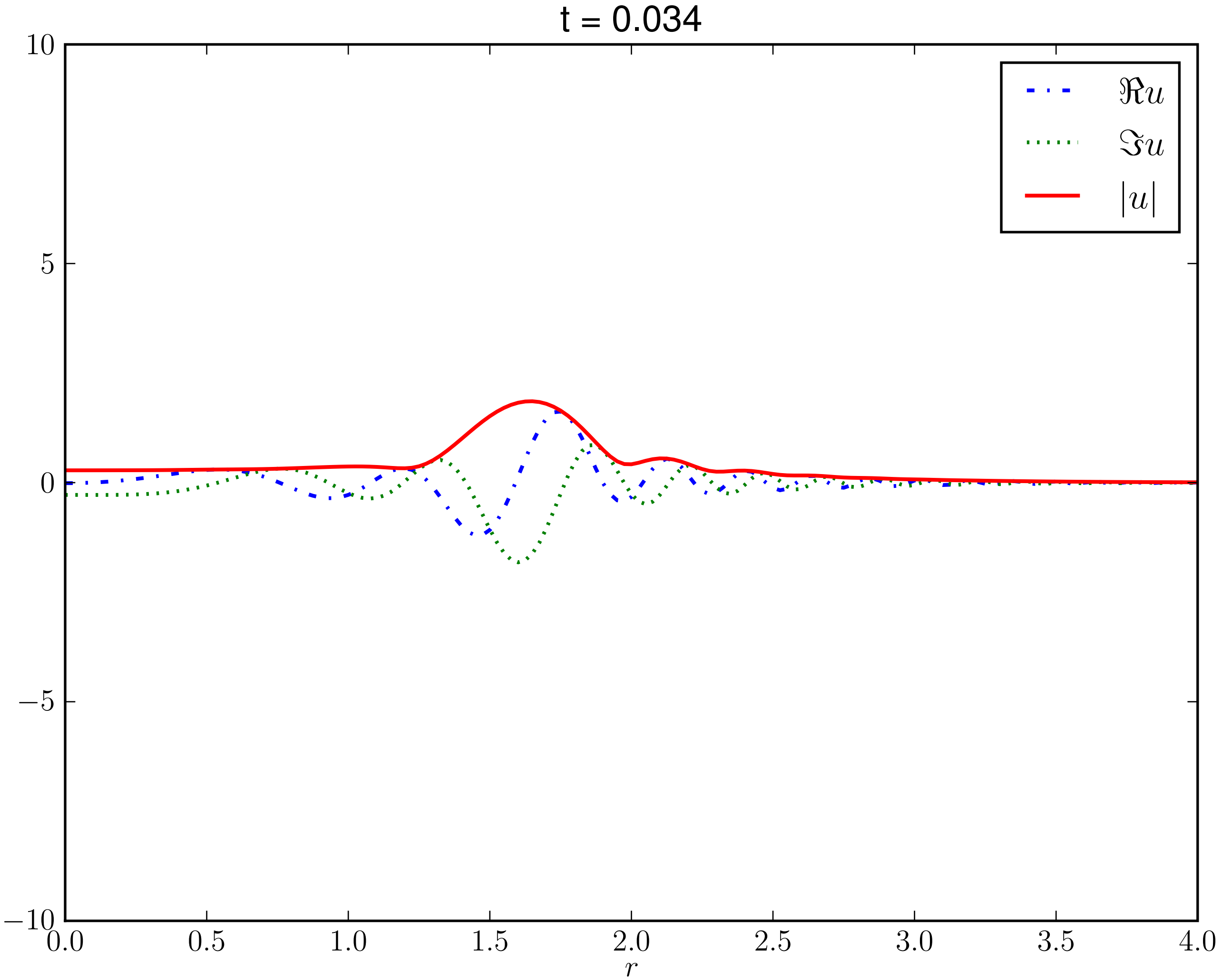
t = 0.028
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

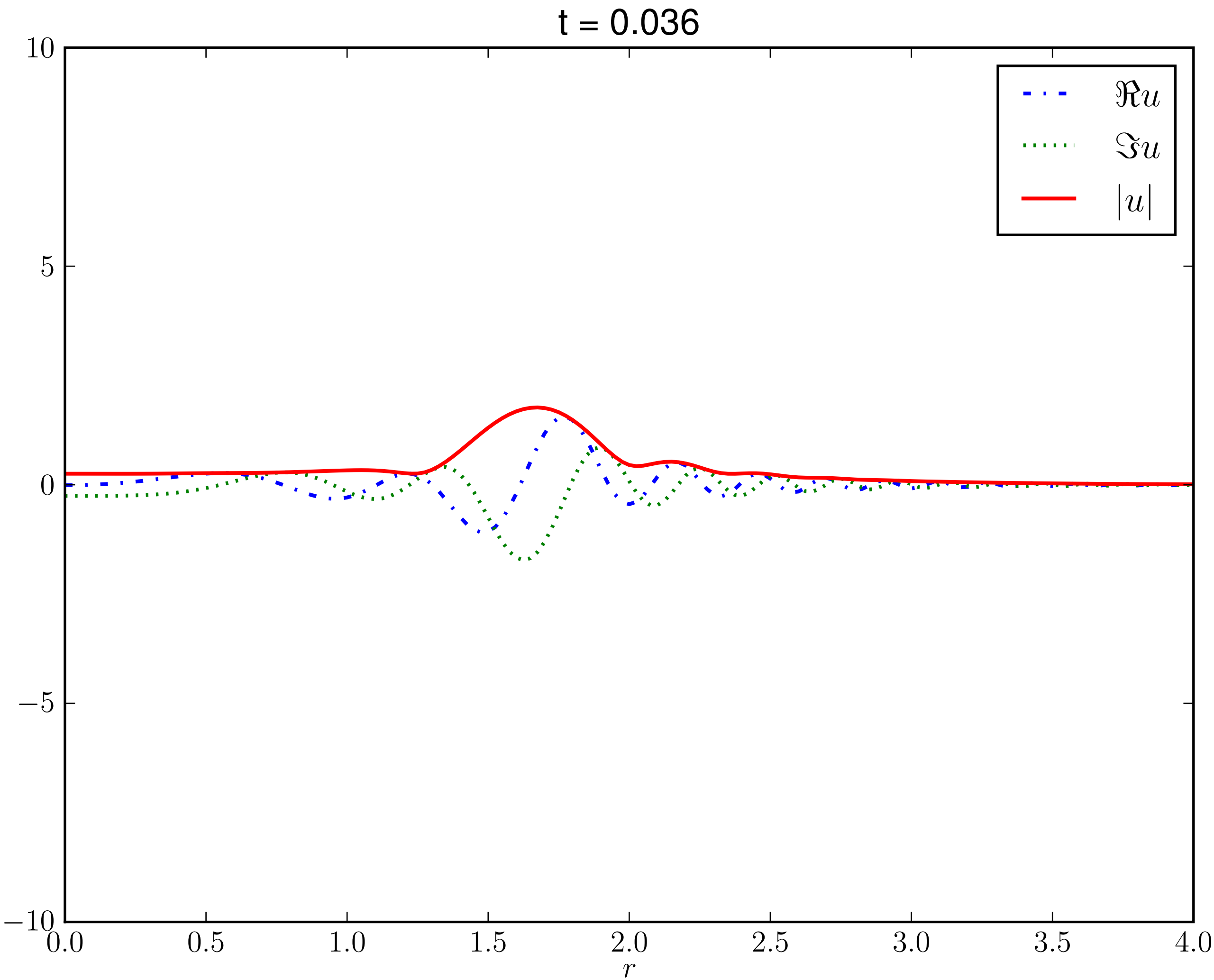


t = 0.03
$\Re u$
$\Im u$
$|u|$
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

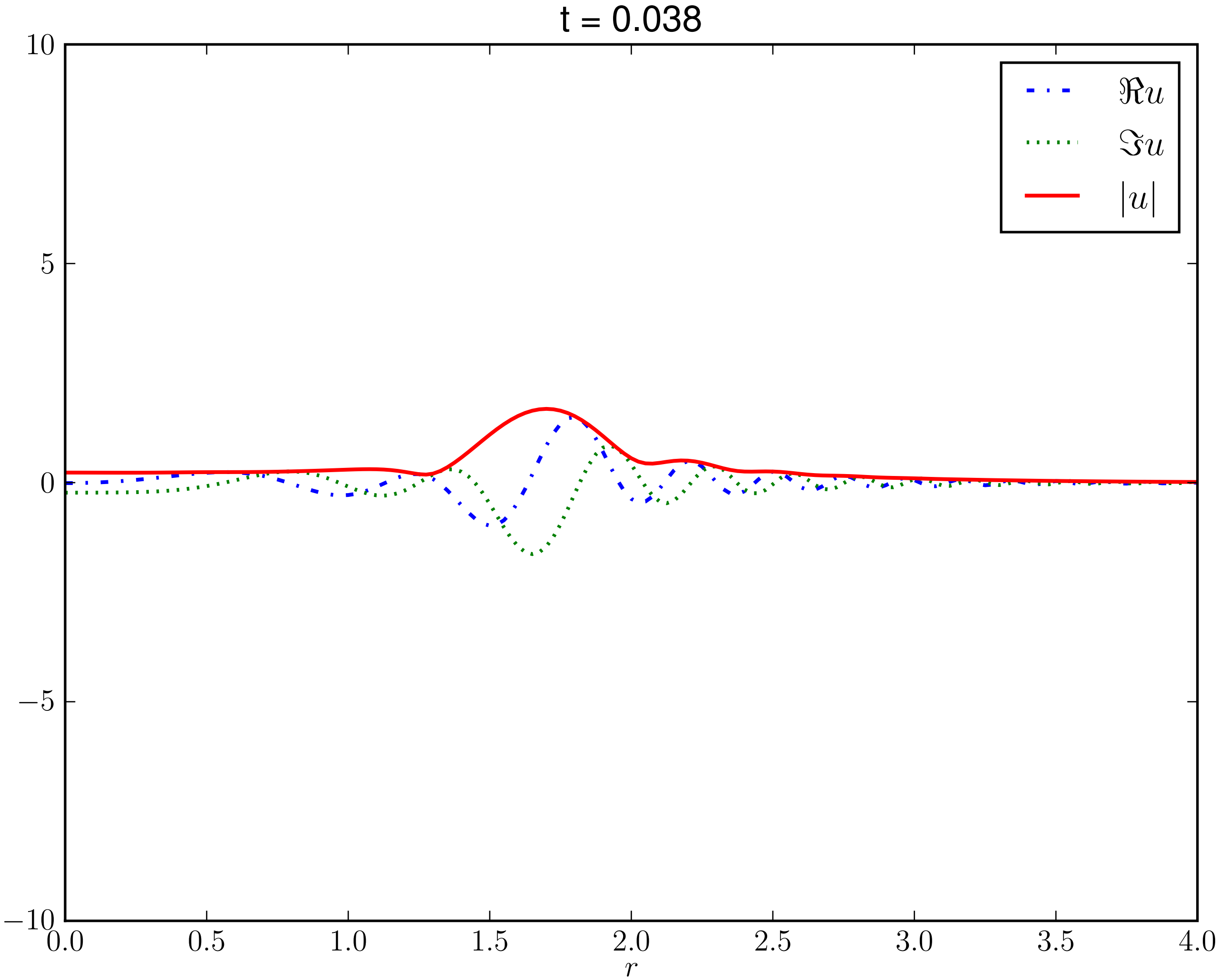


t = 0.032
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

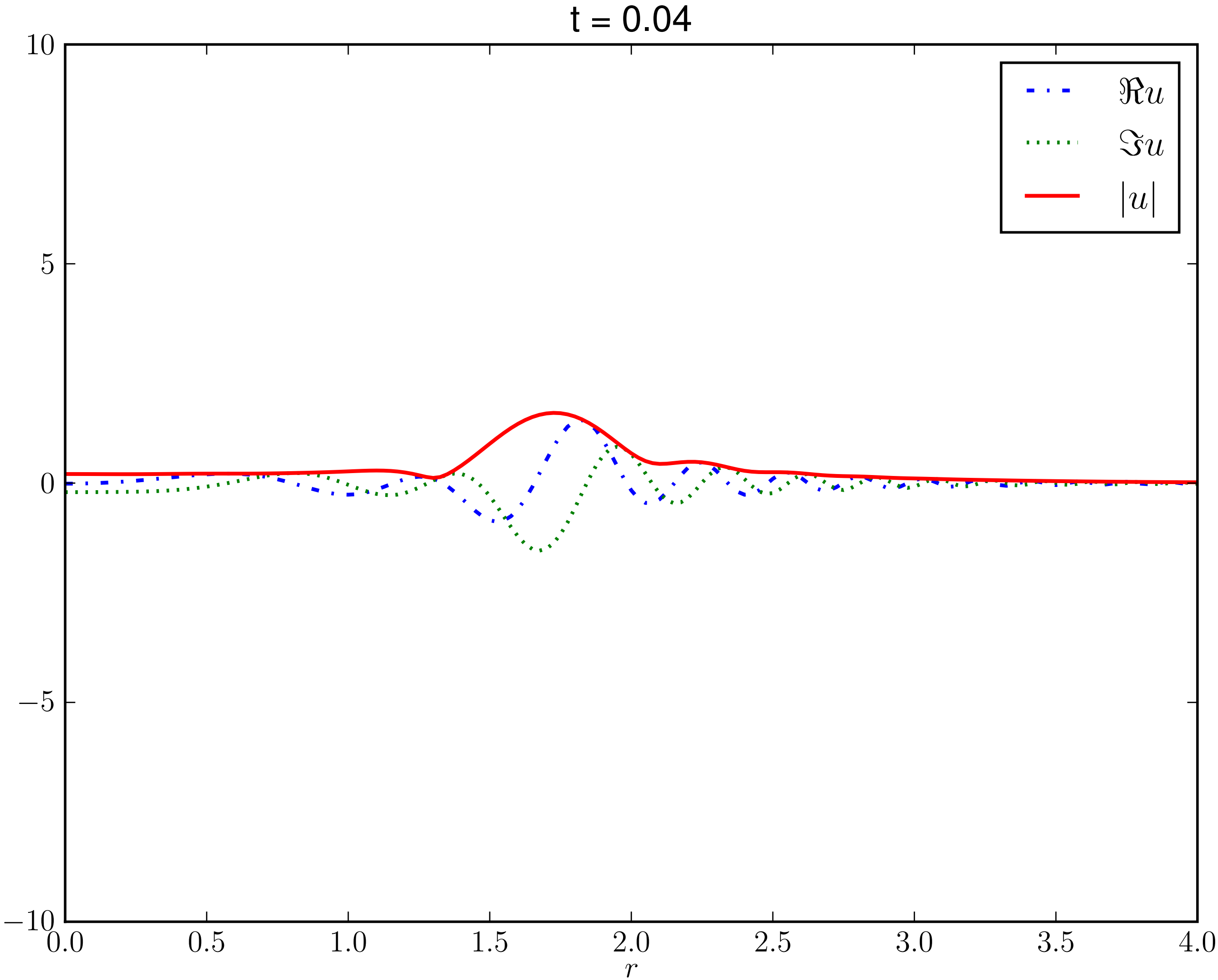

t = 0.034
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



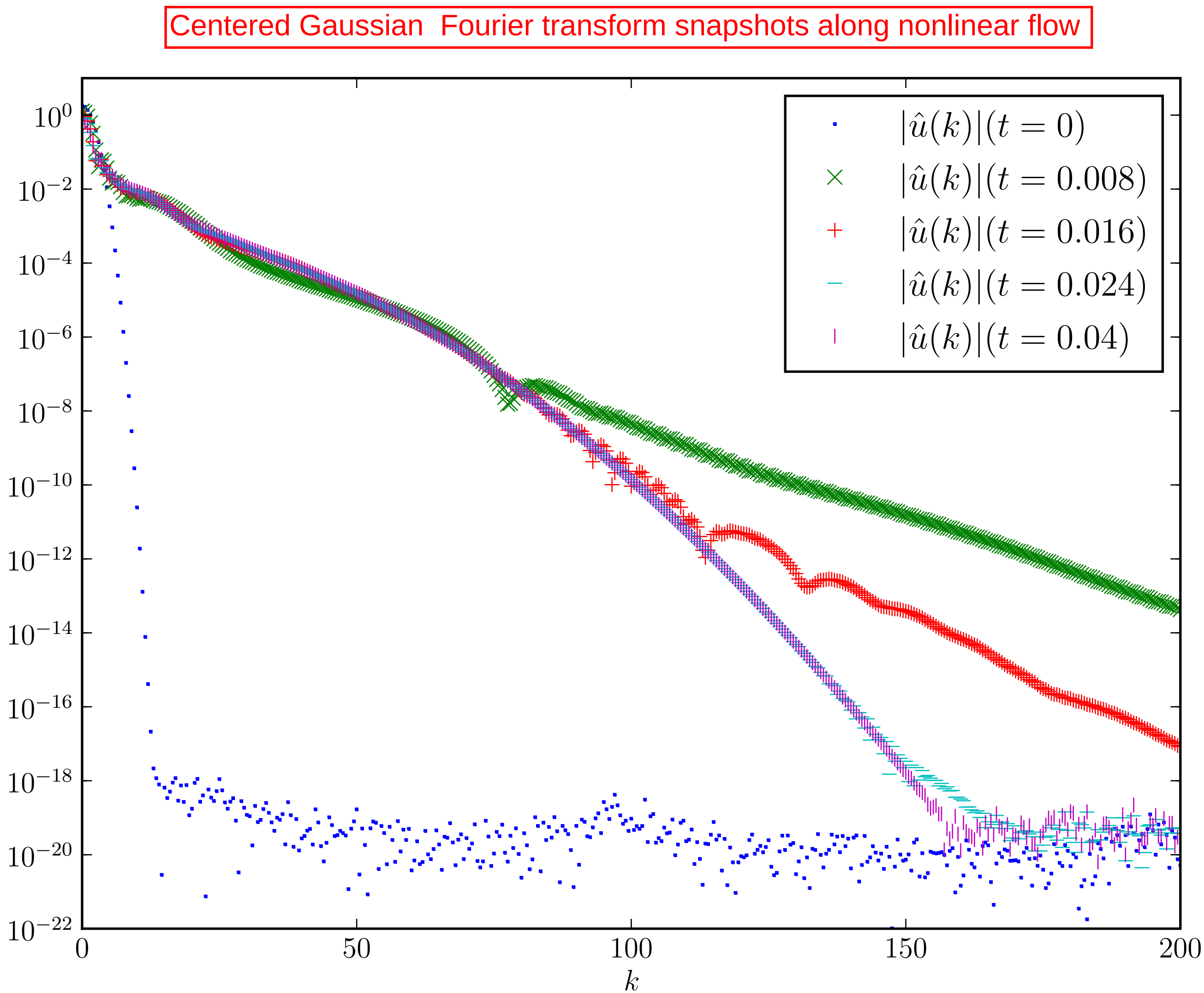
t = 0.036
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

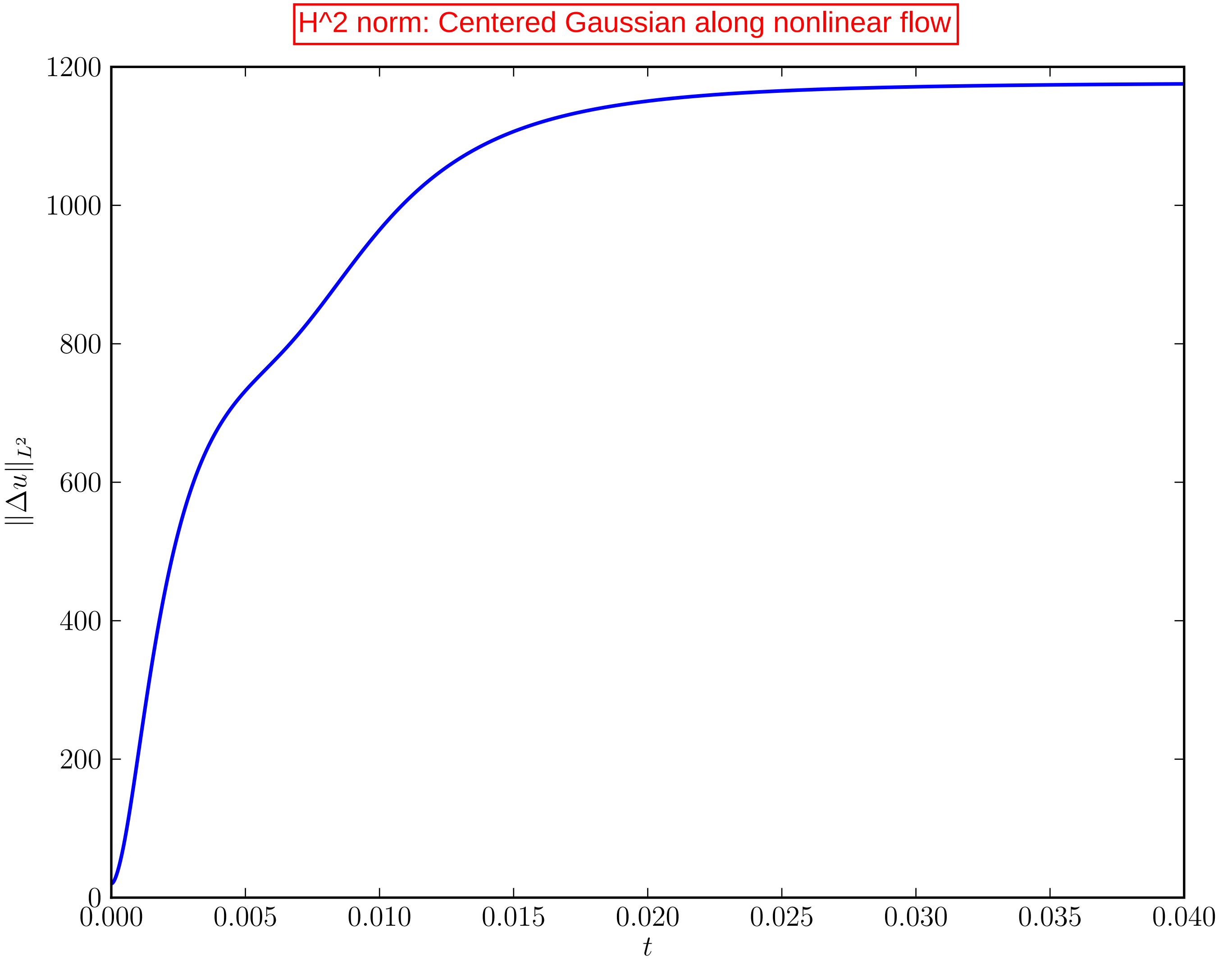


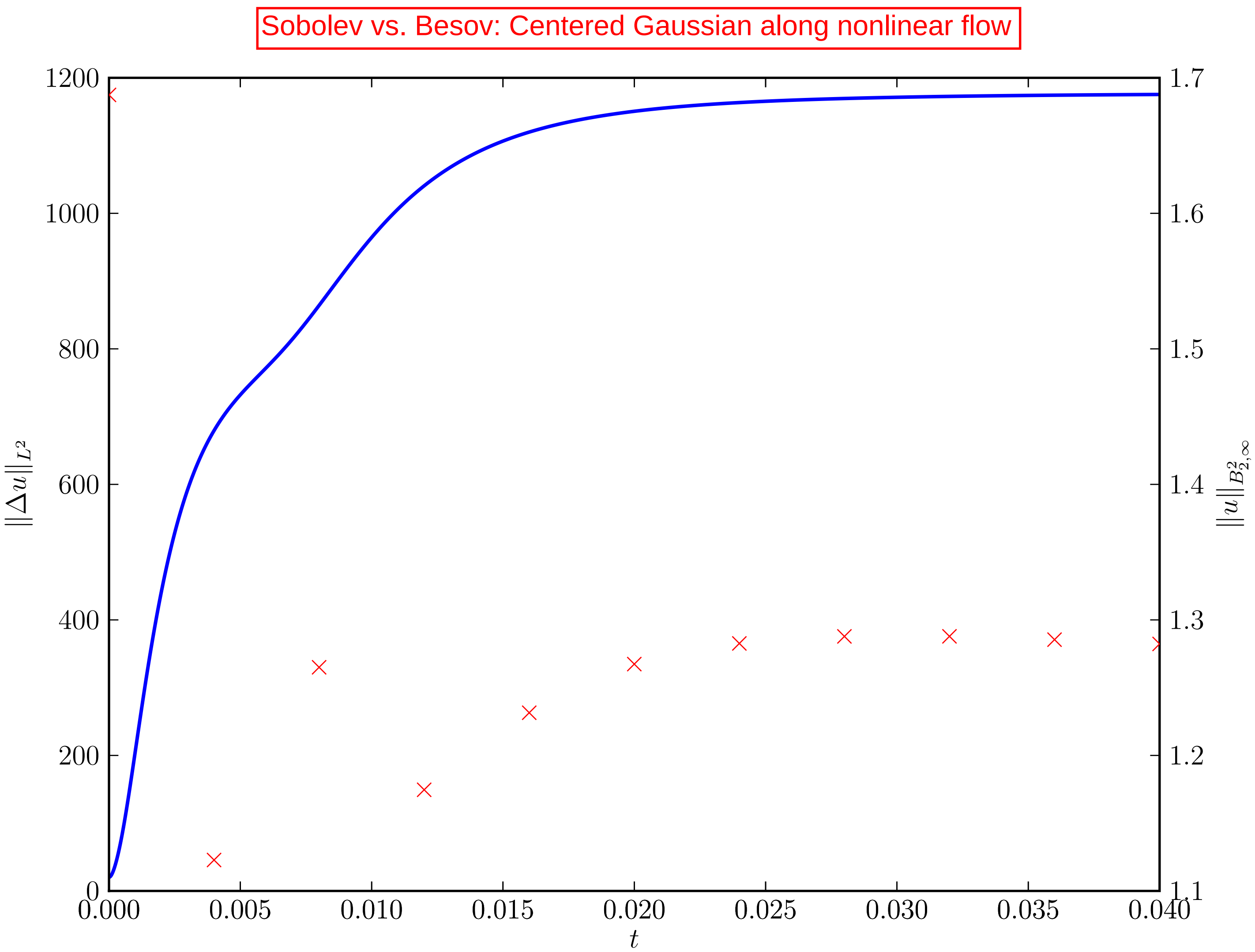
t = 0.038
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



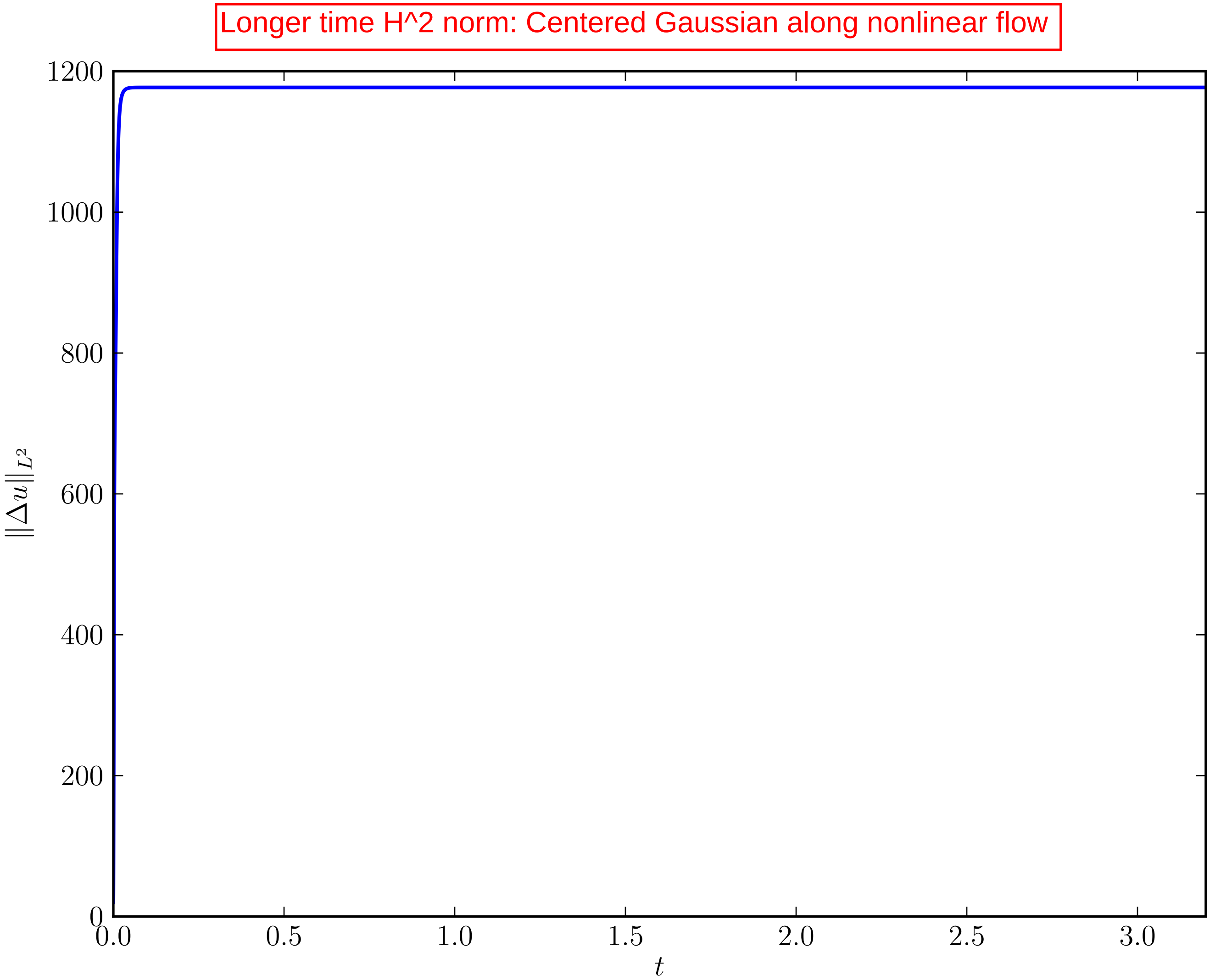
t = 0.04
10
5
0
−5
−10
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

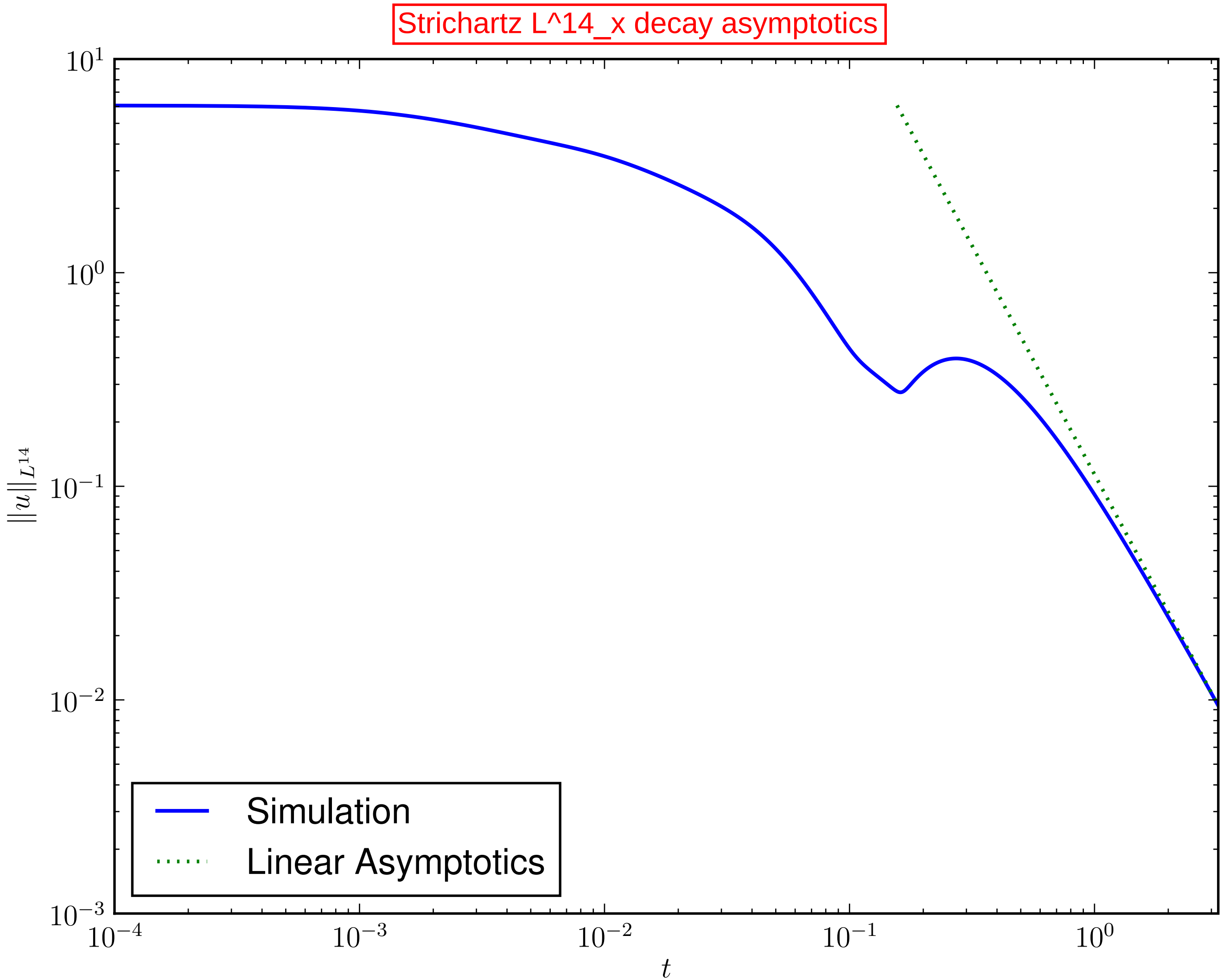

Centered Gaussian Fourier transform snapshots along nonlinear flow
$|\hat{u}(k)|(t = 0)$
$|\hat{u}(k)|(t = 0.008)$
$|\hat{u}(k)|(t = 0.016)$
$|\hat{u}(k)|(t = 0.024)$
$|\hat{u}(k)|(t = 0.04)$
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
0
50
100
150
200
k

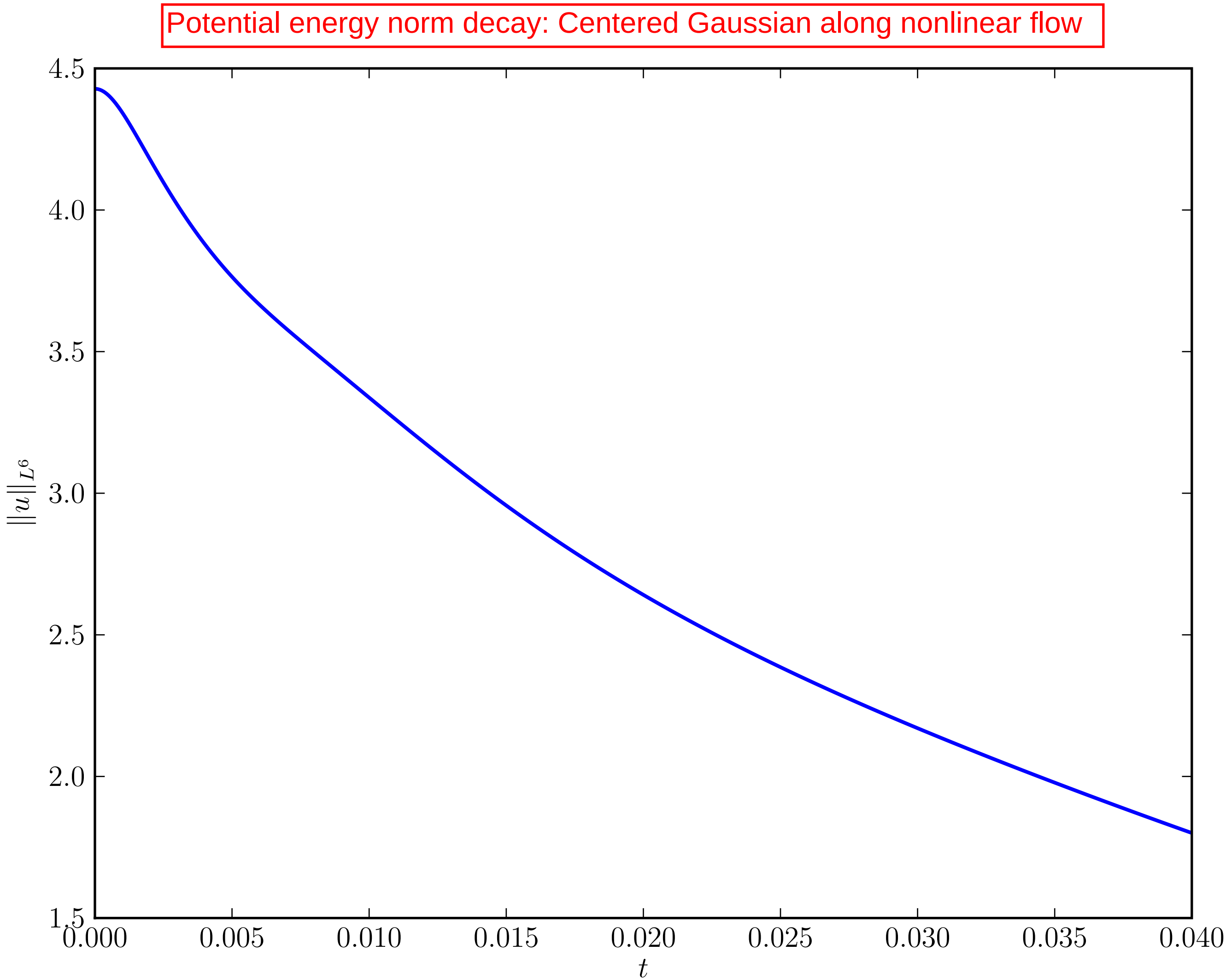

H^2 norm: Centered Gaussian along nonlinear flow
1200
1000
800
600
400
200
0
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
t
$\|\Delta u\|_{L^2}$



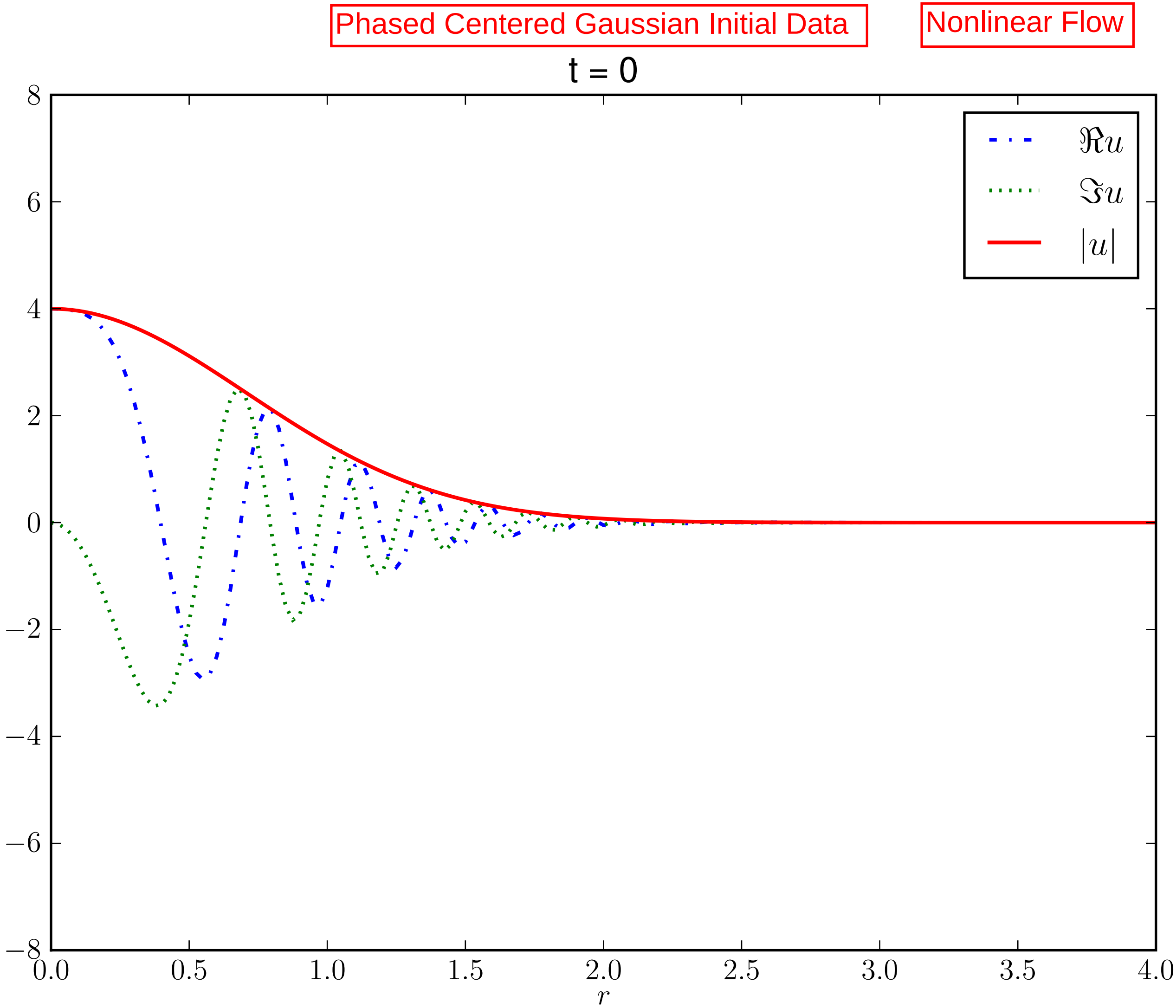
Sobolev vs. Besov: Centered Gaussian along nonlinear flow
1200
1000
800
600
400
200
0
1.7
1.6
1.5
1.4
1.3
1.2
1.1
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
t
$\|\Delta u\|_{L^2}$
$\|u\|_{B^2_{2,\infty}}$


Longer time H^2 norm: Centered Gaussian along nonlinear flow
$\|\Delta u\|_{L^2}$
t
0
200
400
600
800
1000
1200
0.0
0.5
1.0
1.5
2.0
2.5
3.0

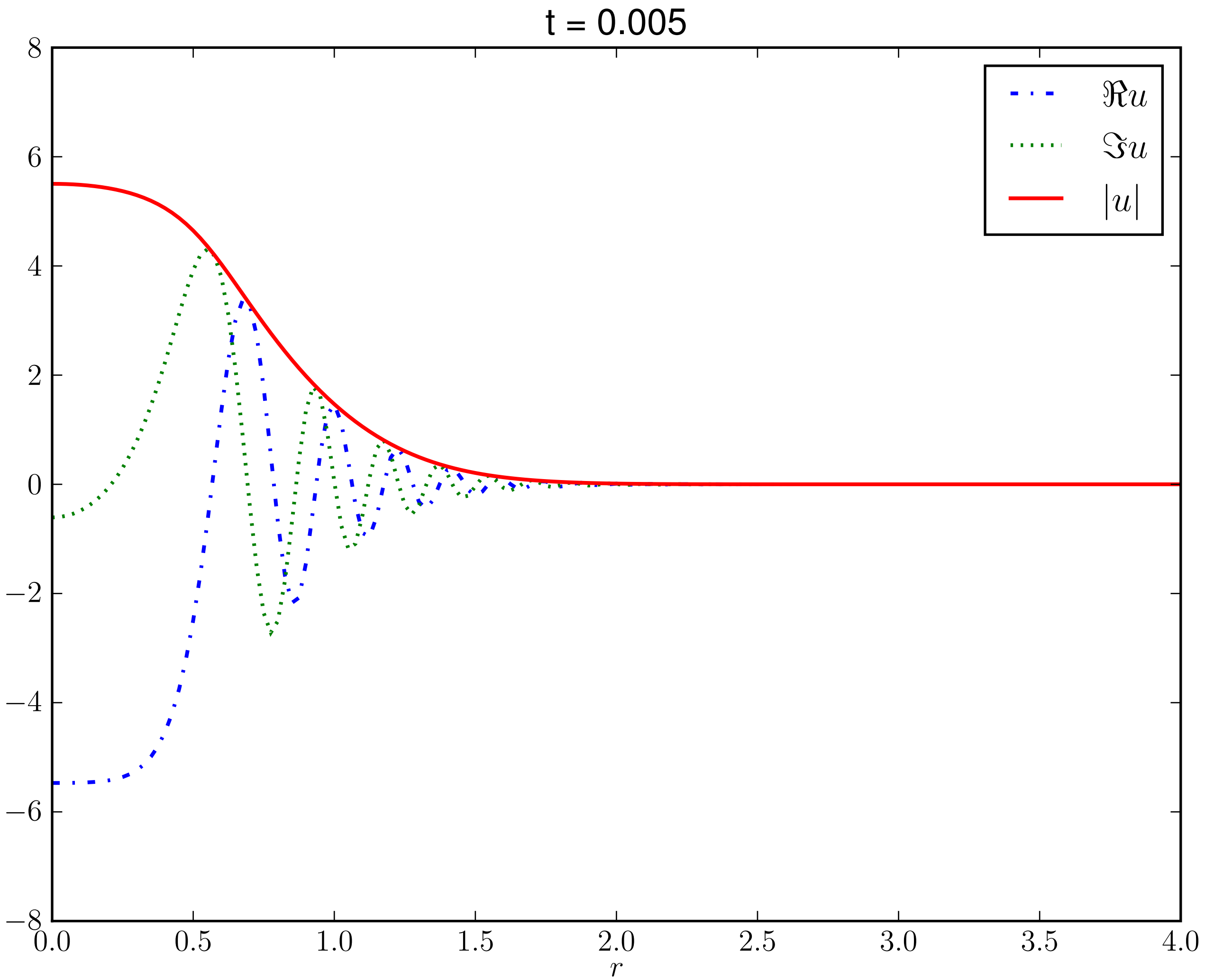

Strichartz L^14_x decay asymptotics
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
$\|u\|_{L^{14}}$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^0
t
Simulation
Linear Asymptotics



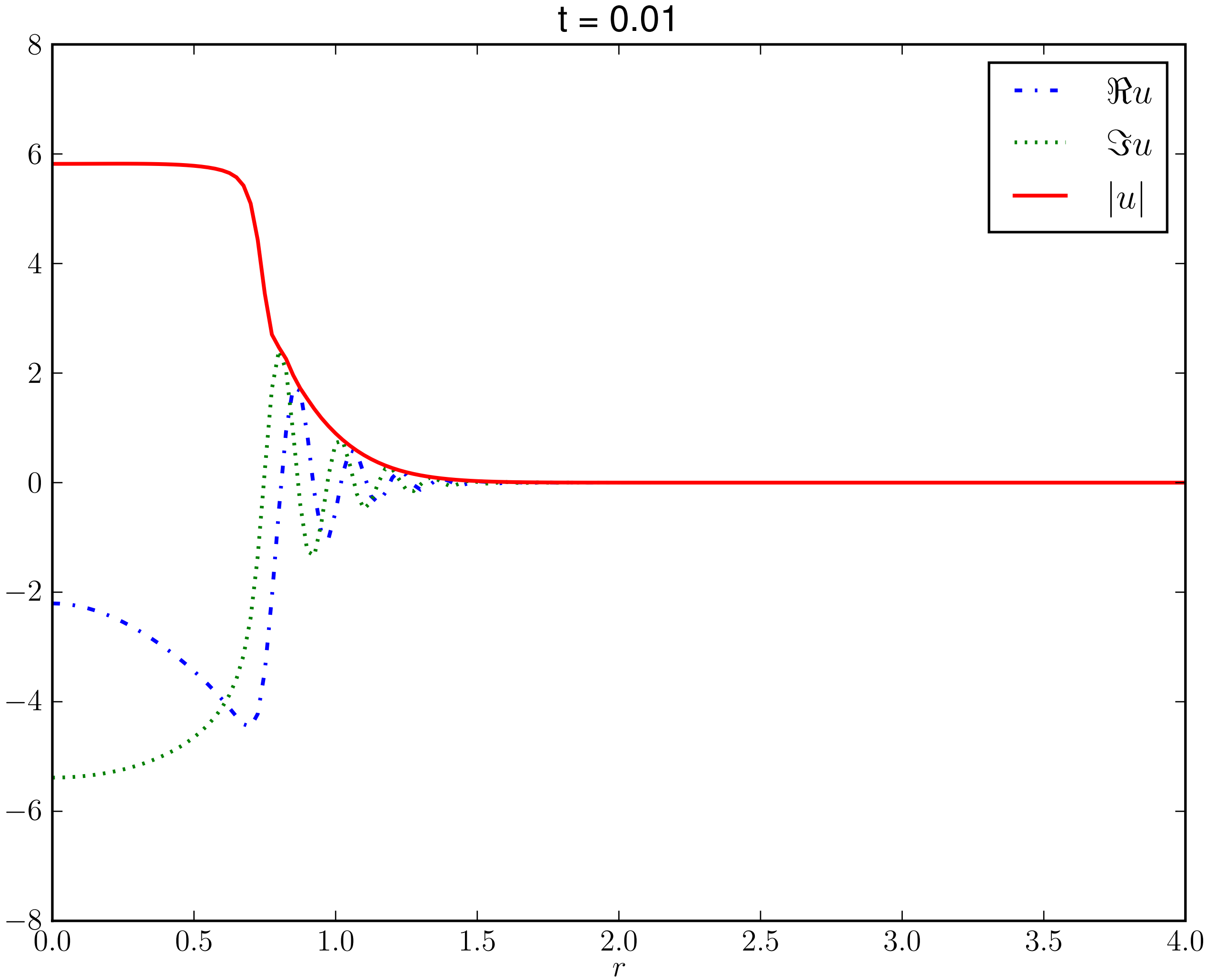
Potential energy norm decay: Centered Gaussian along nonlinear flow
4.5
4.0
3.5
3.0
2.5
2.0
1.5
$\|u\|_{L^6}$
0.000
0.005
0.010
0.015
0.020
0.025
0.030
0.035
0.040
t

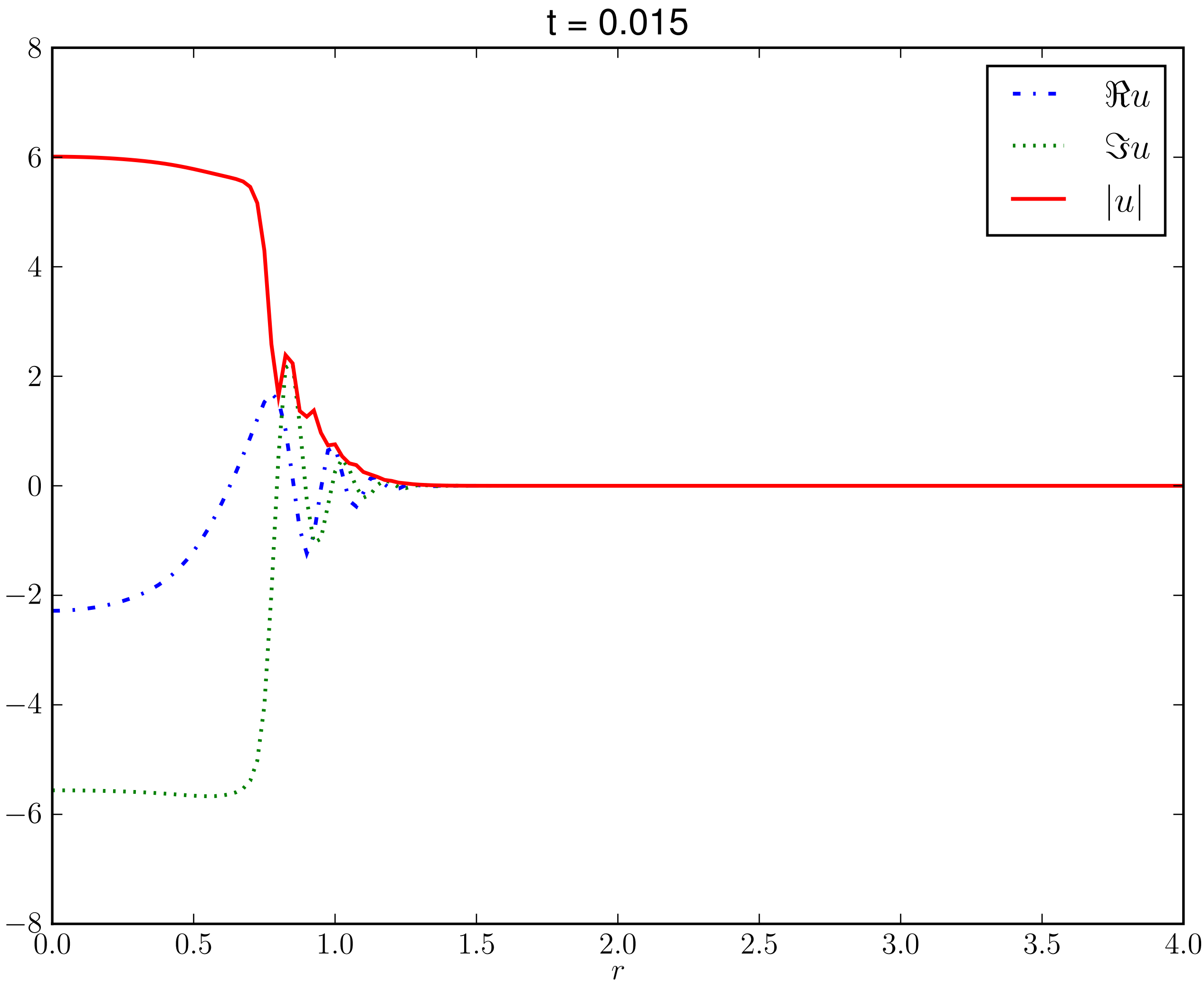


Phased Centered Gaussian Initial Data
Nonlinear Flow
t = 0
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

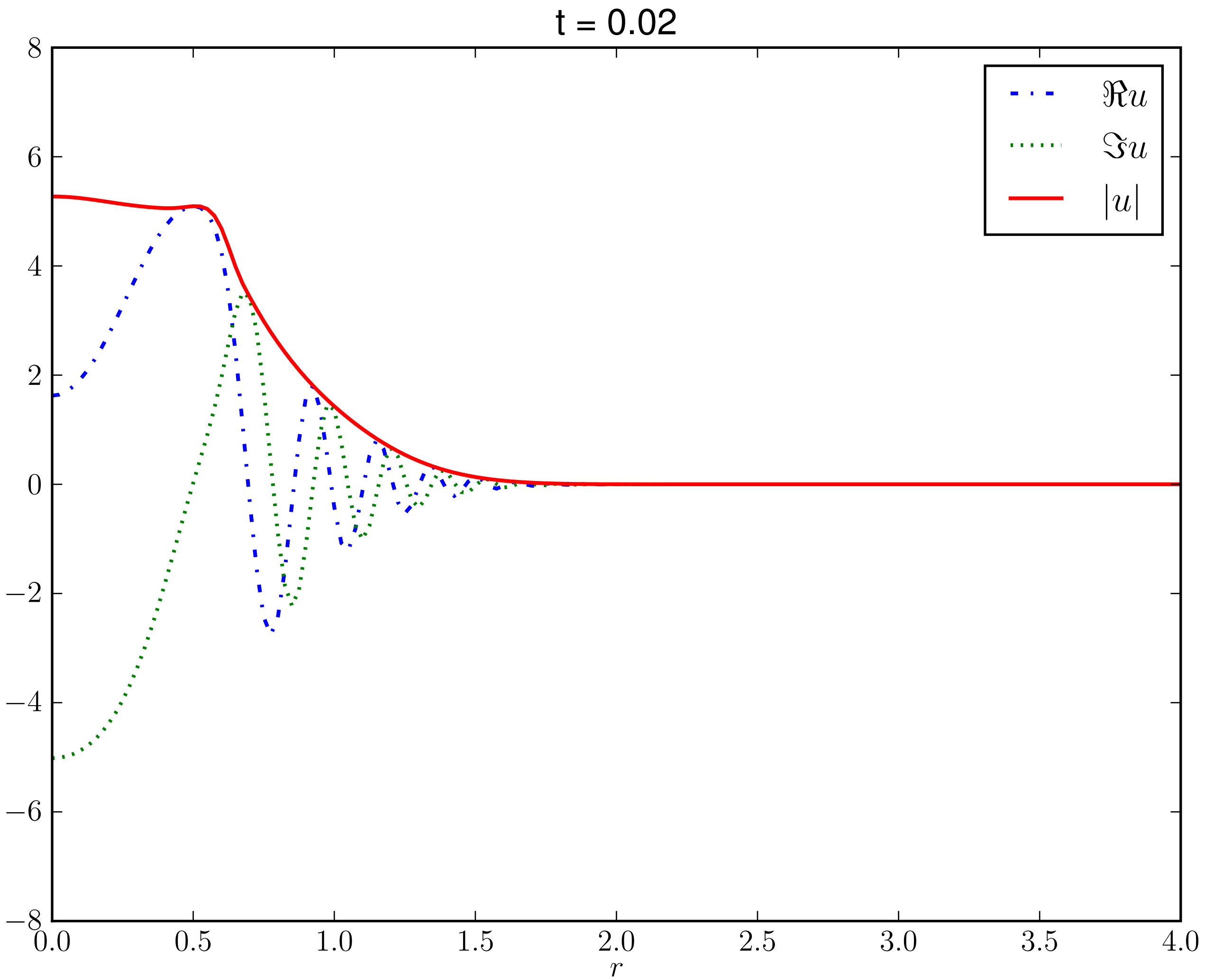


t = 0.005
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

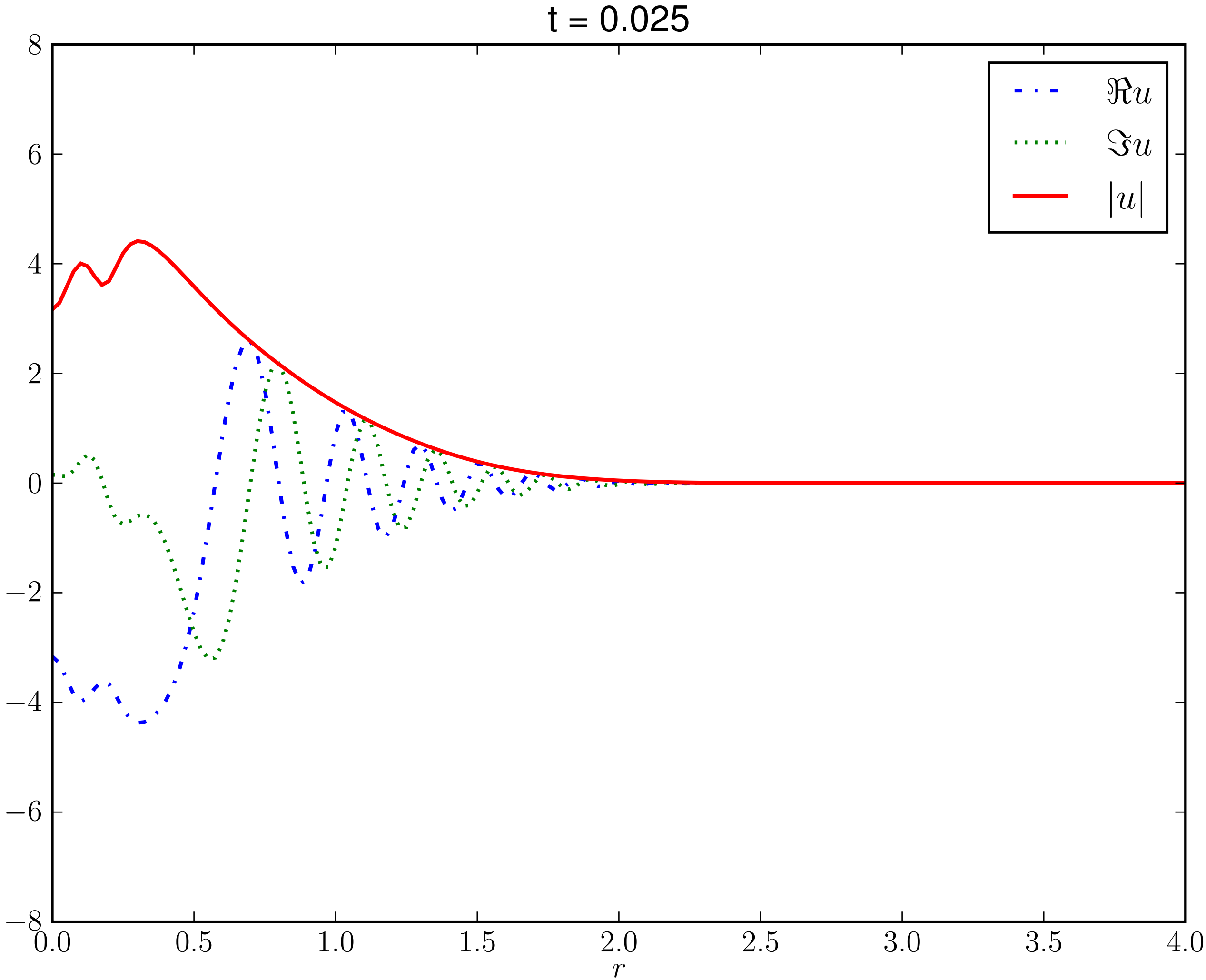

t = 0.01
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

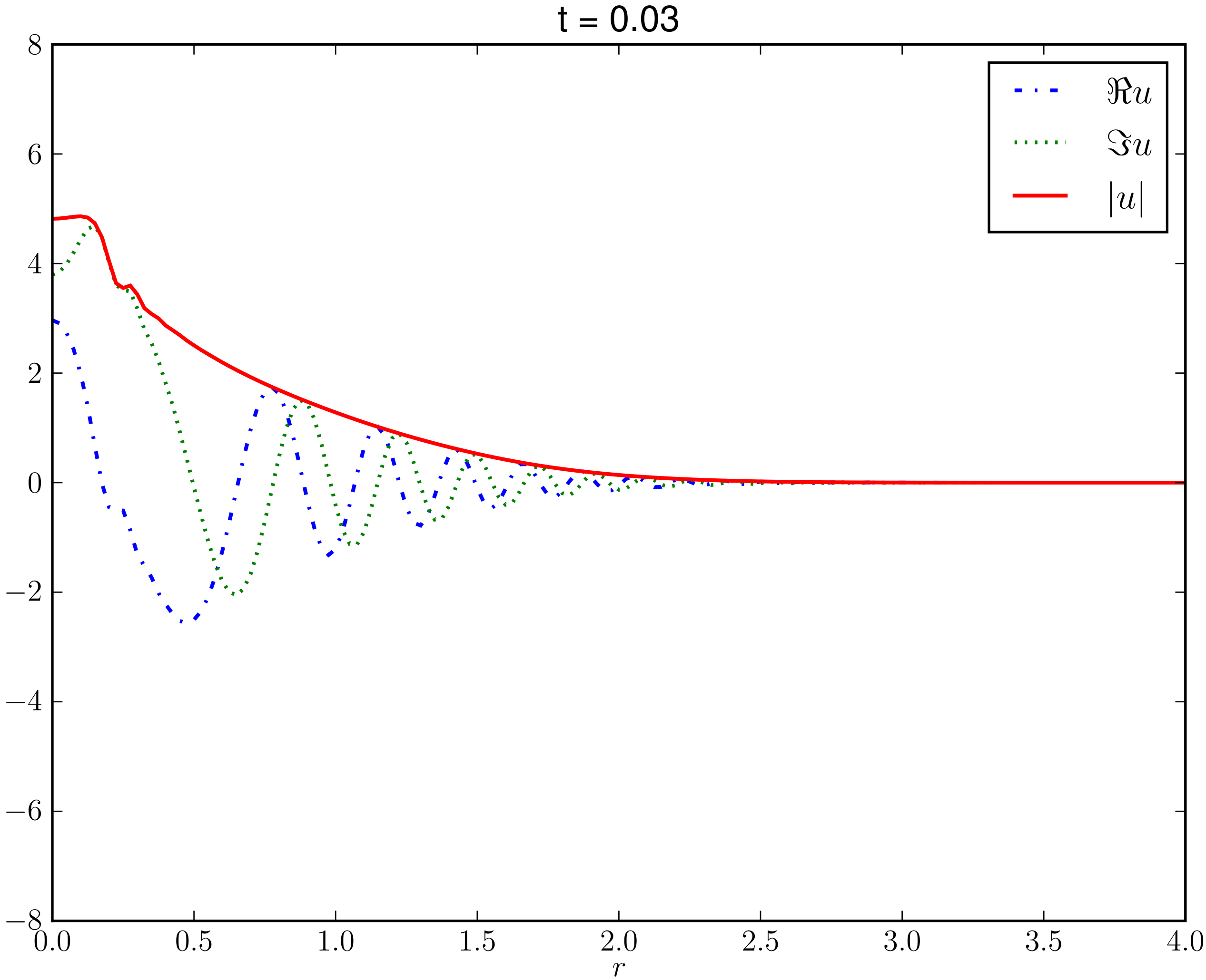


t = 0.015
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

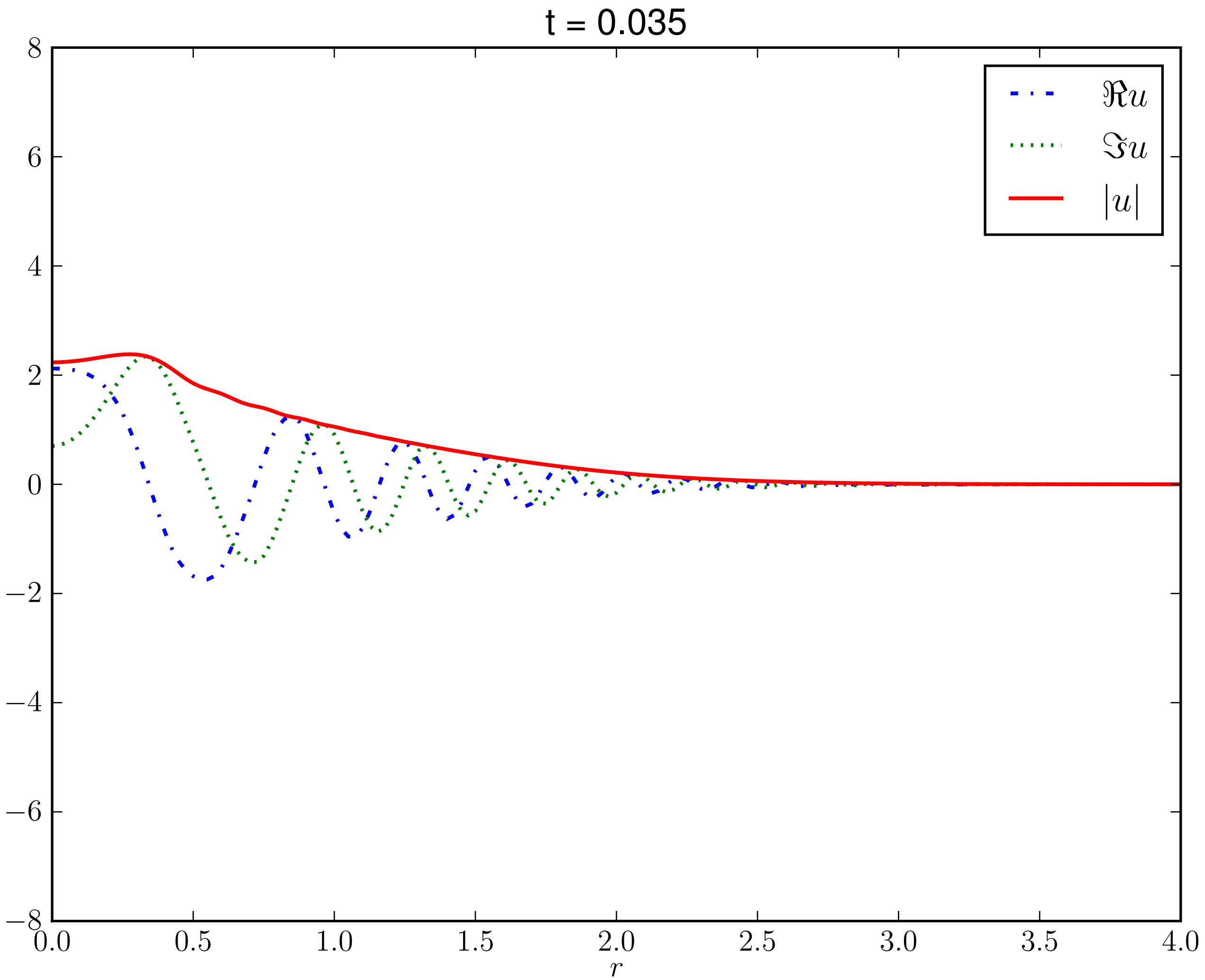


t = 0.02
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

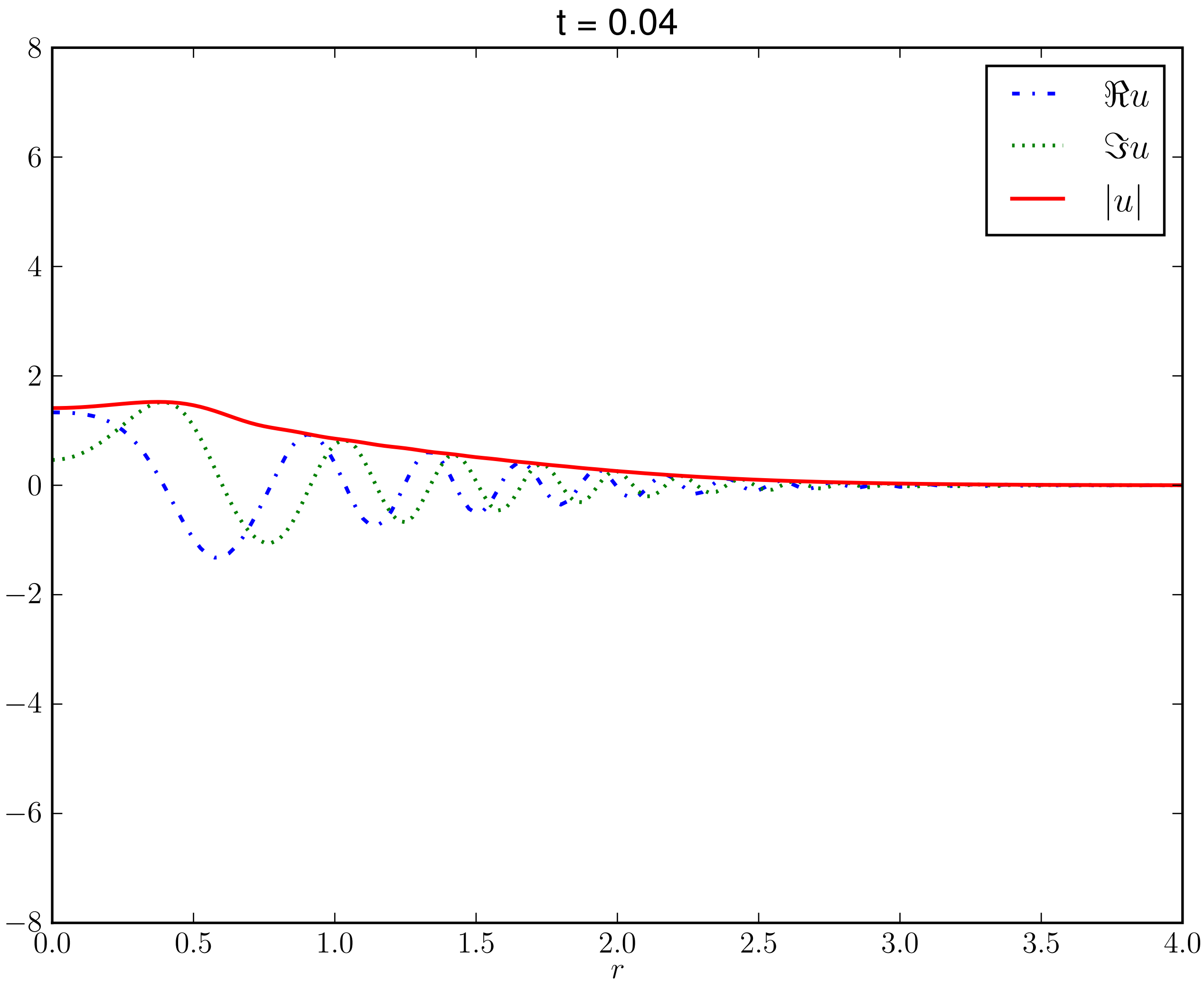

t = 0.025
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r



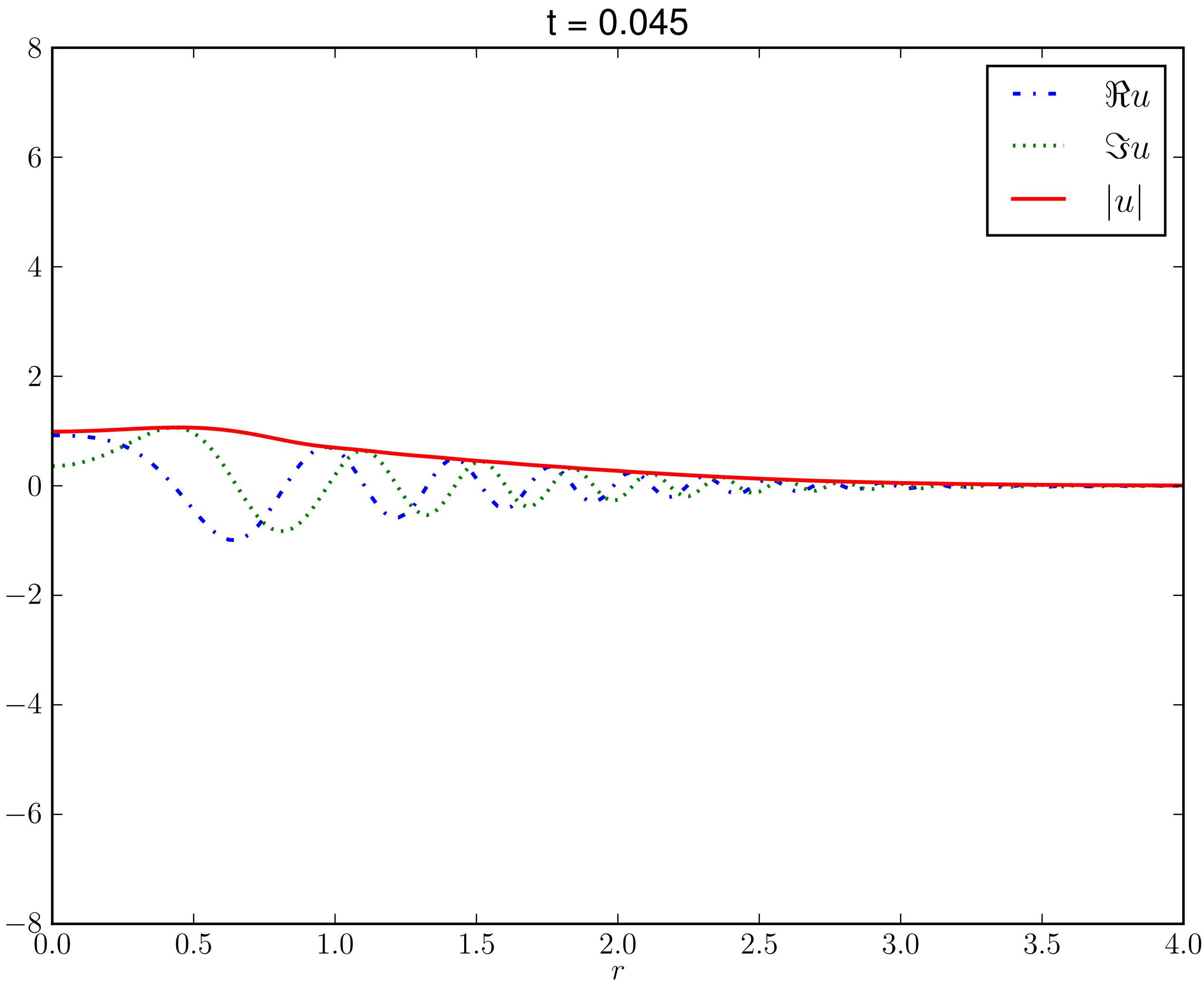
t = 0.03
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



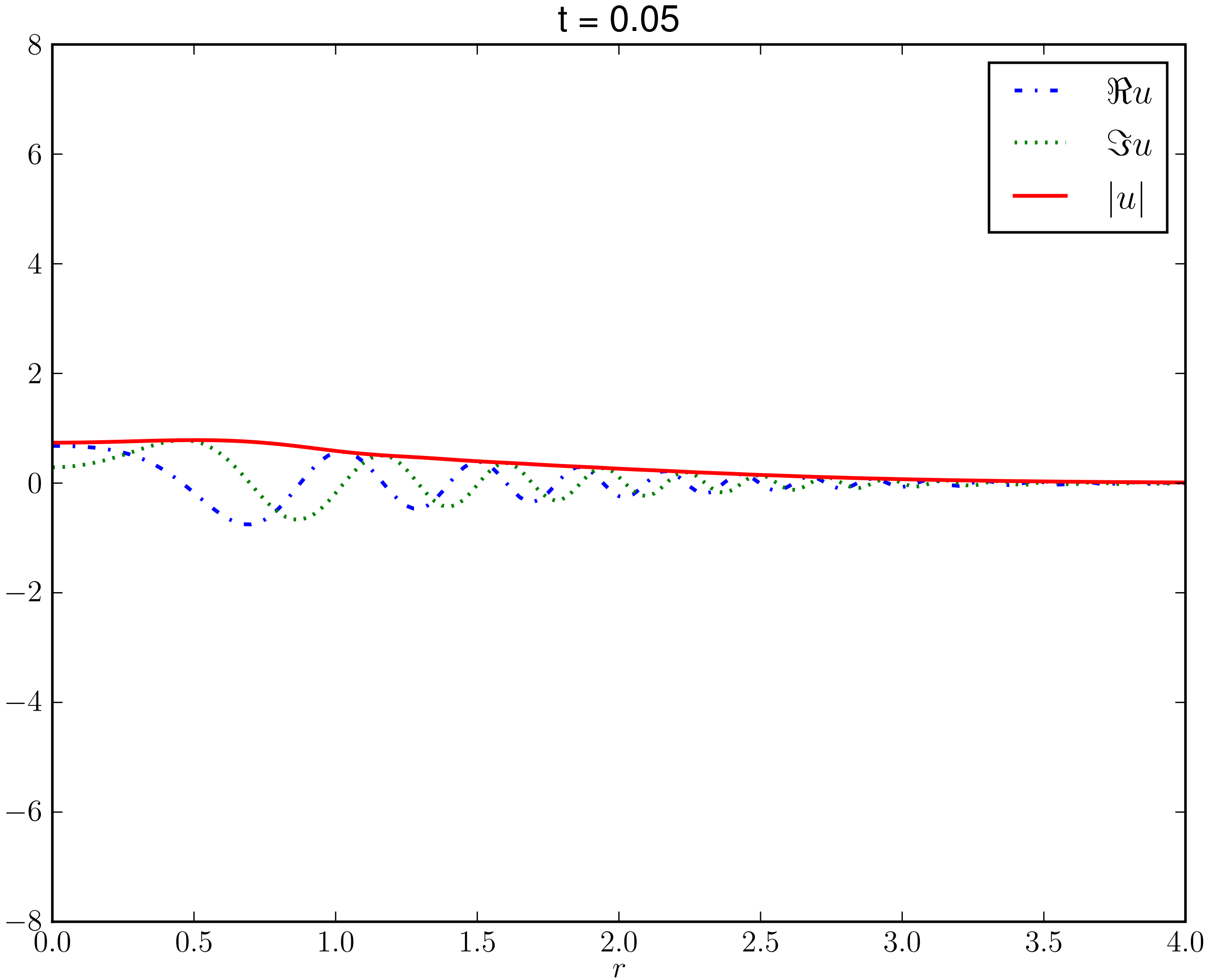
t = 0.035
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

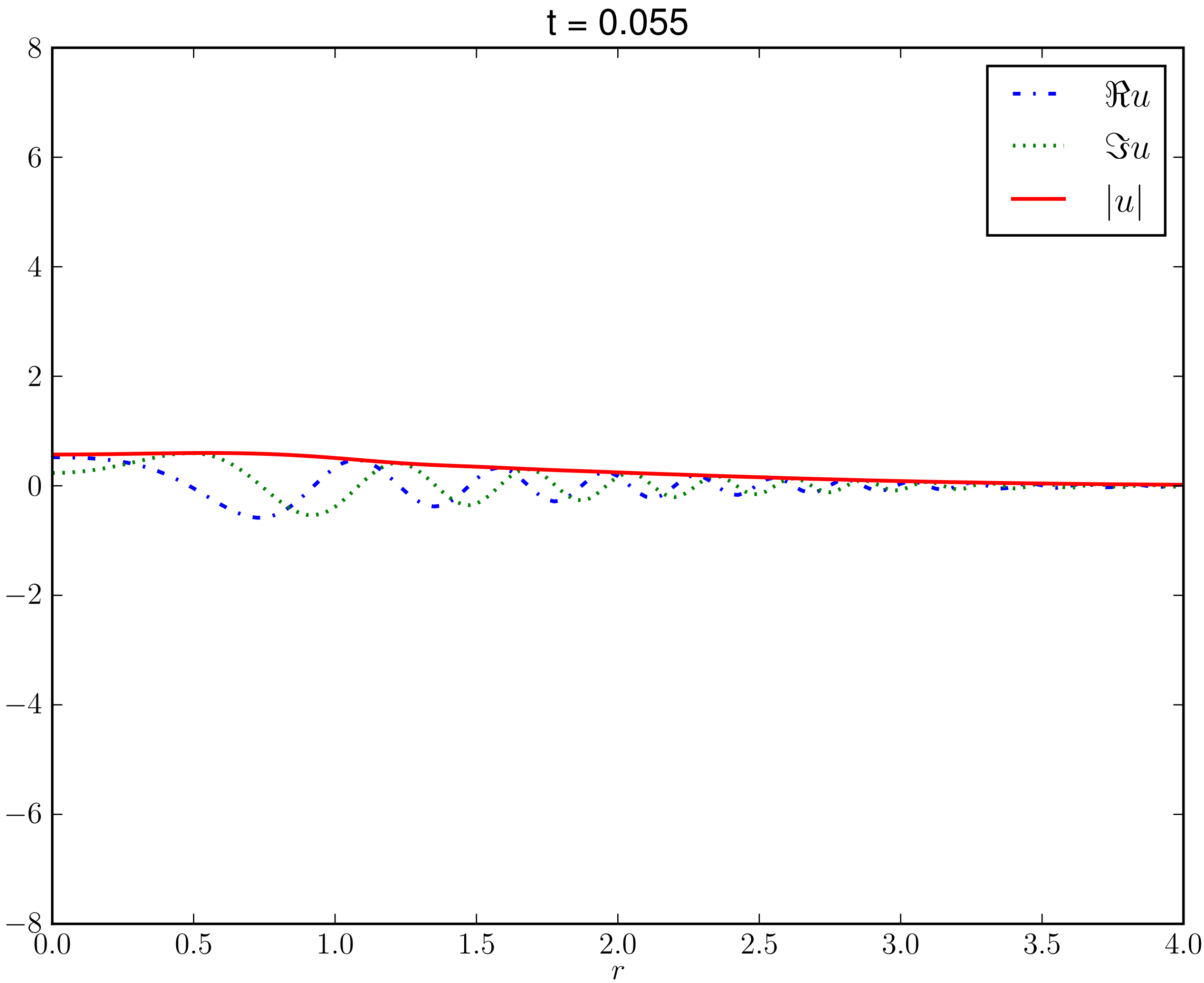


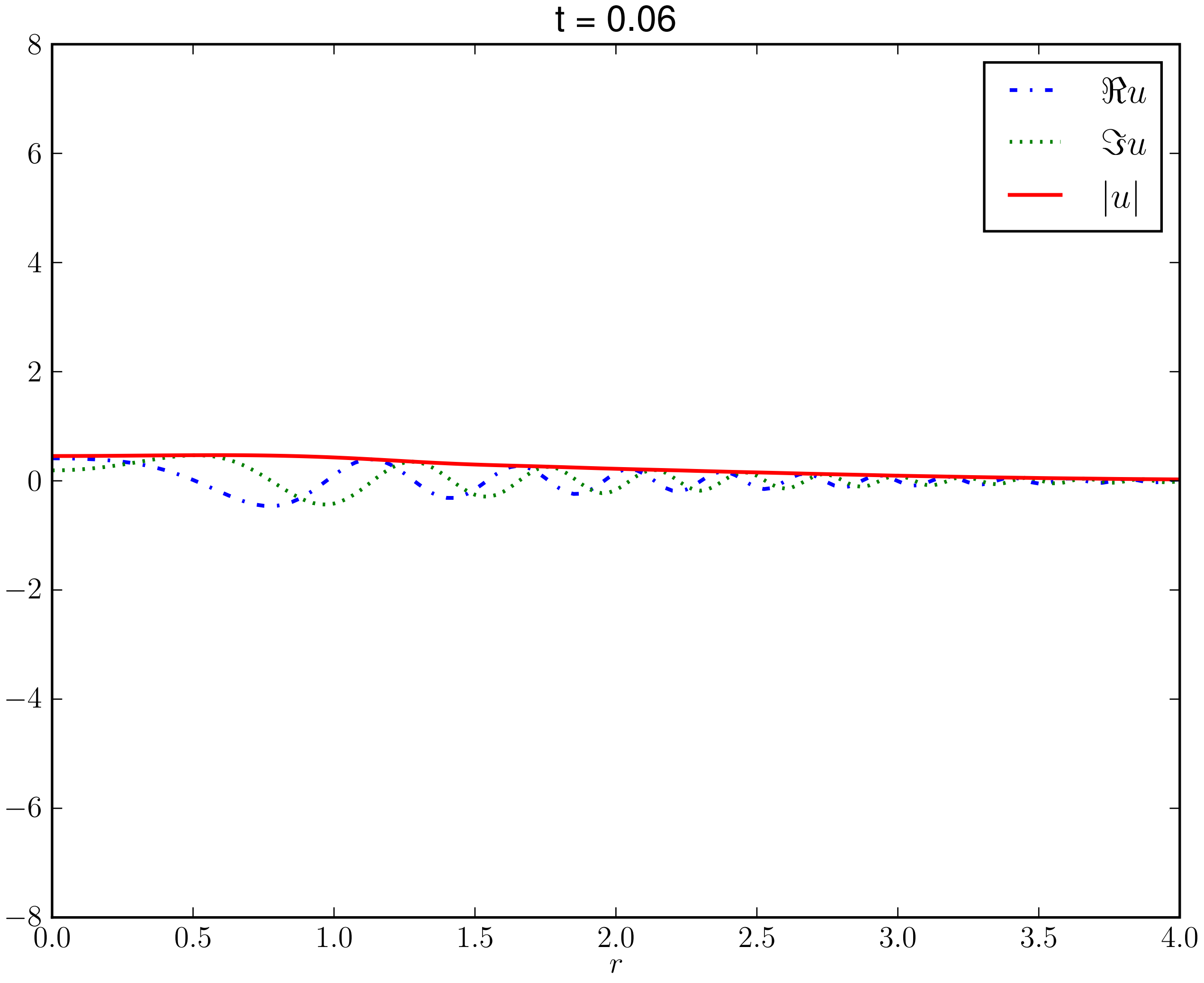
t = 0.04
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$



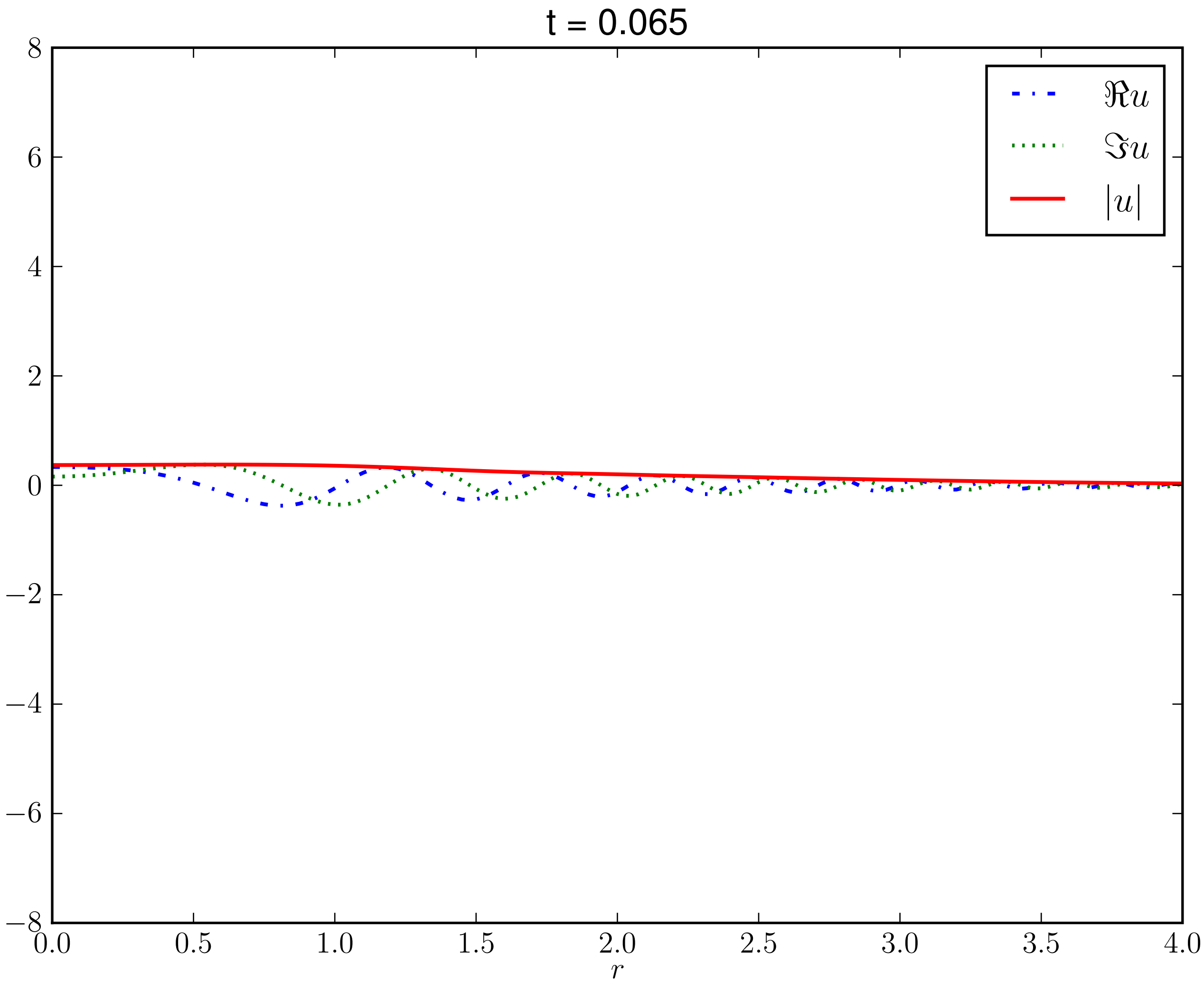
t = 0.045
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.05
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

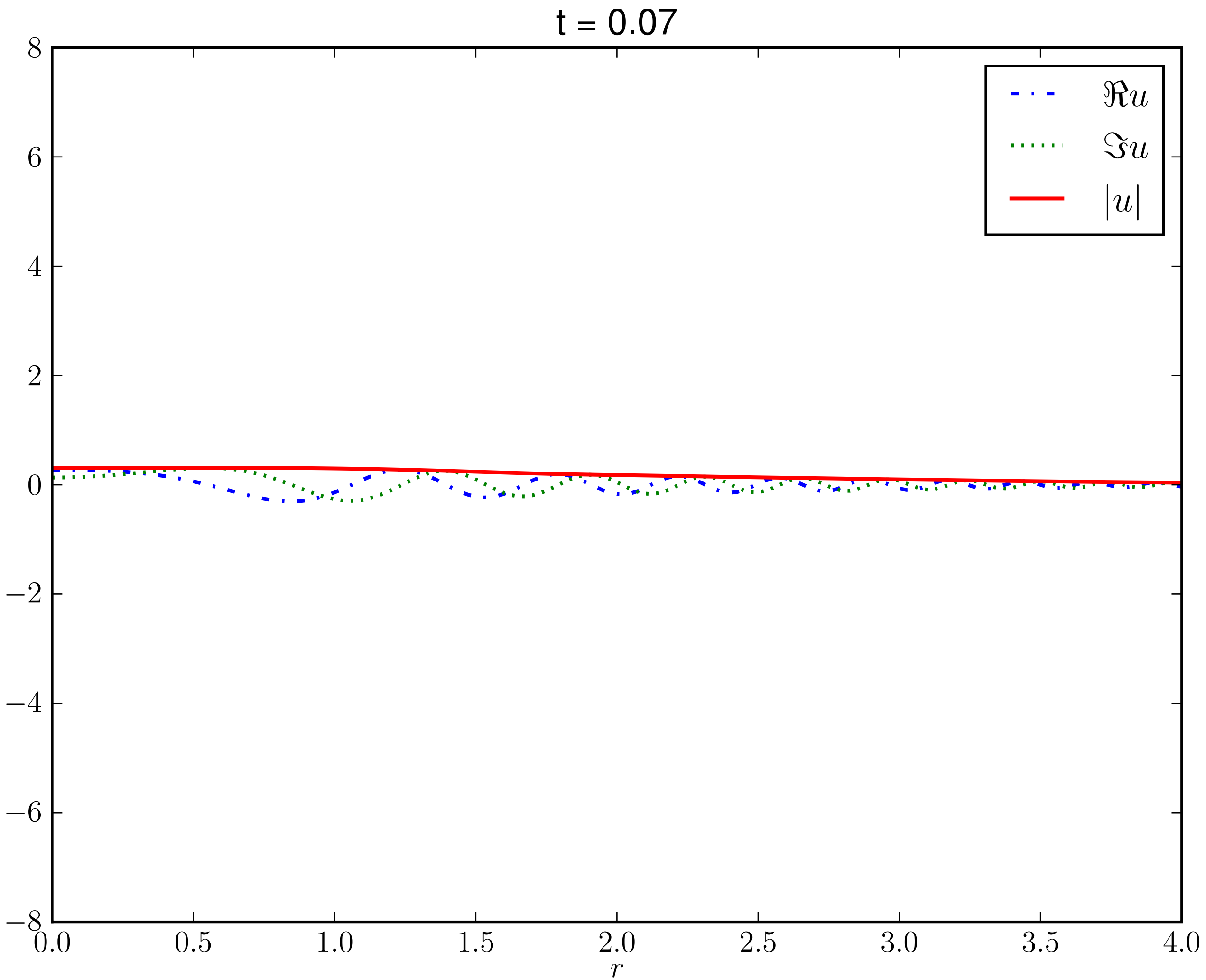

t = 0.055
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



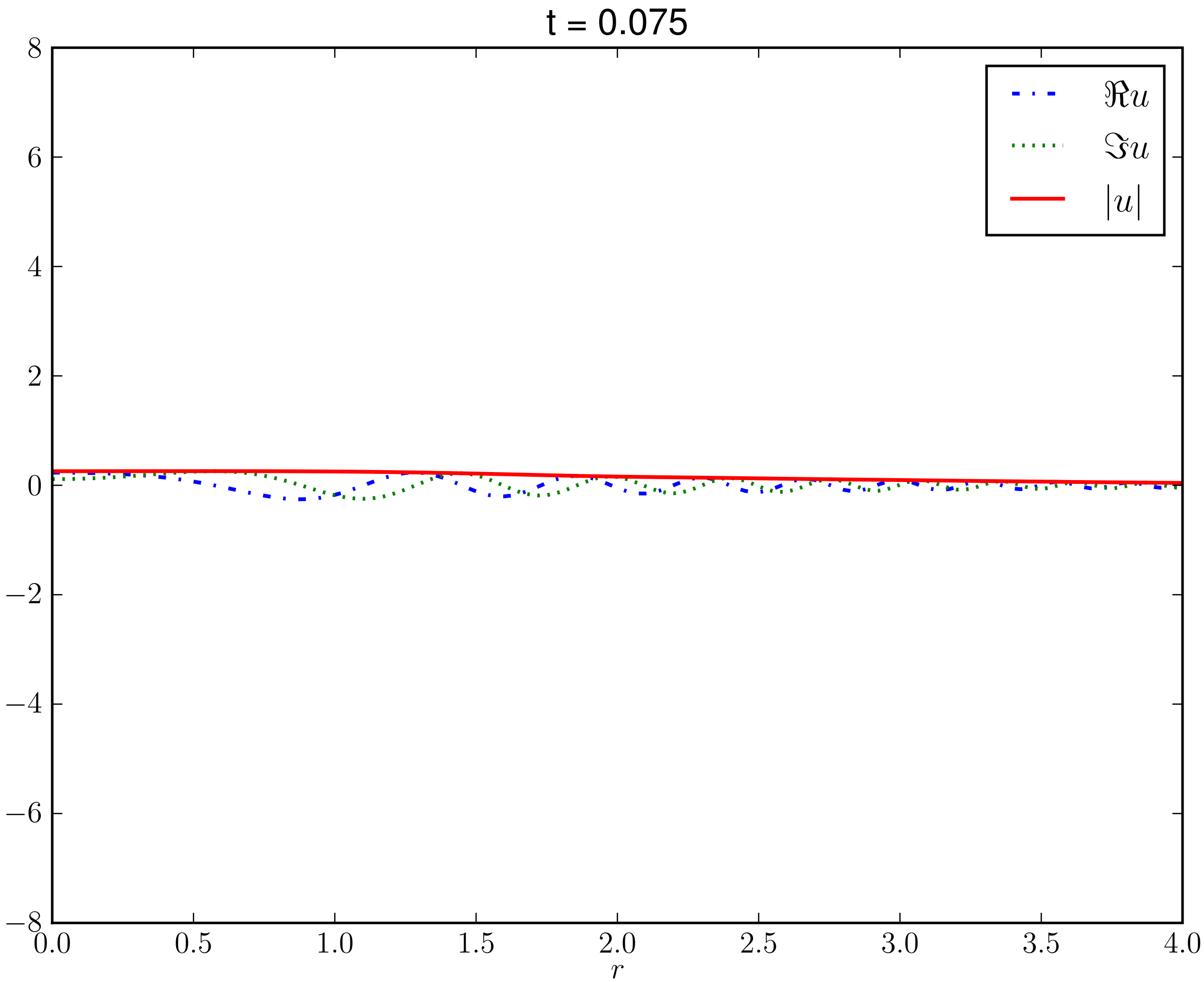
t = 0.06
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



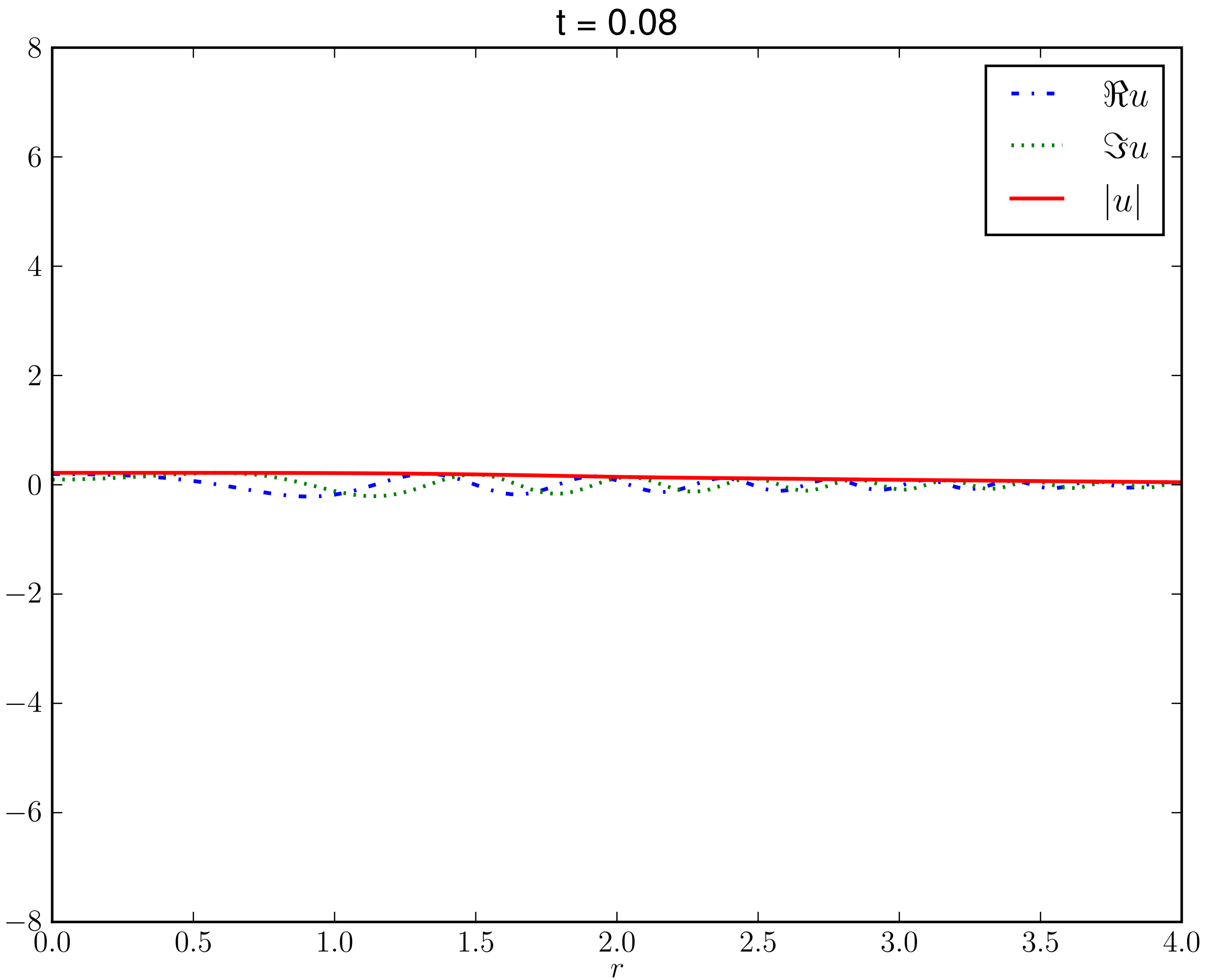
t = 0.065
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$



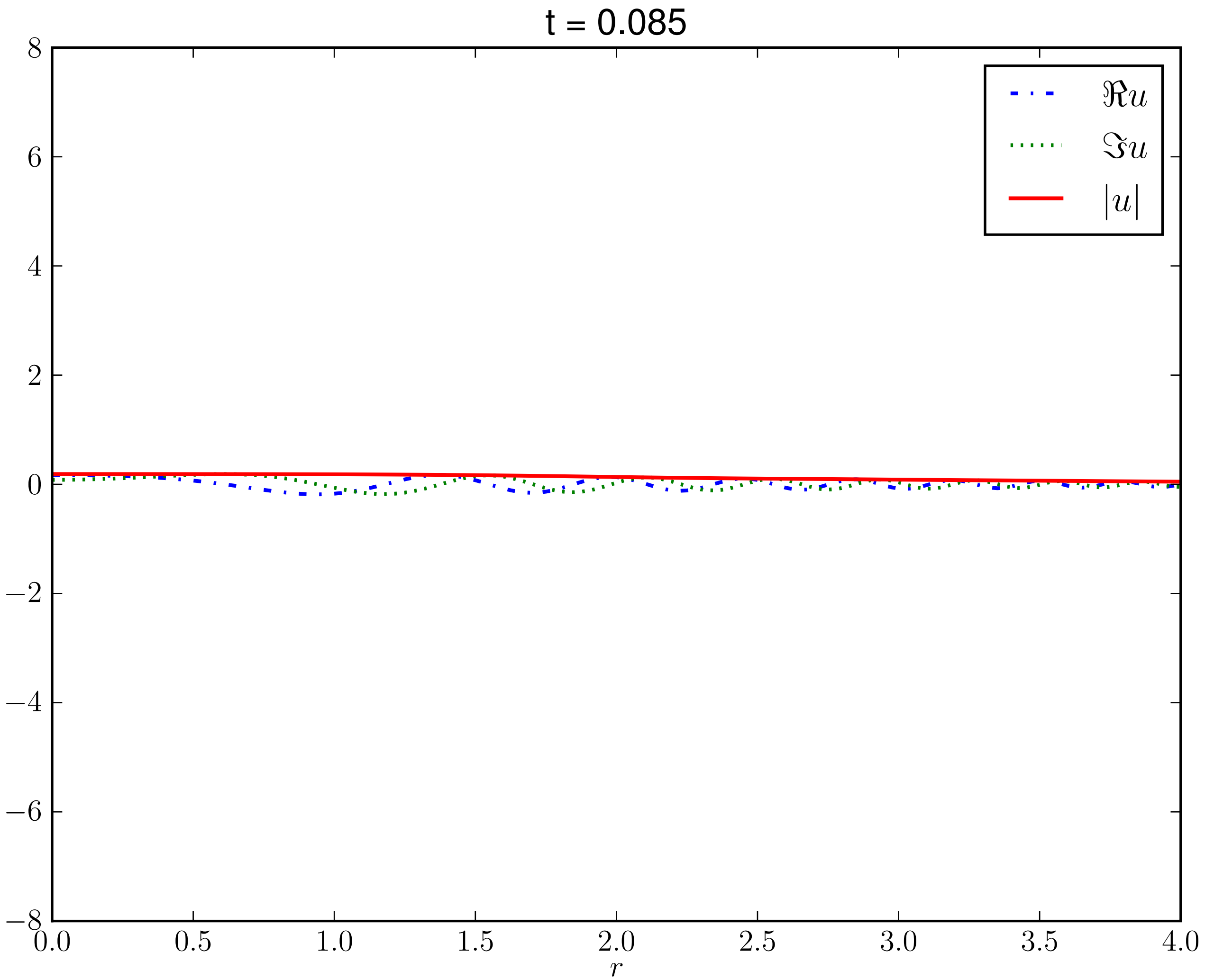
t = 0.07
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$



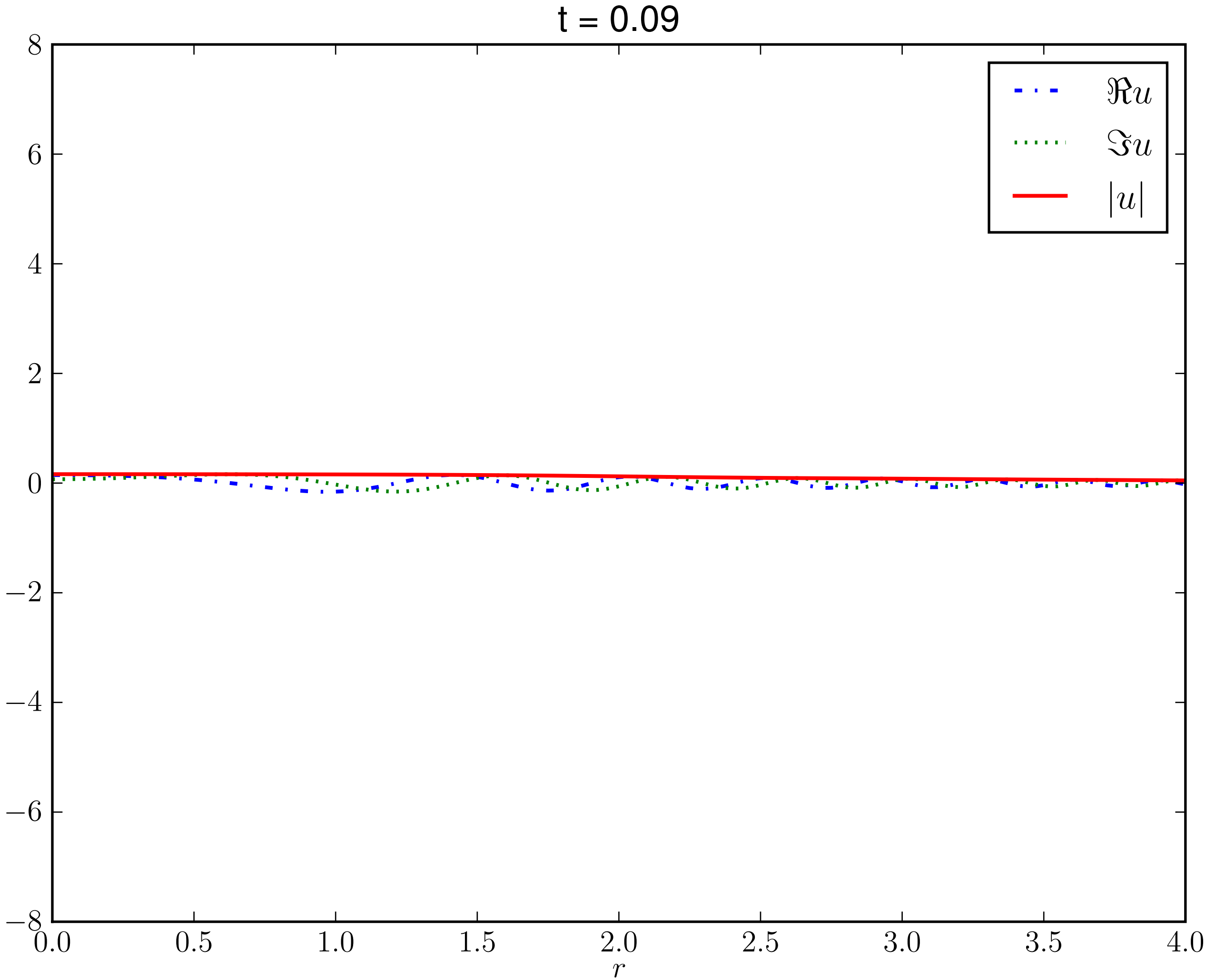
t = 0.075
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r



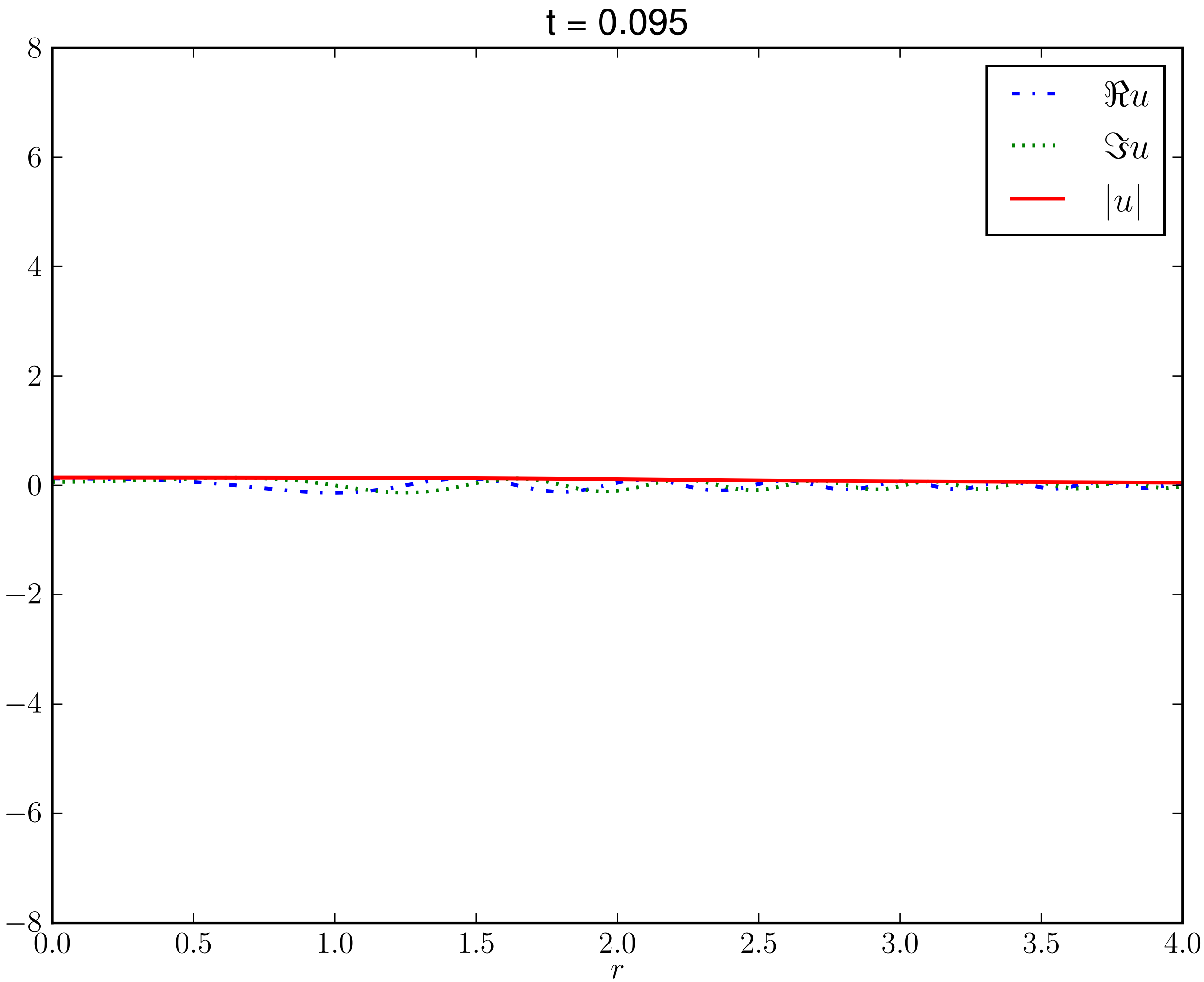
t = 0.08
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$



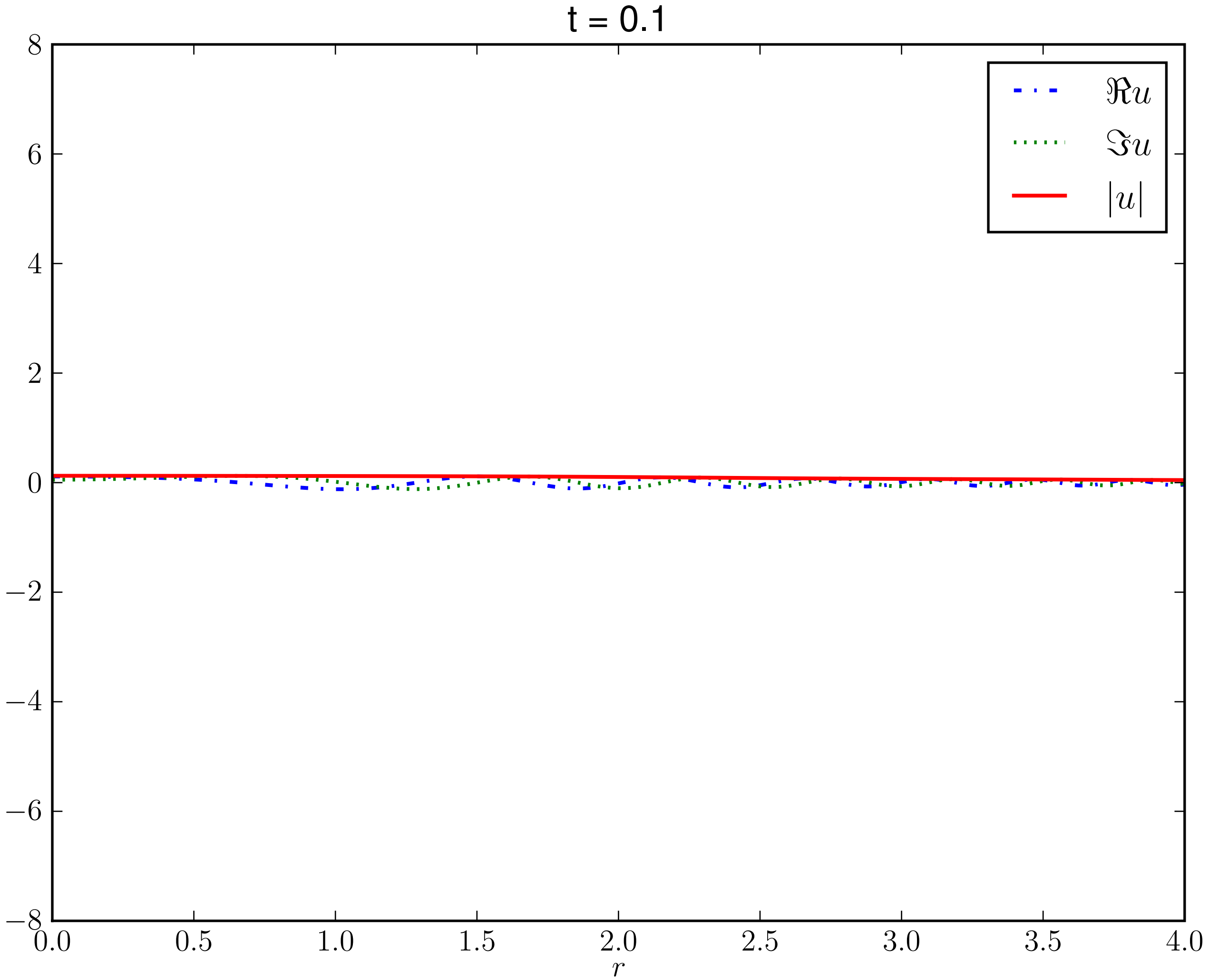
t = 0.085
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



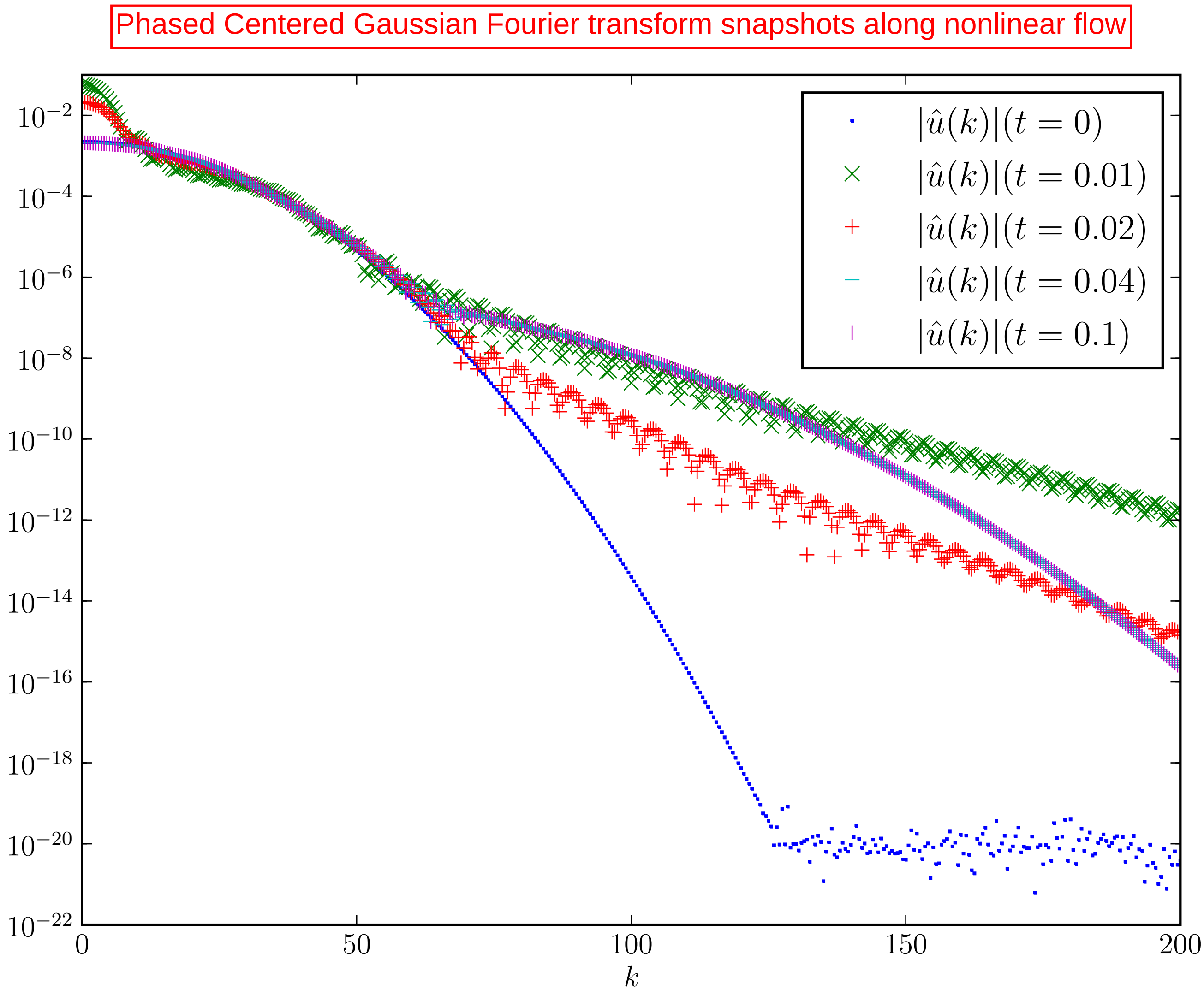
t = 0.09
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

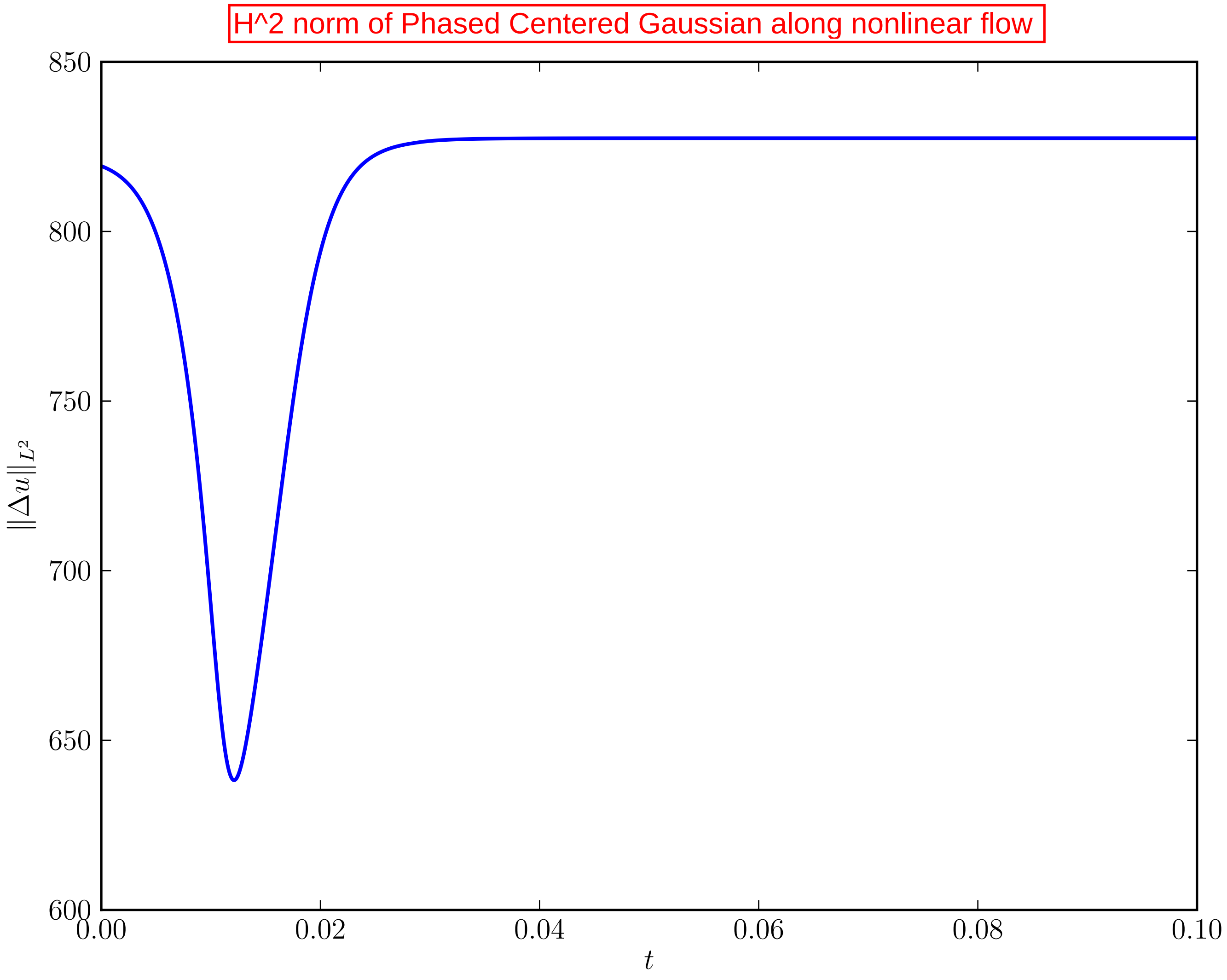


t = 0.095
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

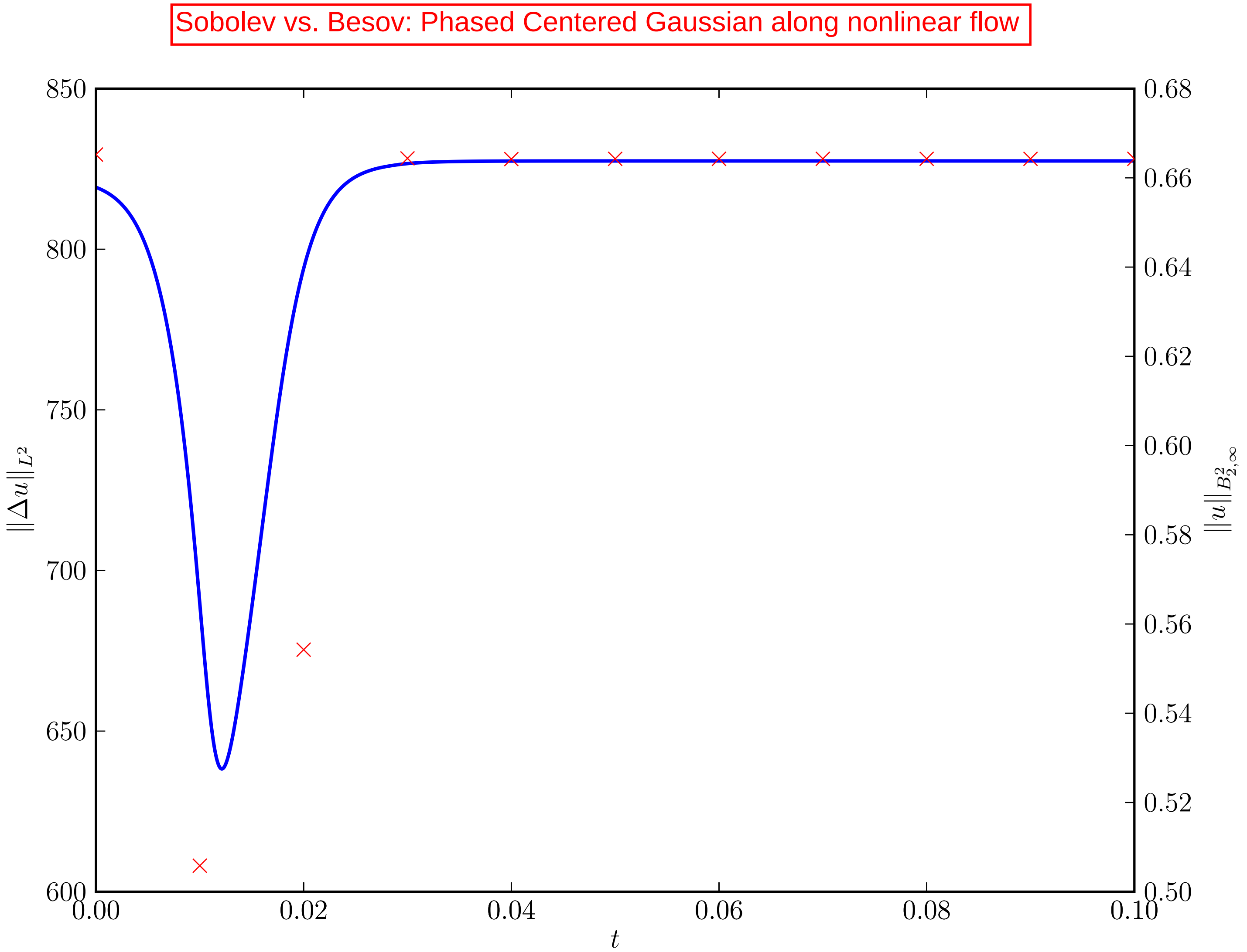


t = 0.1
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

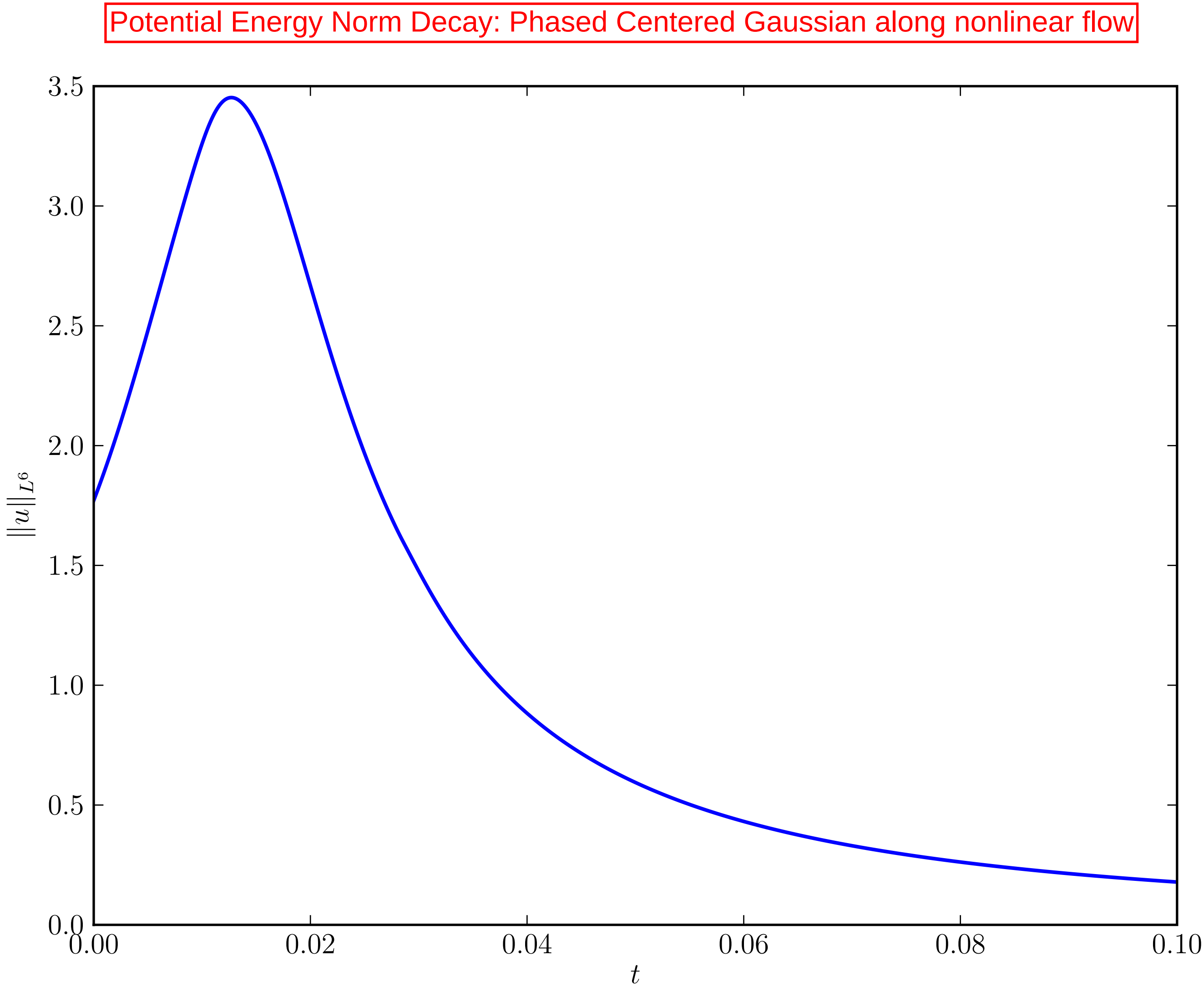

Phased Centered Gaussian Fourier transform snapshots along nonlinear flow
$|\hat{u}(k)|(t=0)$
$|\hat{u}(k)|(t=0.01)$
$|\hat{u}(k)|(t=0.02)$
$|\hat{u}(k)|(t=0.04)$
$|\hat{u}(k)|(t=0.1)$
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
0
50
100
150
200
k

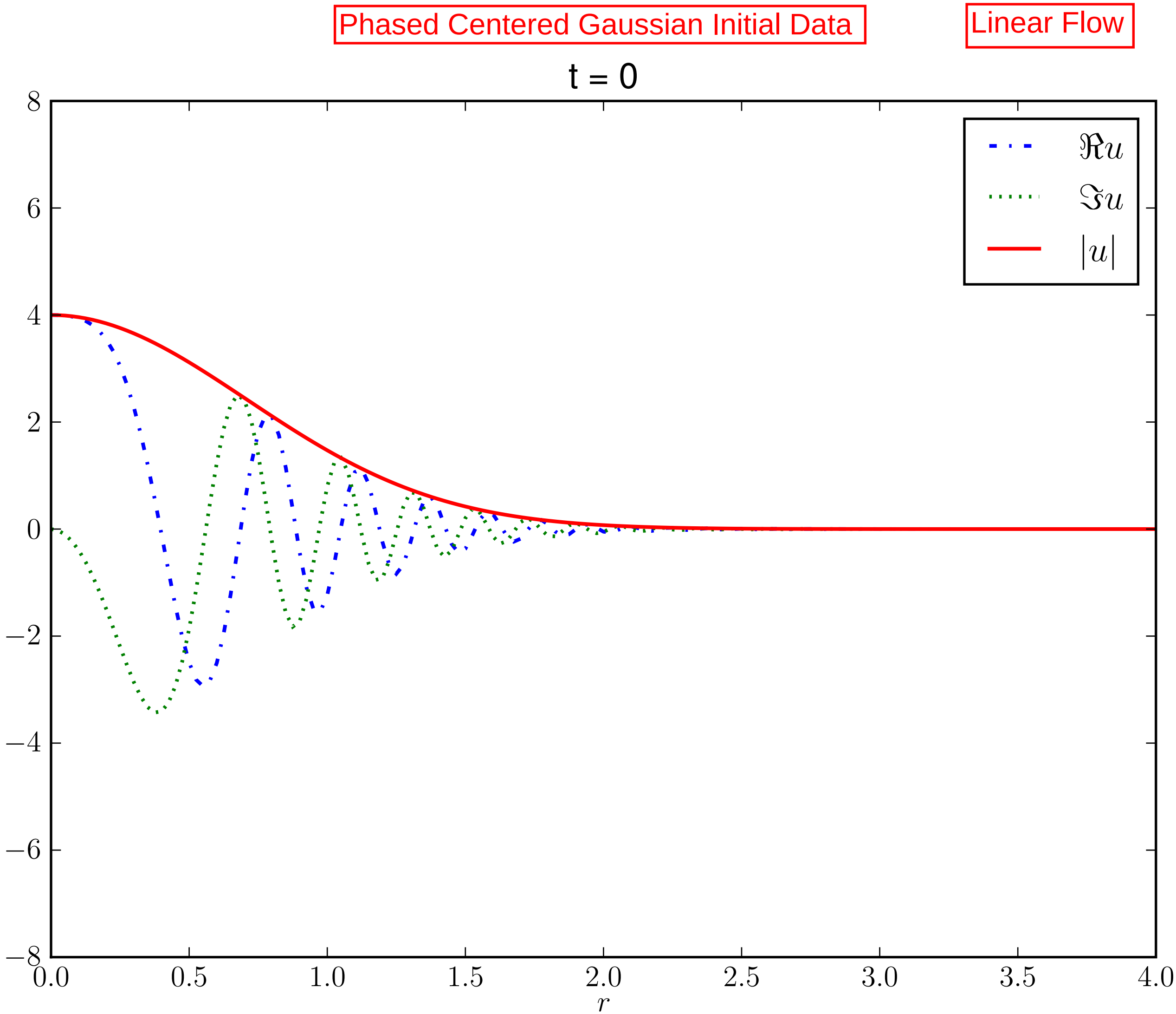


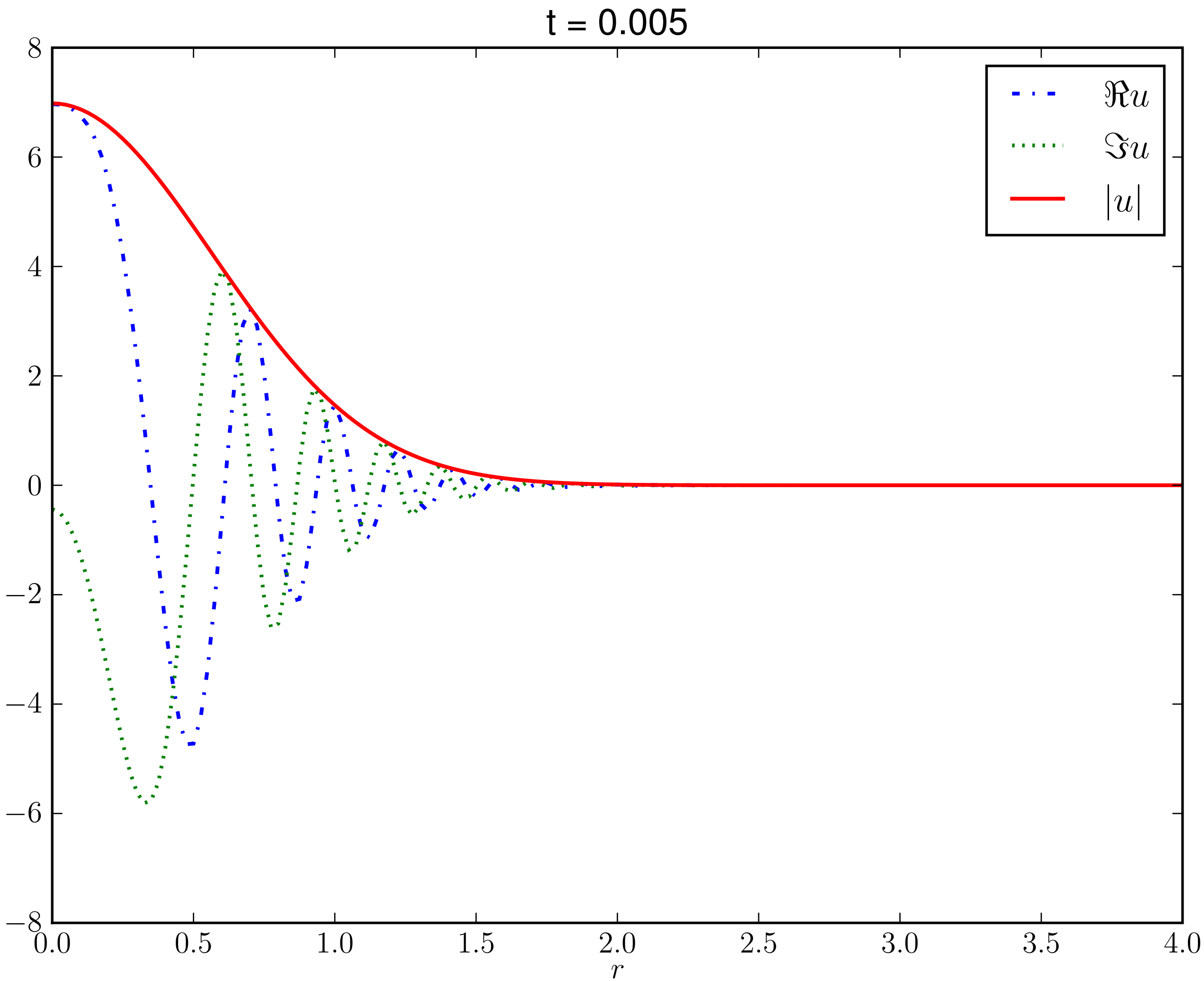
H^2 norm of Phased Centered Gaussian along nonlinear flow
850
800
750
700
650
600
$\|\Delta u\|_{L^2}$
0.00
0.02
0.04
0.06
0.08
0.10
t

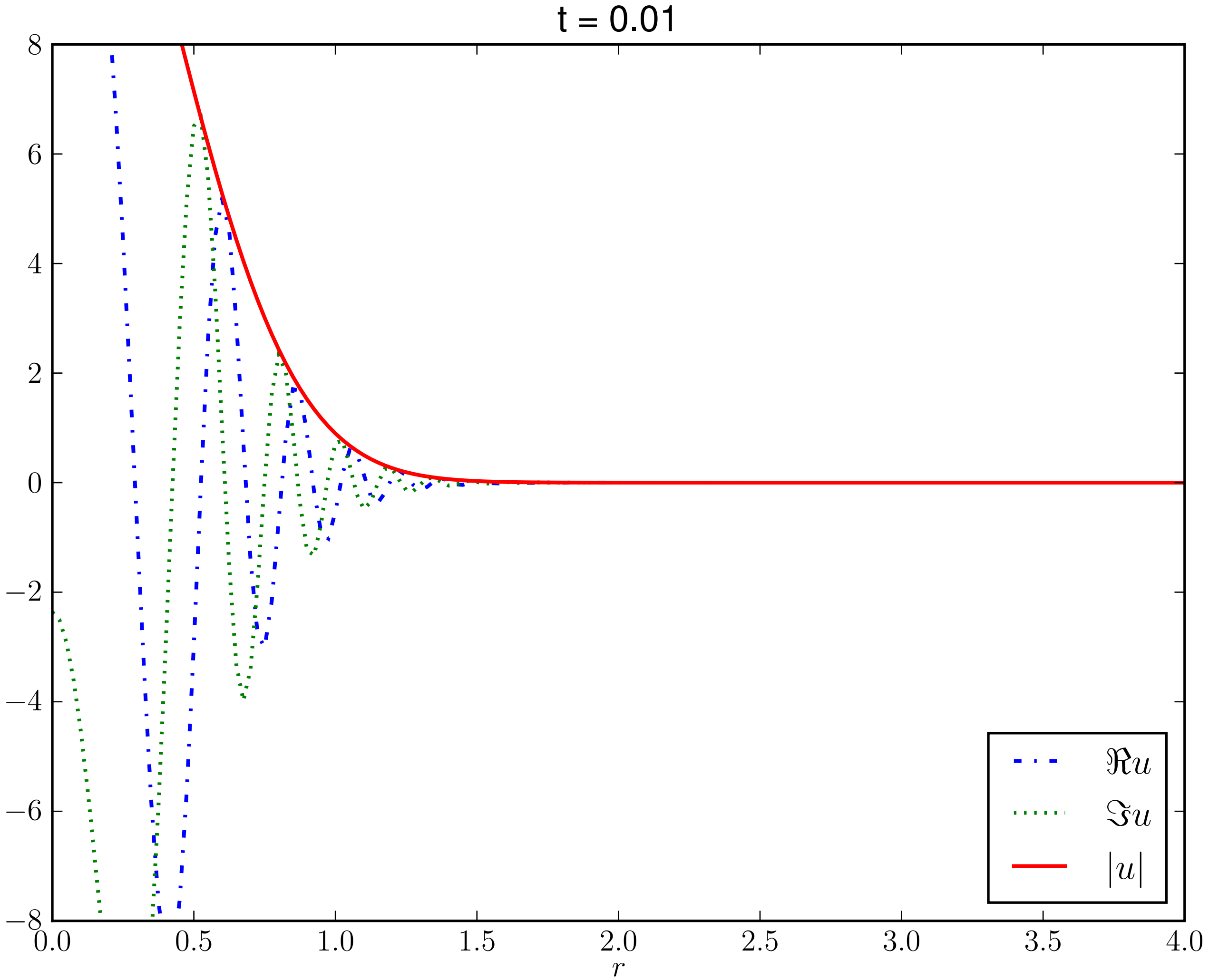


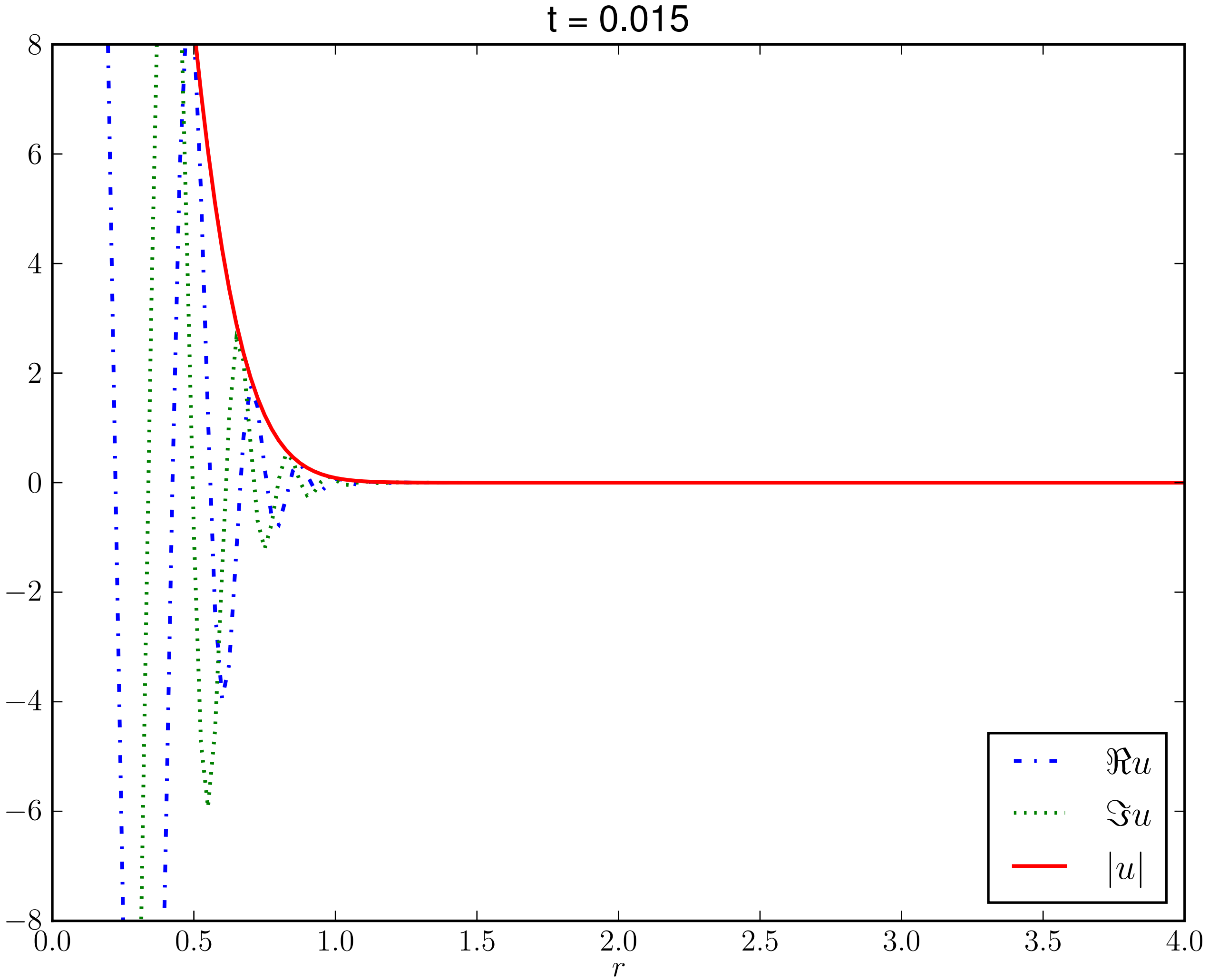
Sobolev vs. Besov: Phased Centered Gaussian along nonlinear flow
850
800
750
700
650
600
0.68
0.66
0.64
0.62
0.60
0.58
0.56
0.54
0.52
0.50
0.00
0.02
0.04
0.06
0.08
0.10
t
$\|\Delta u\|_{L^2}$
$\|u\|_{B^2_{2,\infty}}$

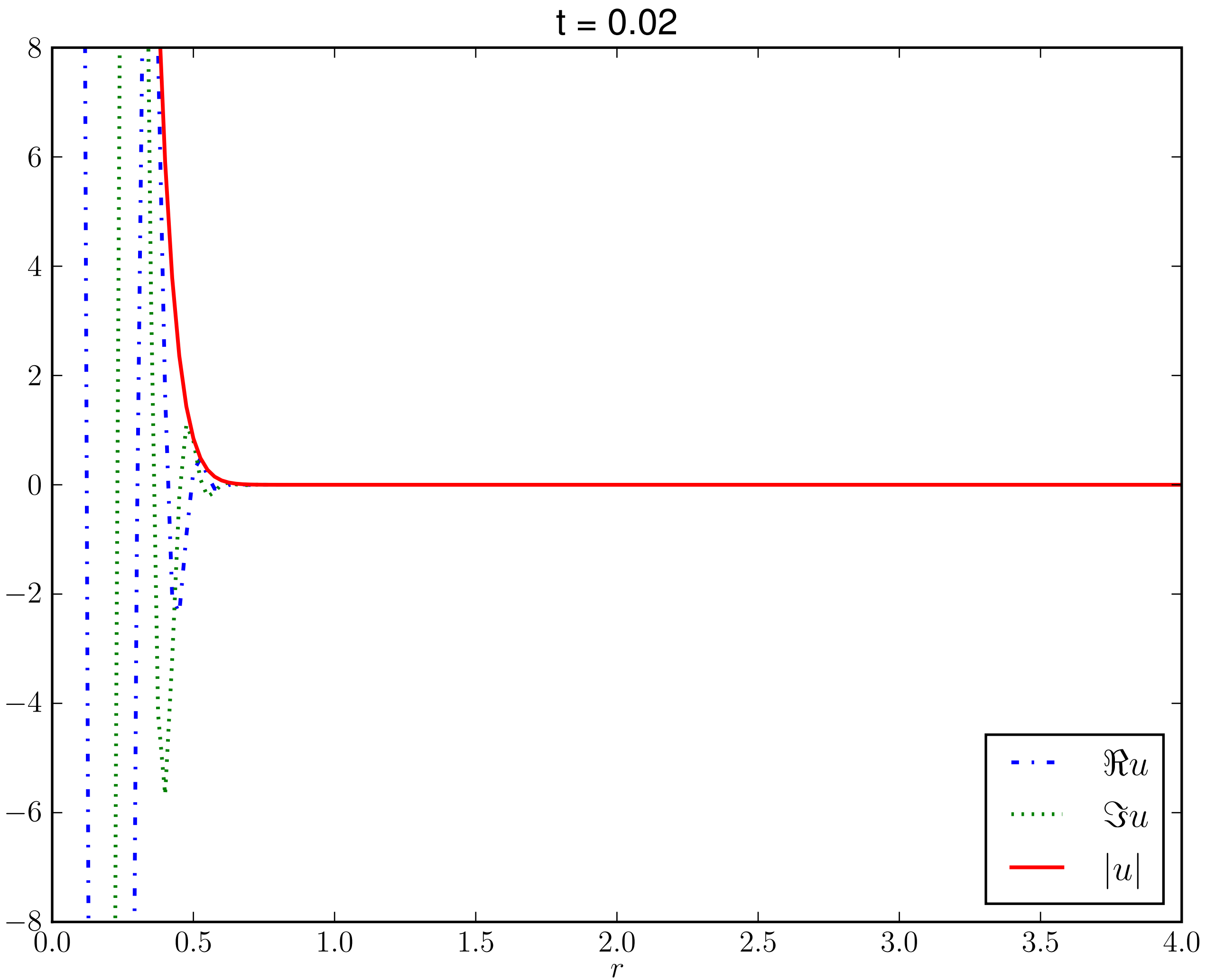

Potential Energy Norm Decay: Phased Centered Gaussian along nonlinear flow
$\|u\|_{L^6}$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.00
0.02
0.04
0.06
0.08
0.10
t

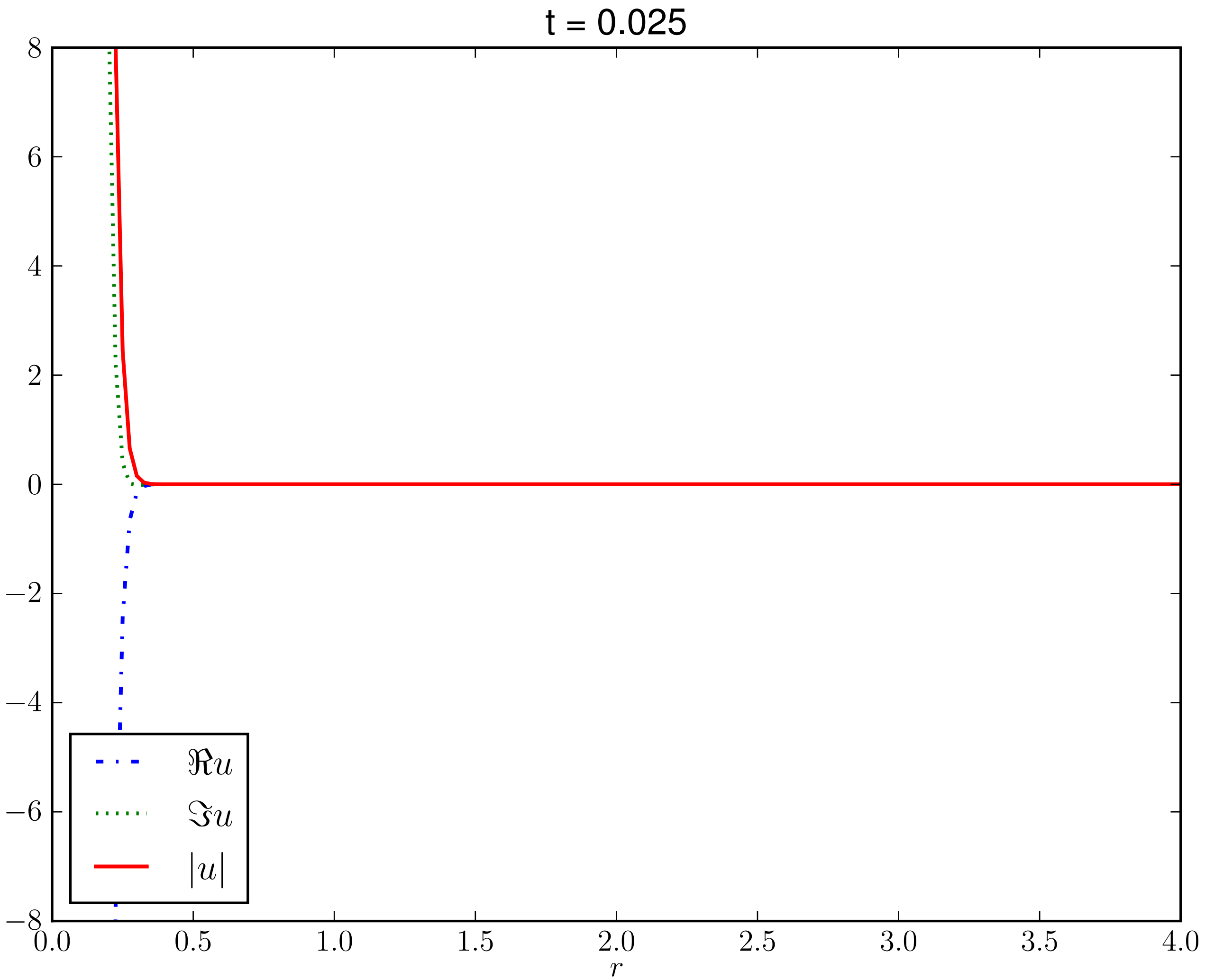

Phased Centered Gaussian Initial Data
Linear Flow
t = 0
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

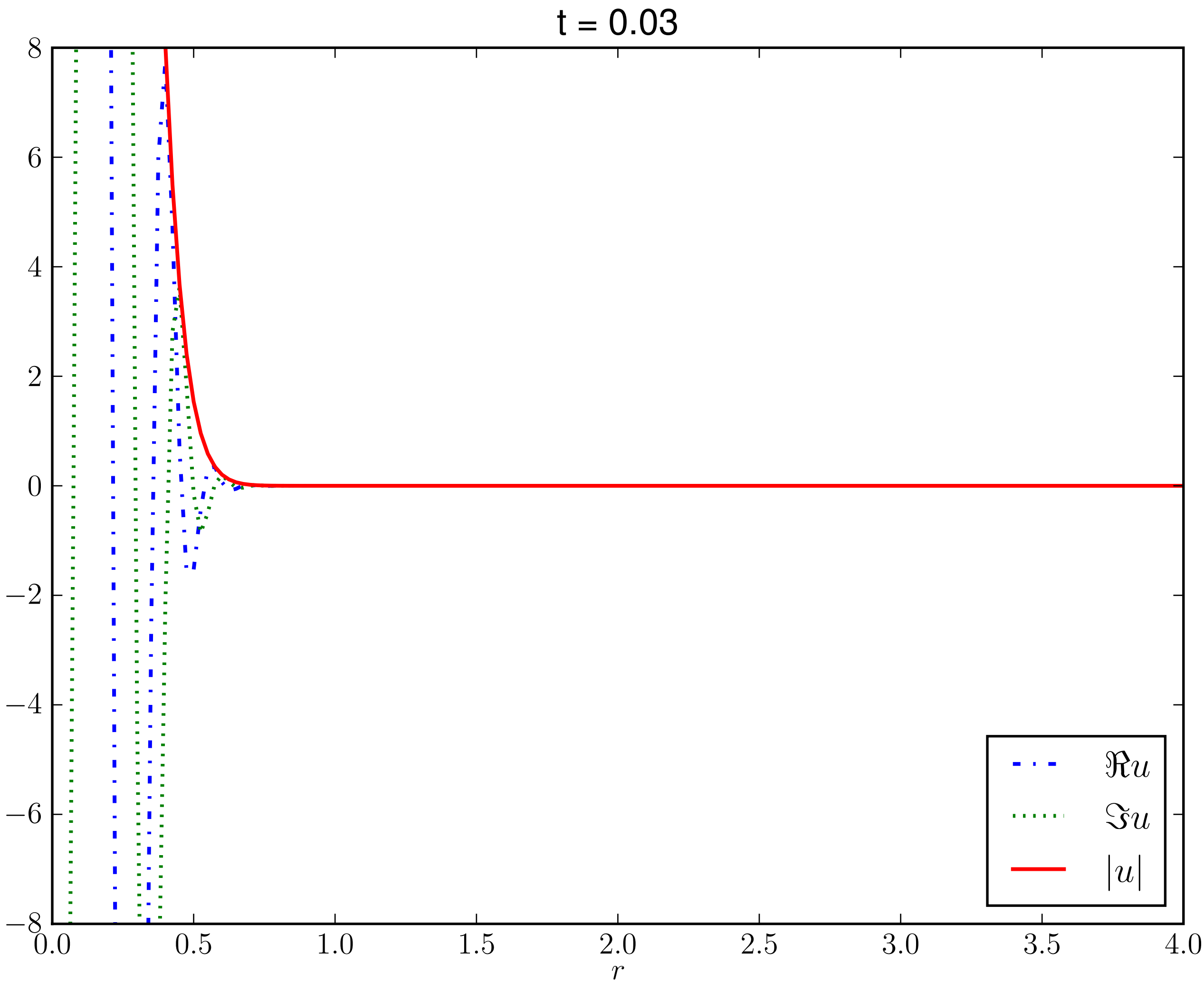

t = 0.005
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.01
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

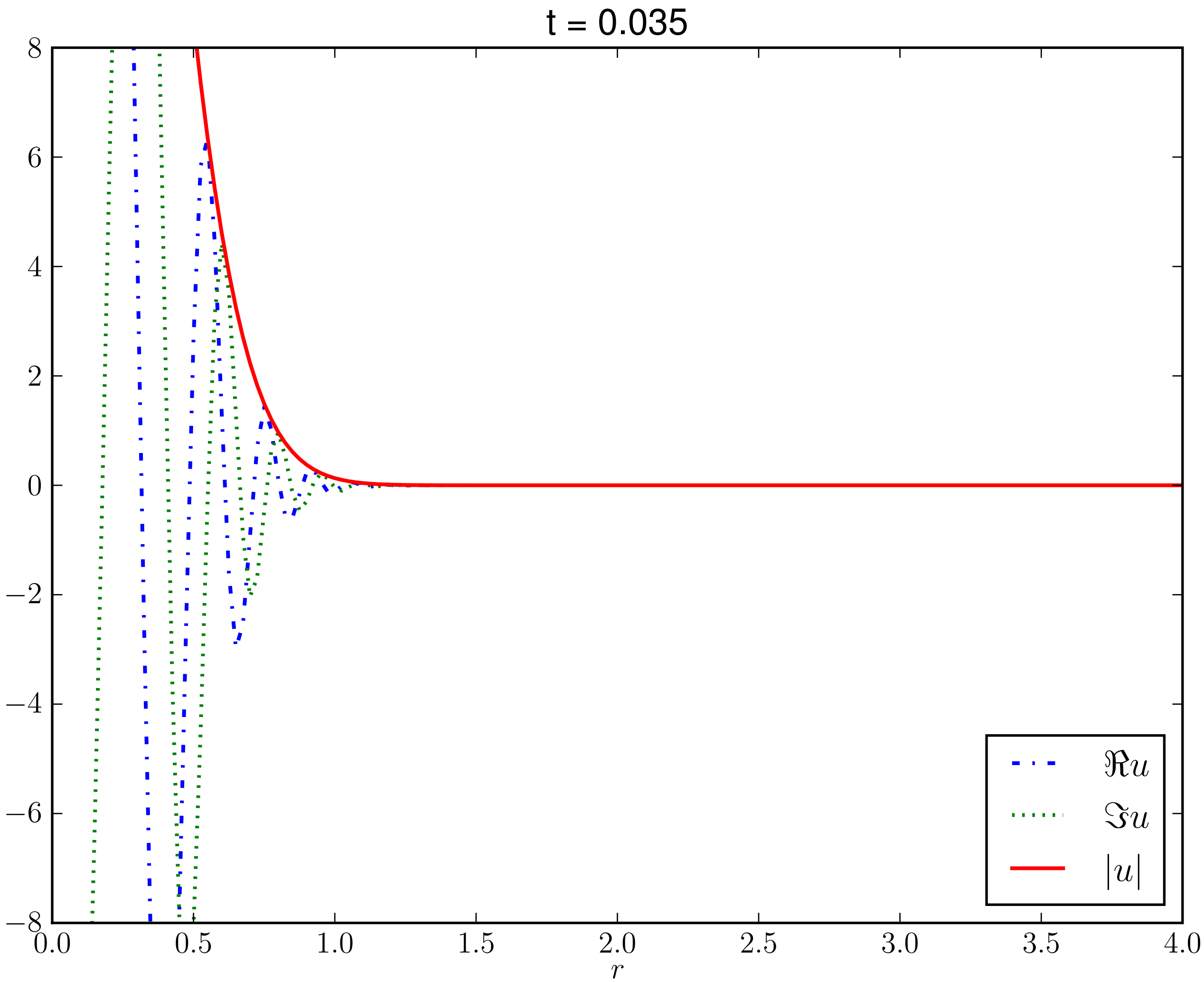

t = 0.015
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.02
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

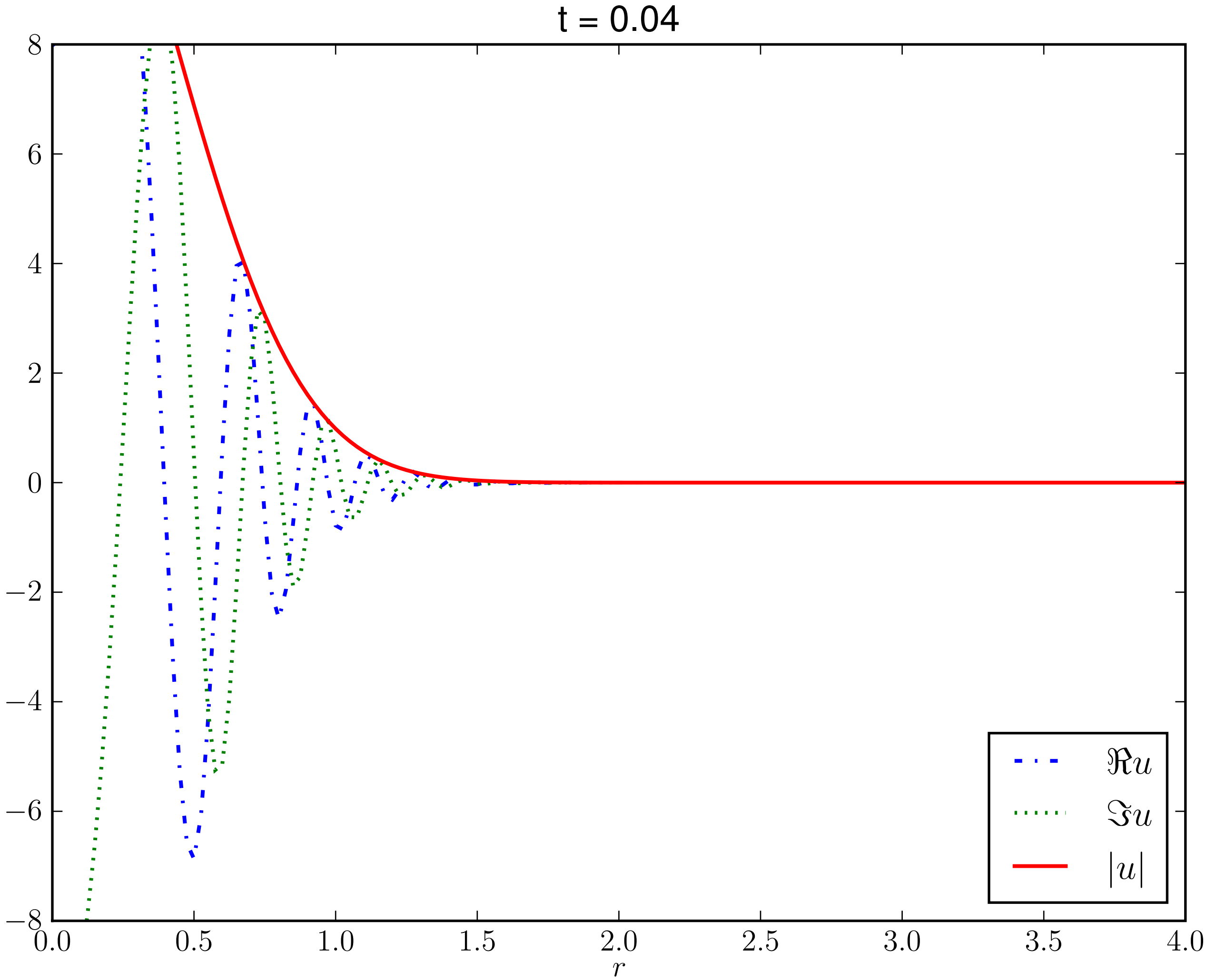

t = 0.025
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



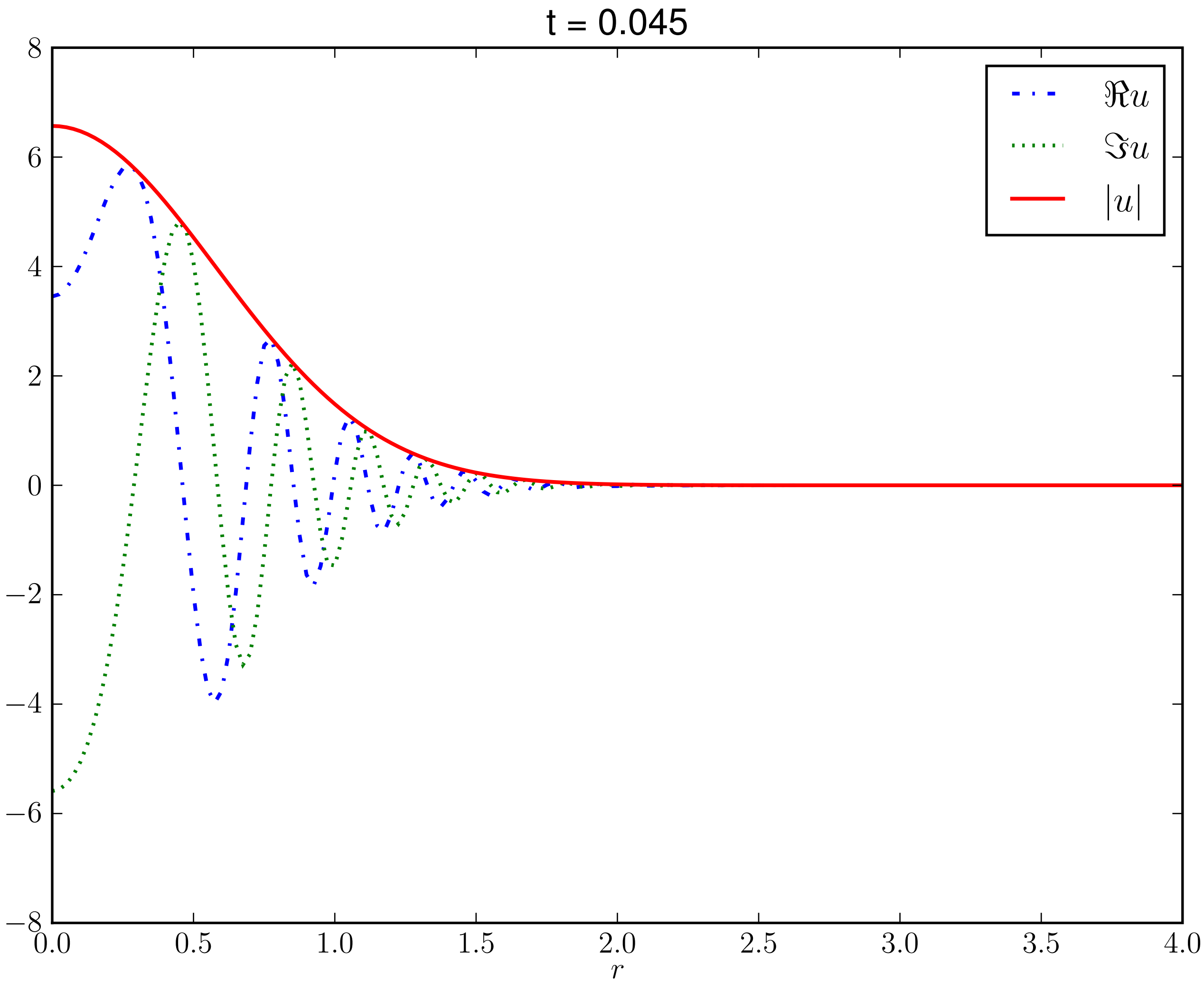
t = 0.03
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



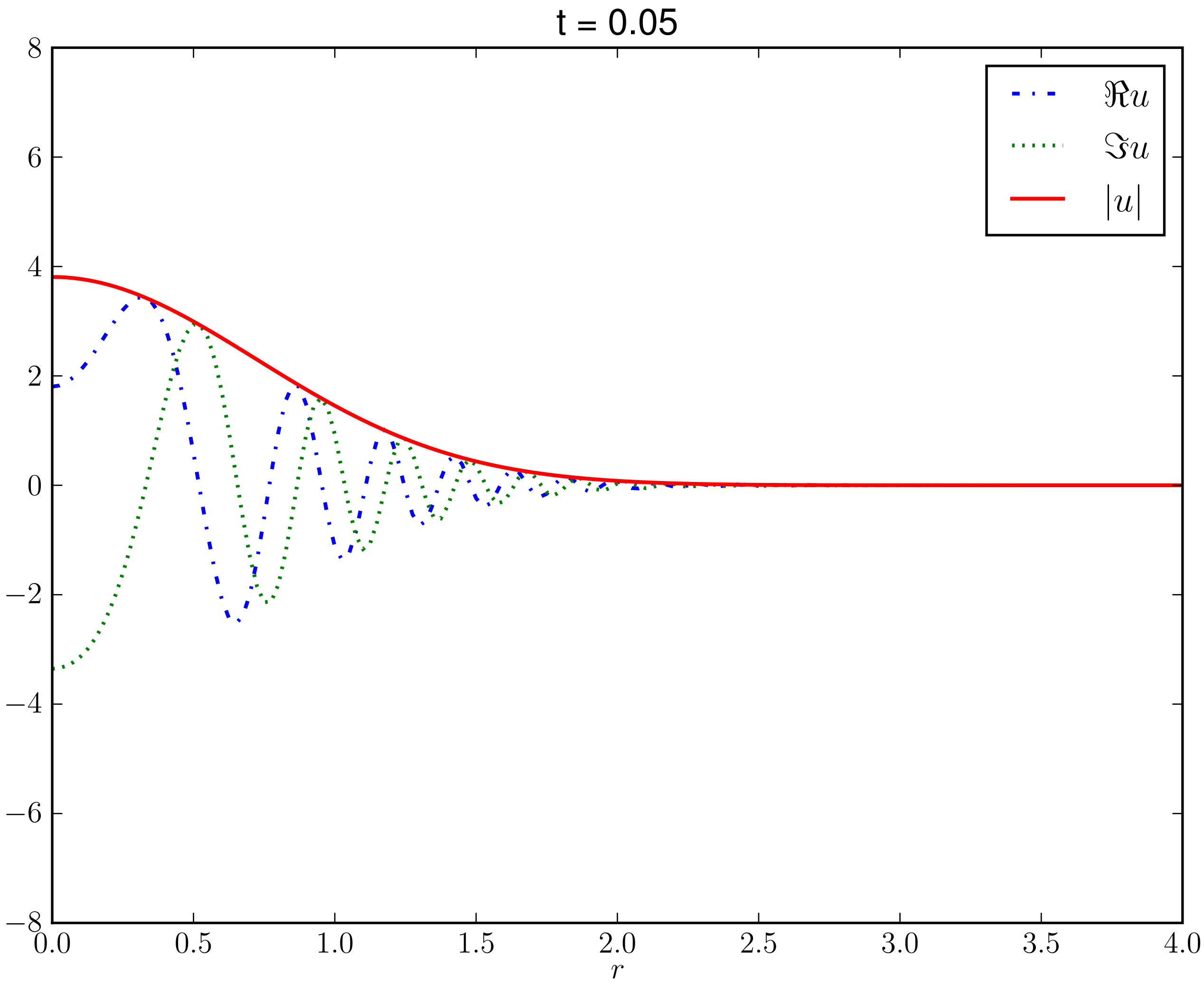
t = 0.035
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



t = 0.04
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



t = 0.045
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



t = 0.05
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
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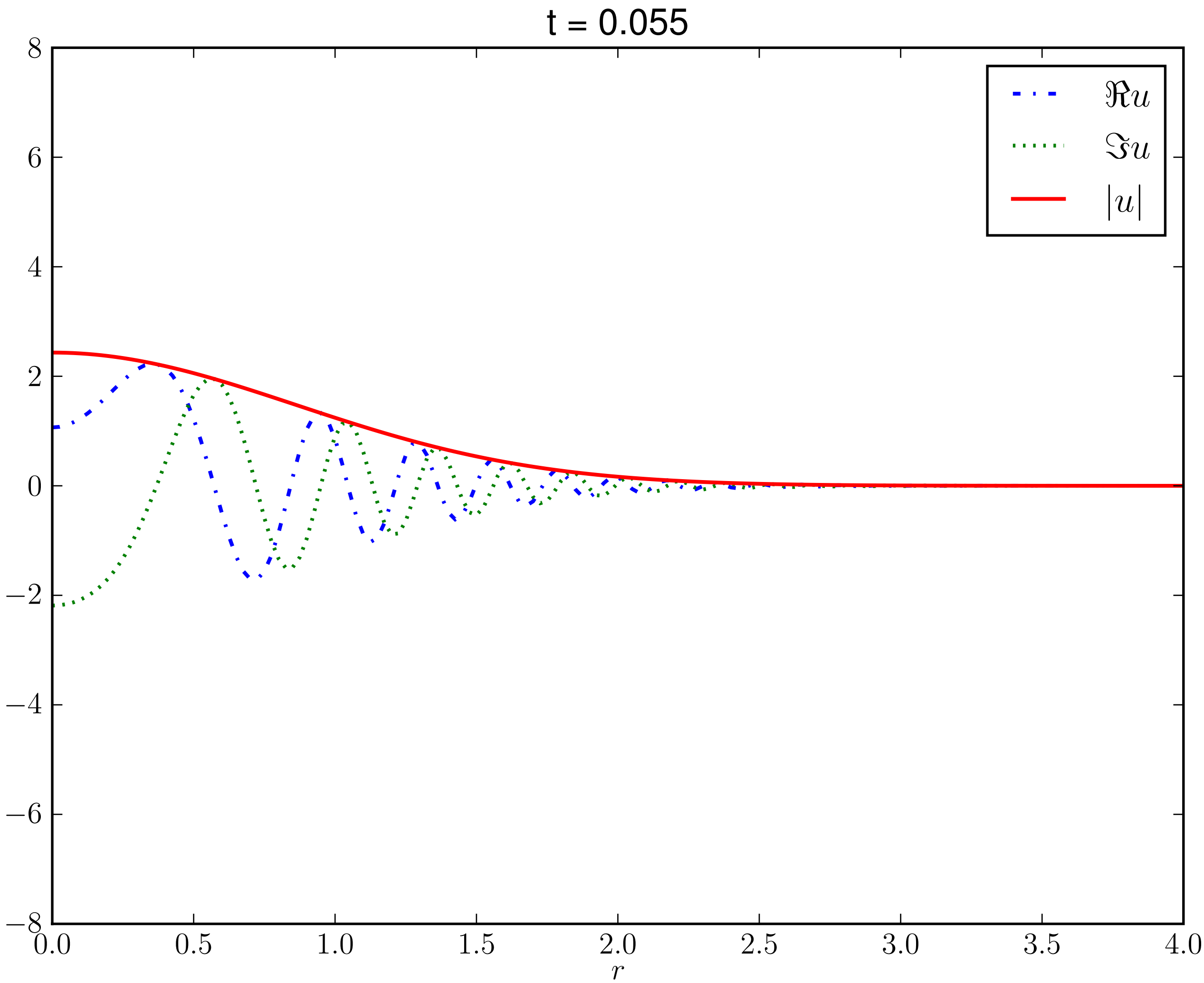
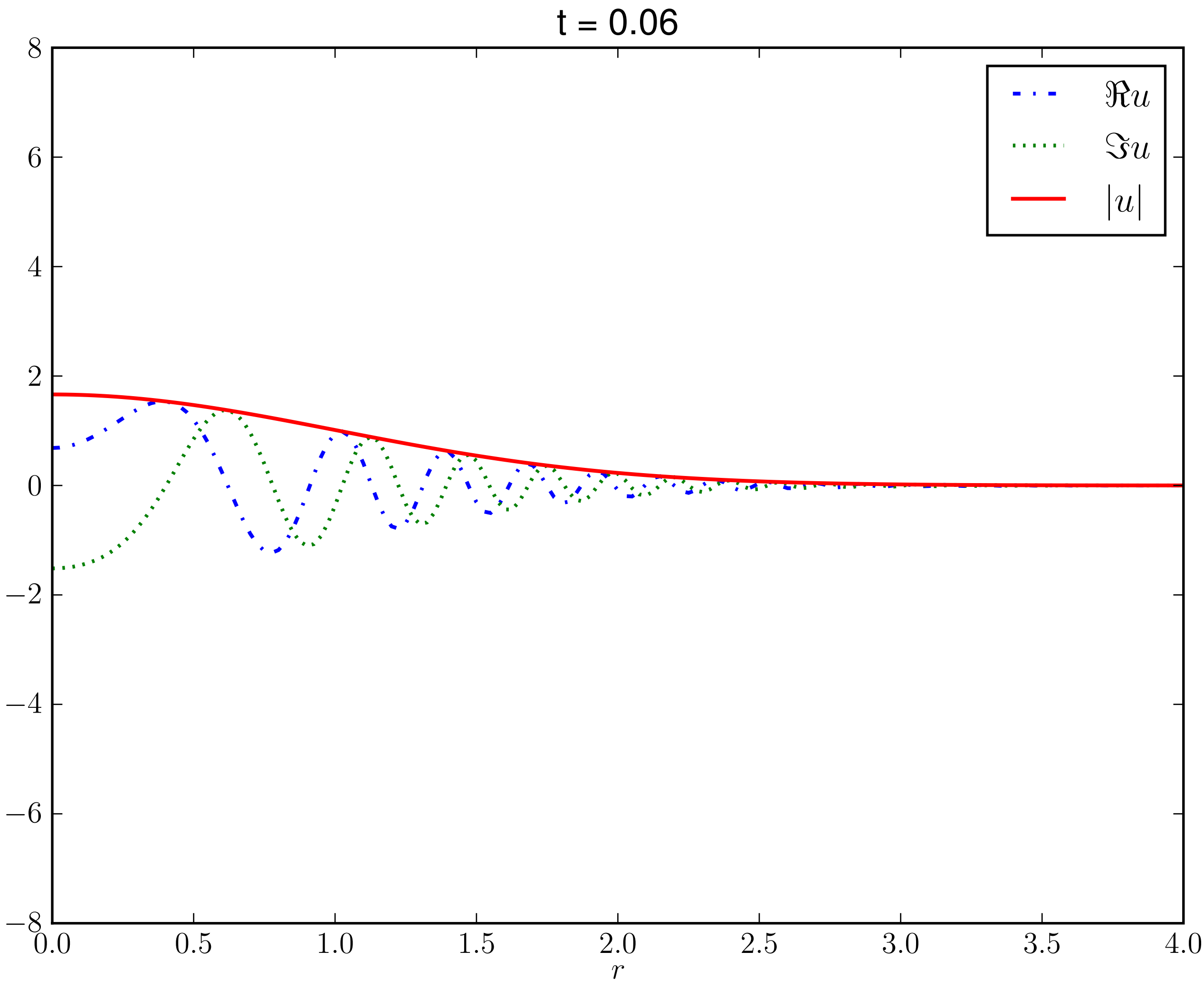
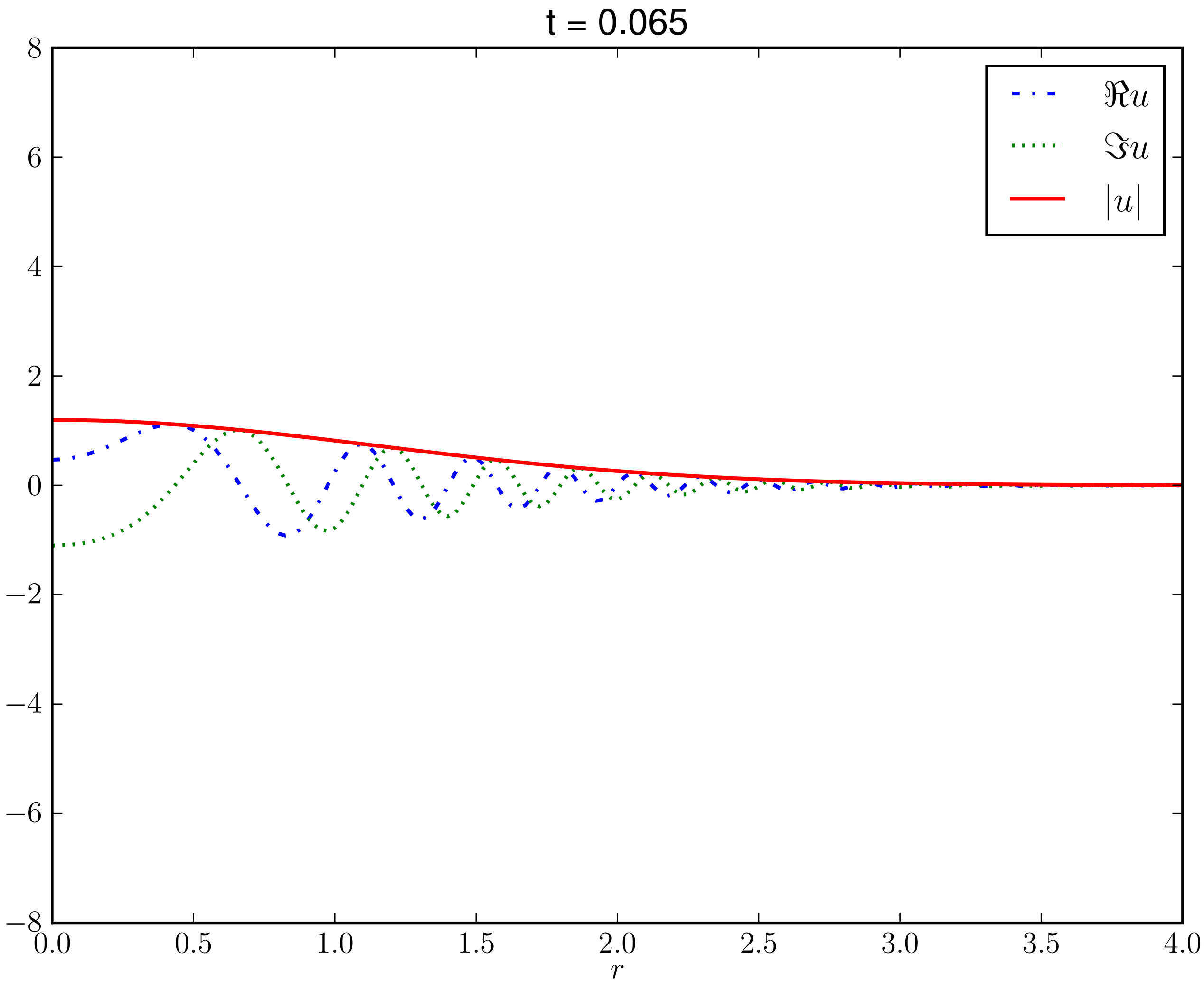
ℑu
|u|



t = 0.055
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.06
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



t = 0.065
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
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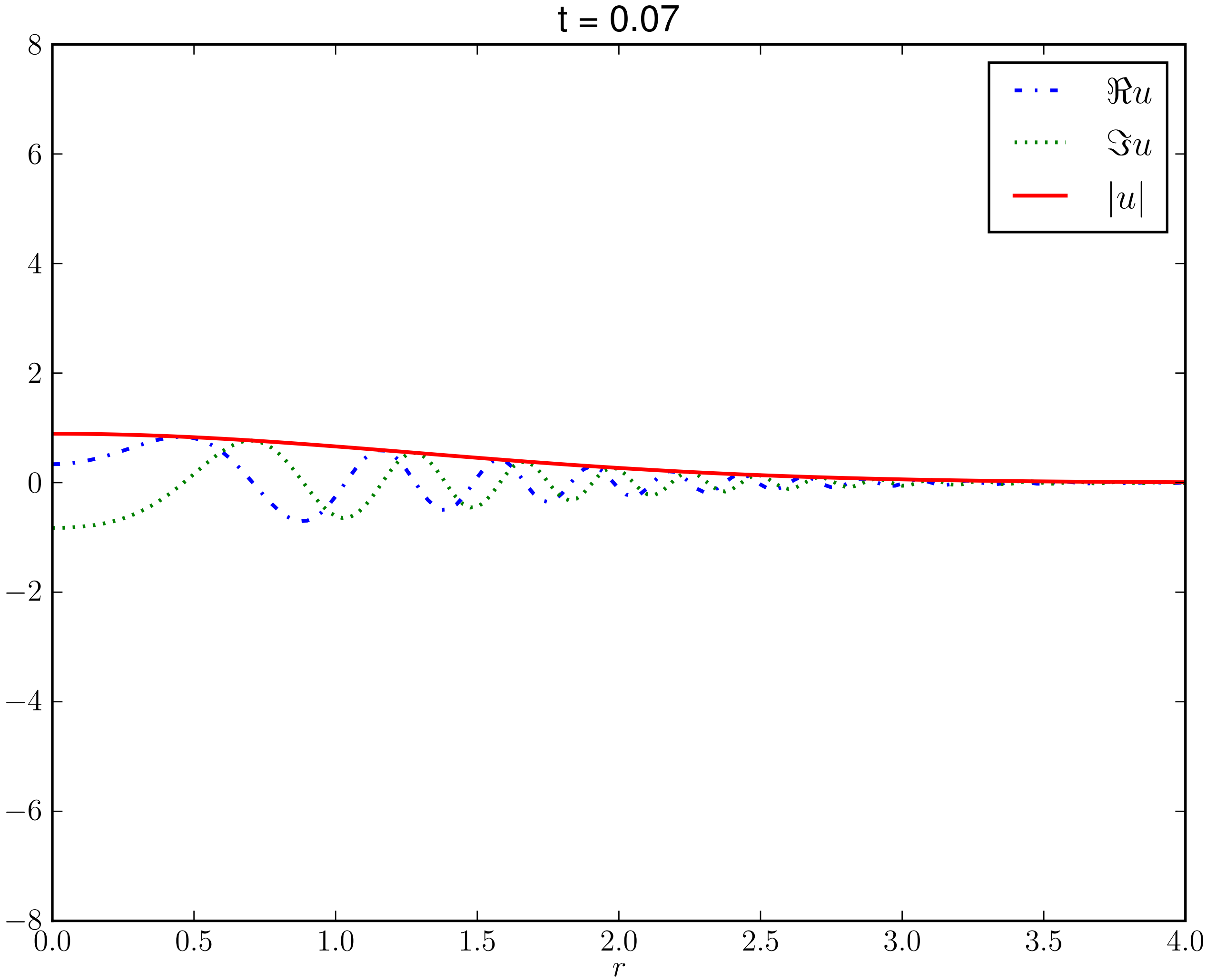
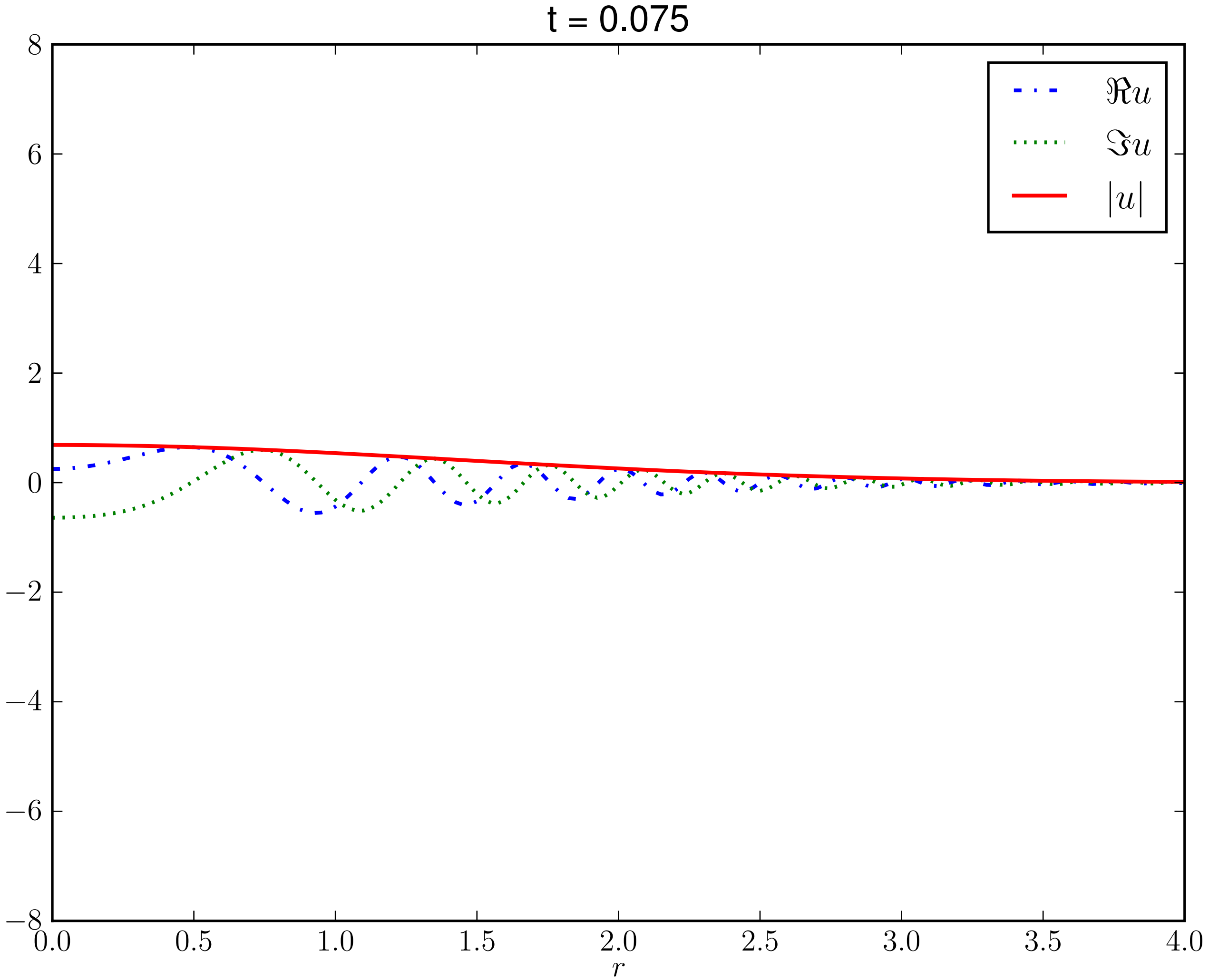
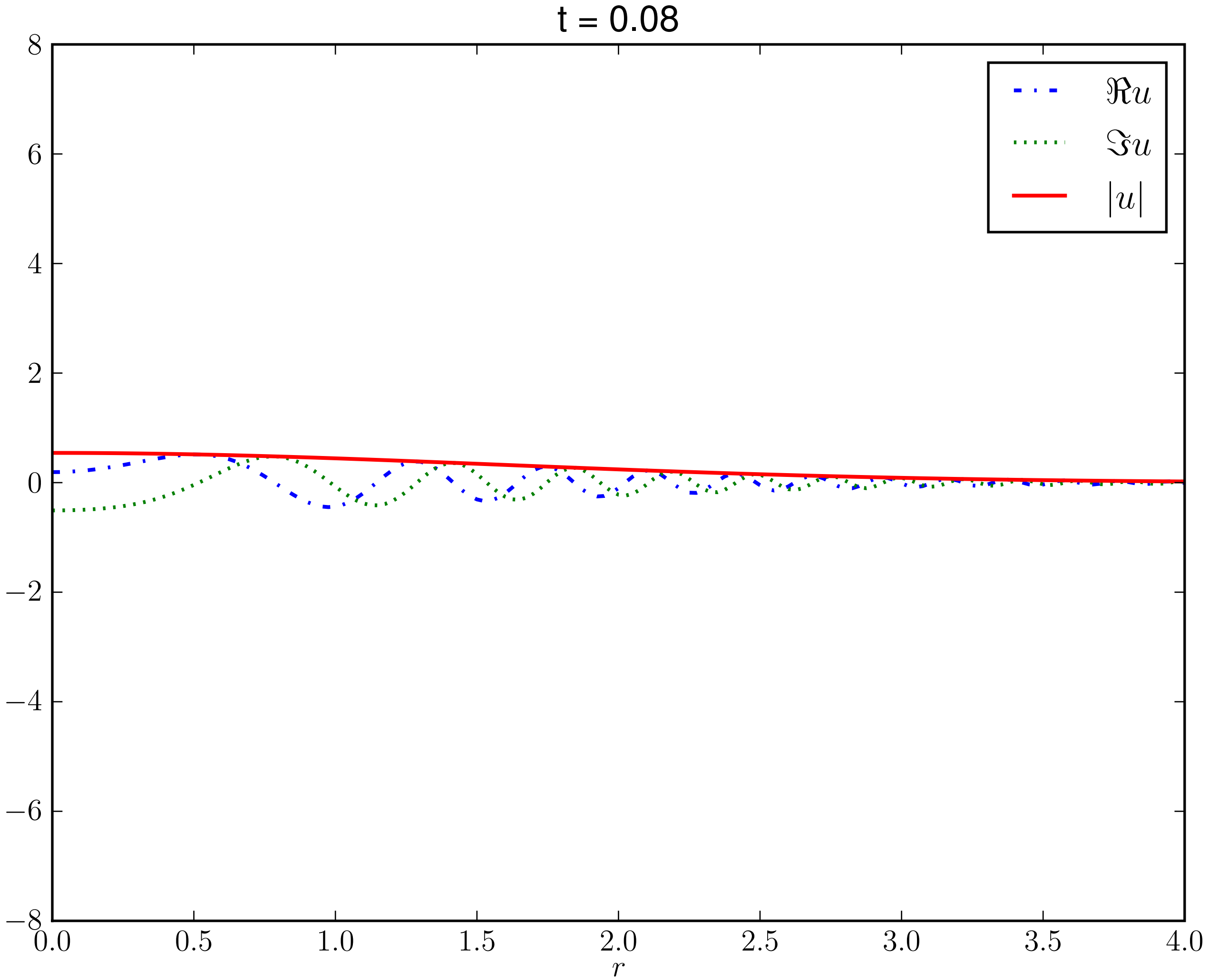
|u|



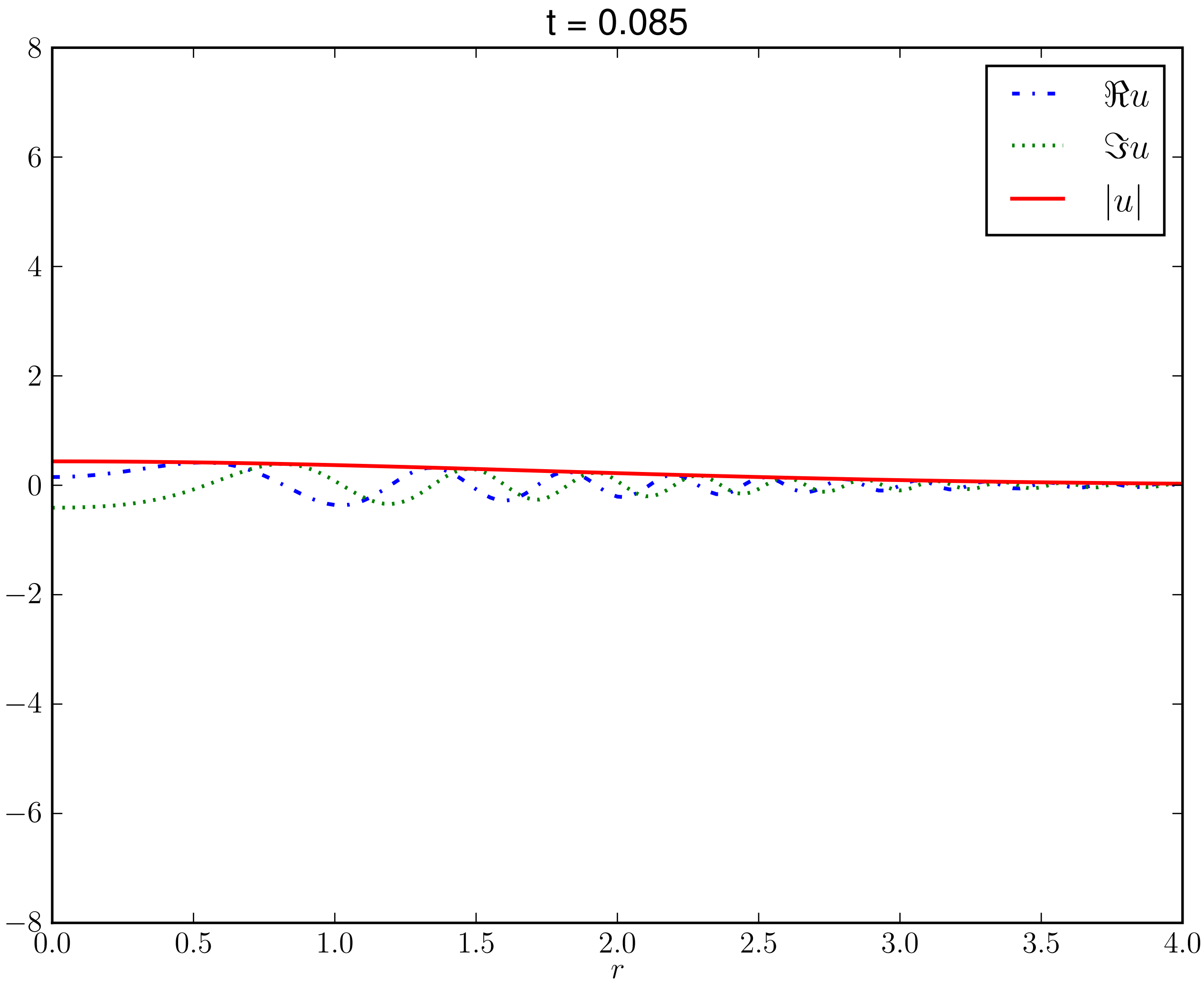
t = 0.07
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

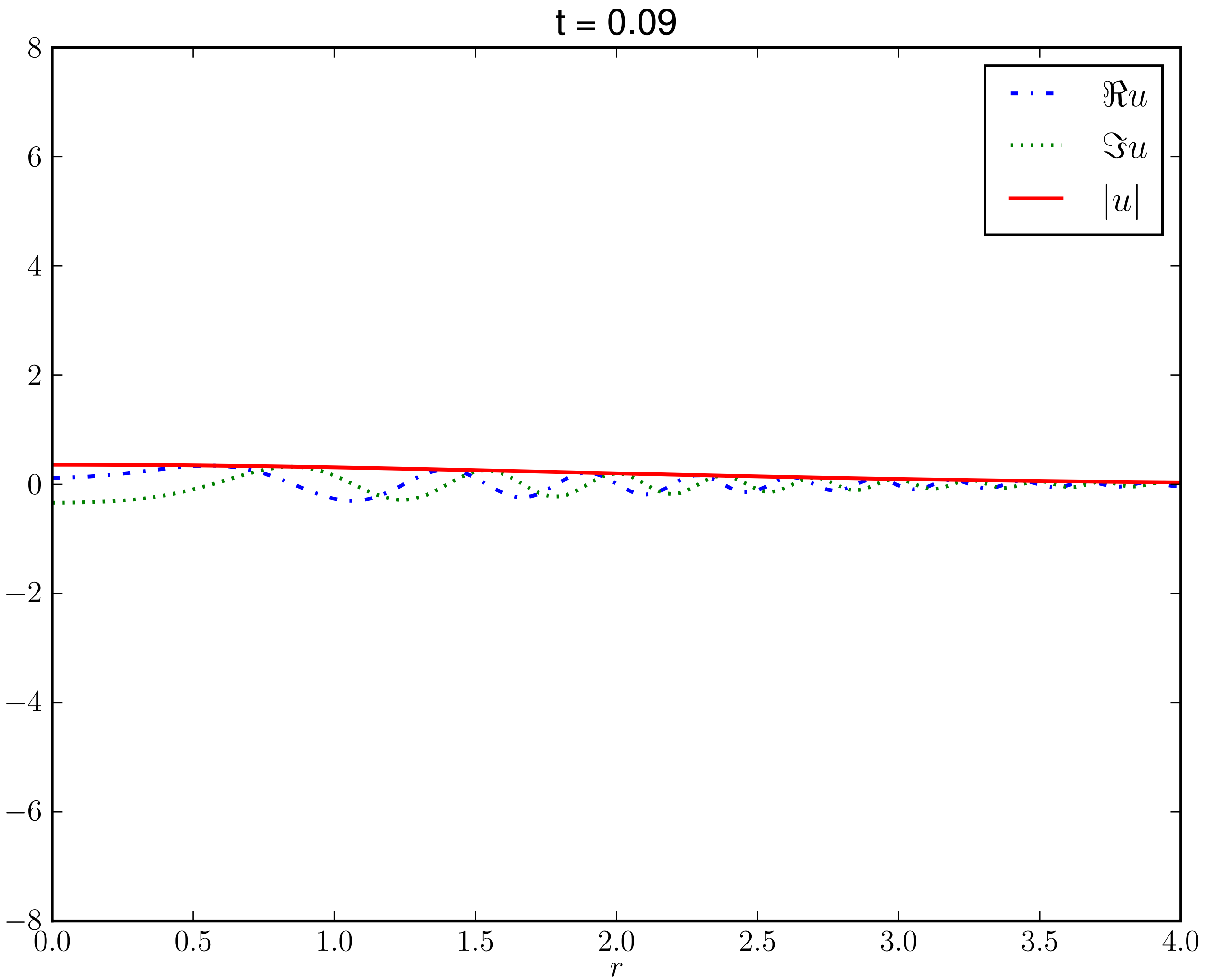


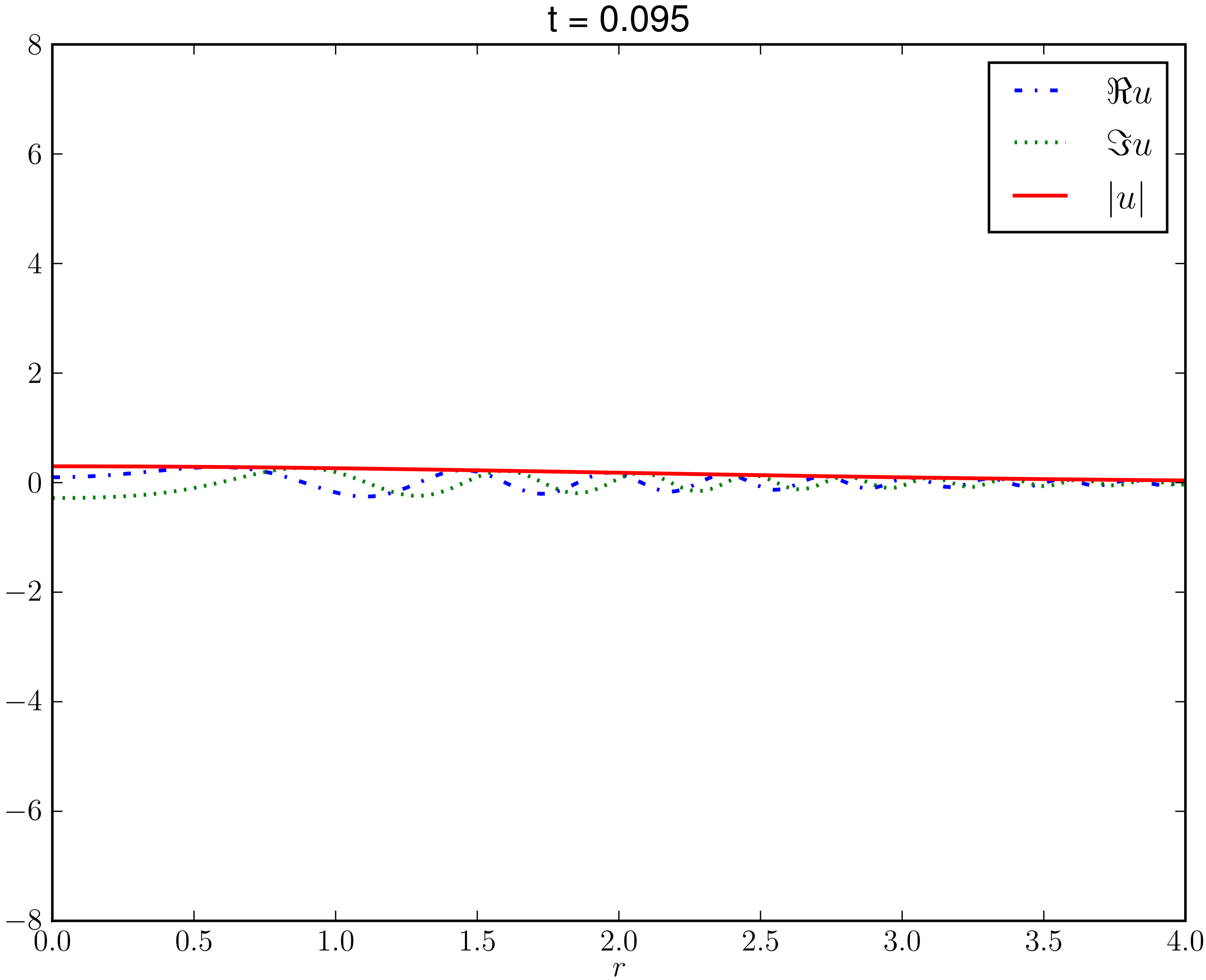
t = 0.075
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



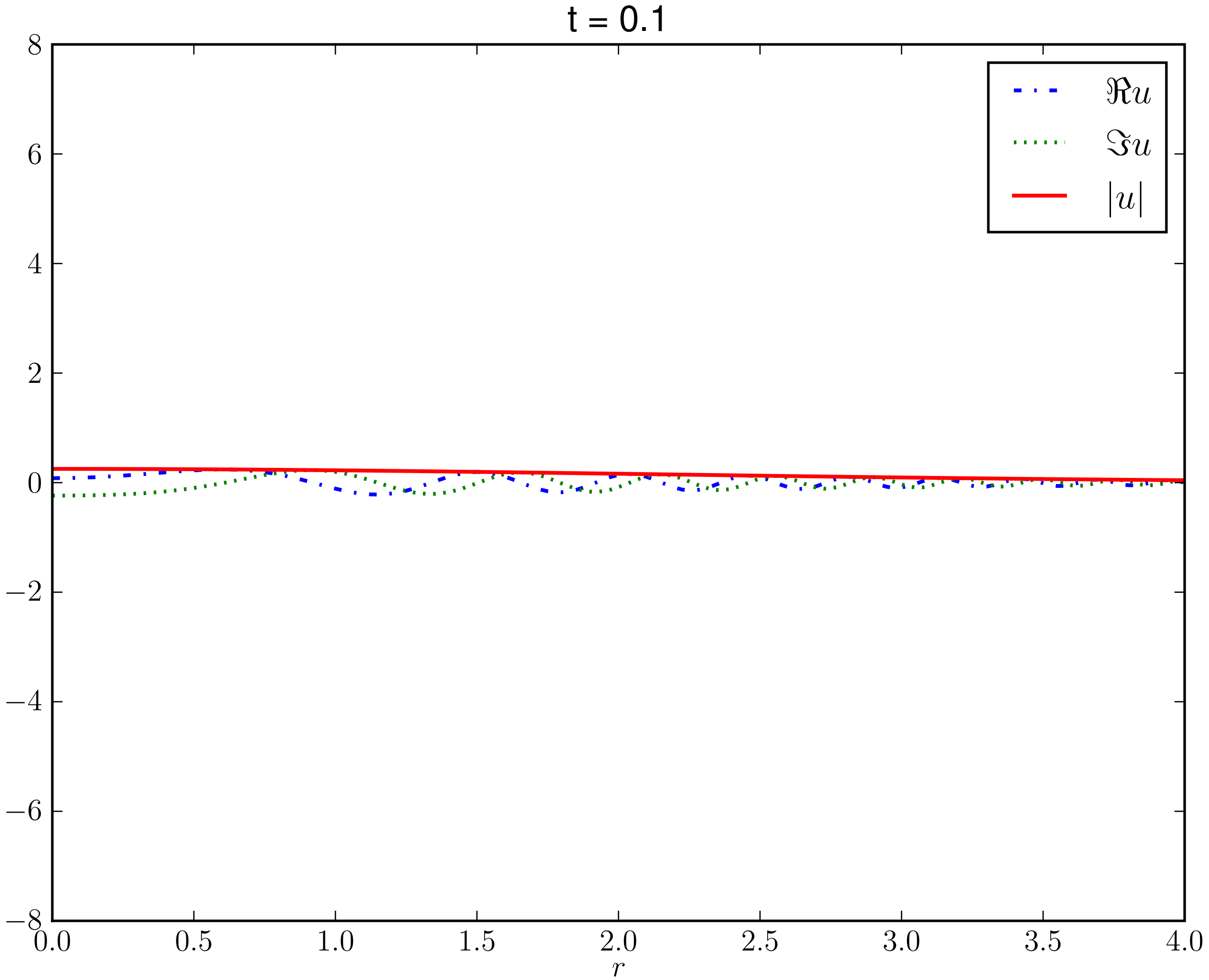
t = 0.08
$\Re u$
$\Im u$
$|u|$
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r

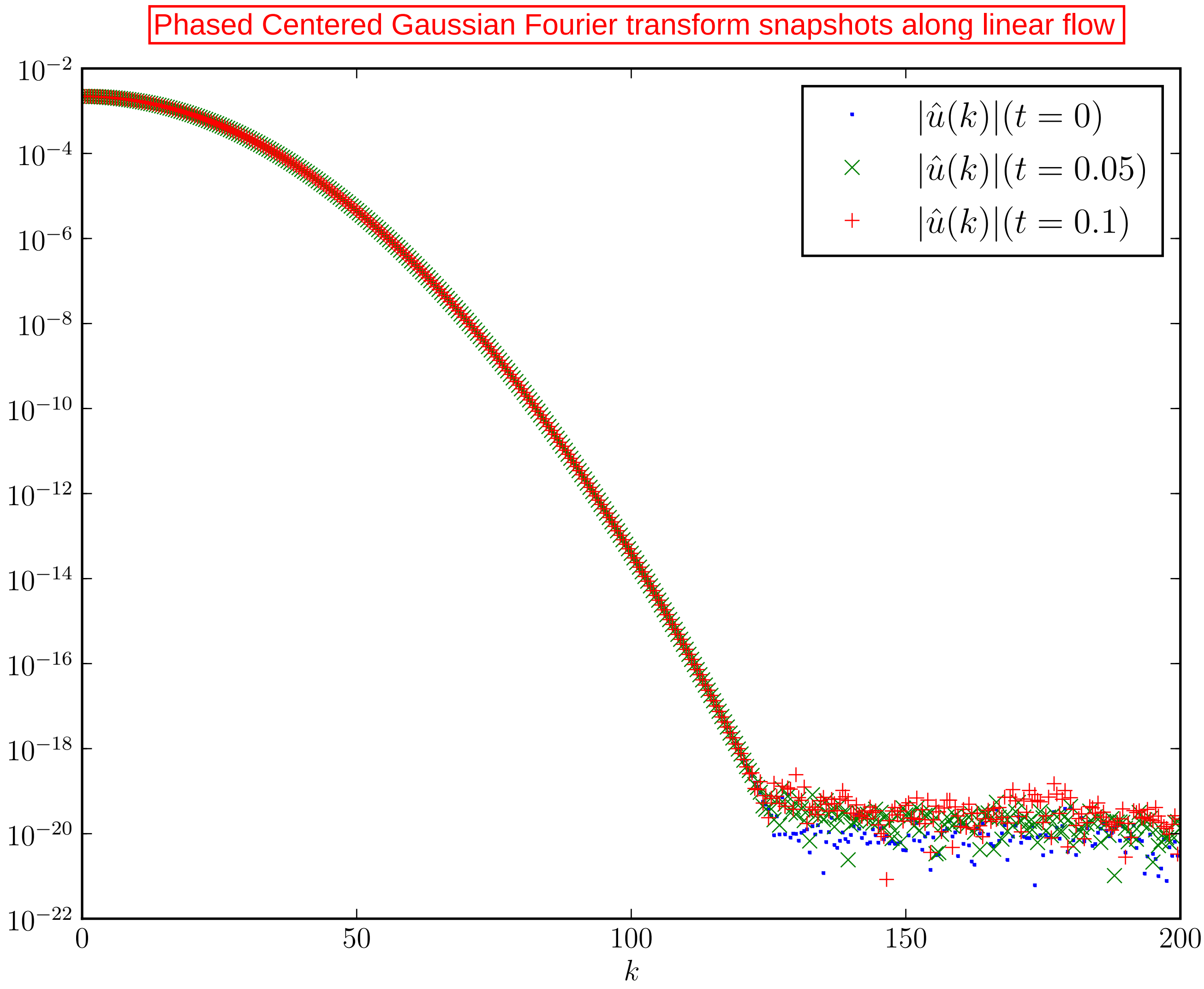

t = 0.085
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

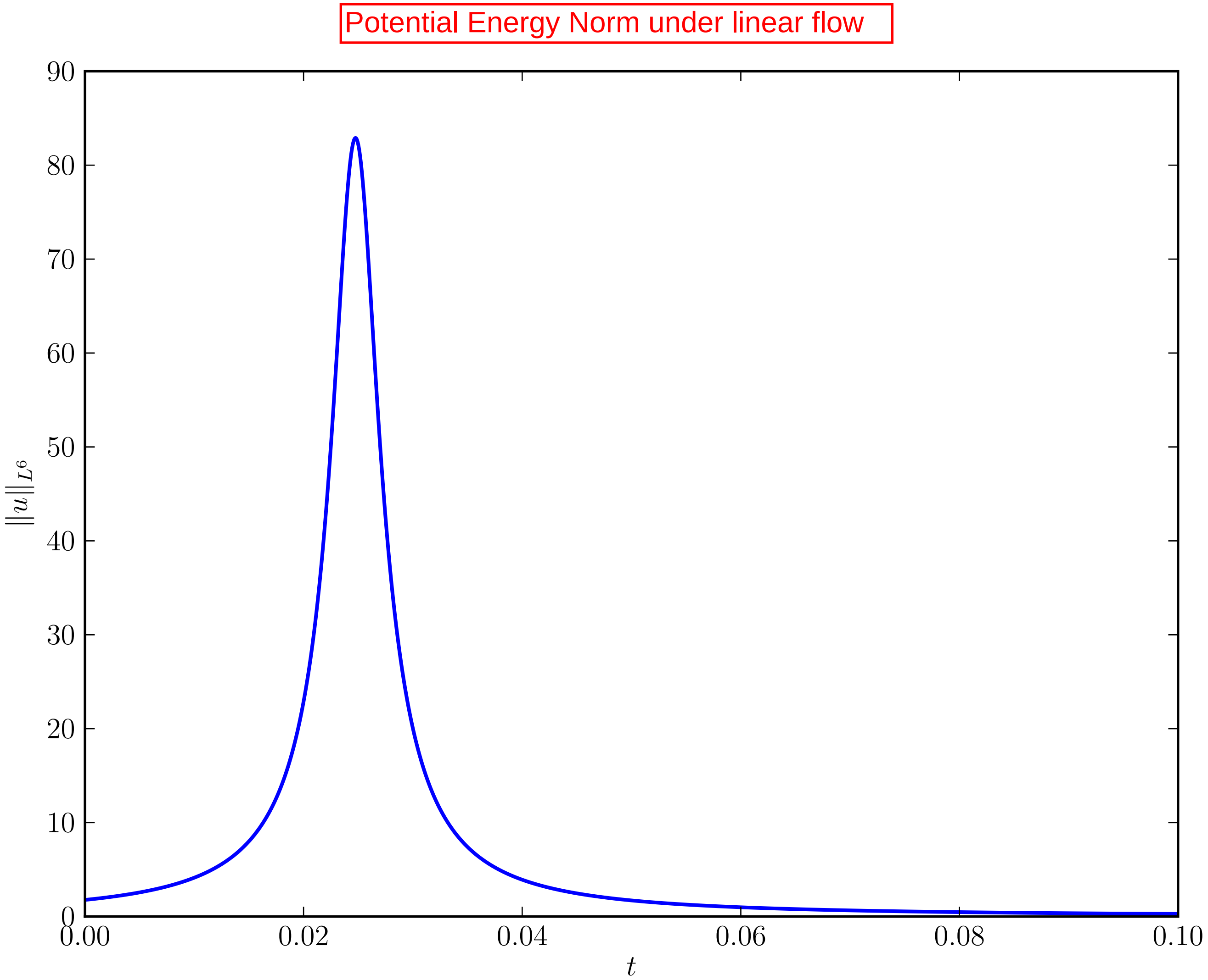

t = 0.09
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$



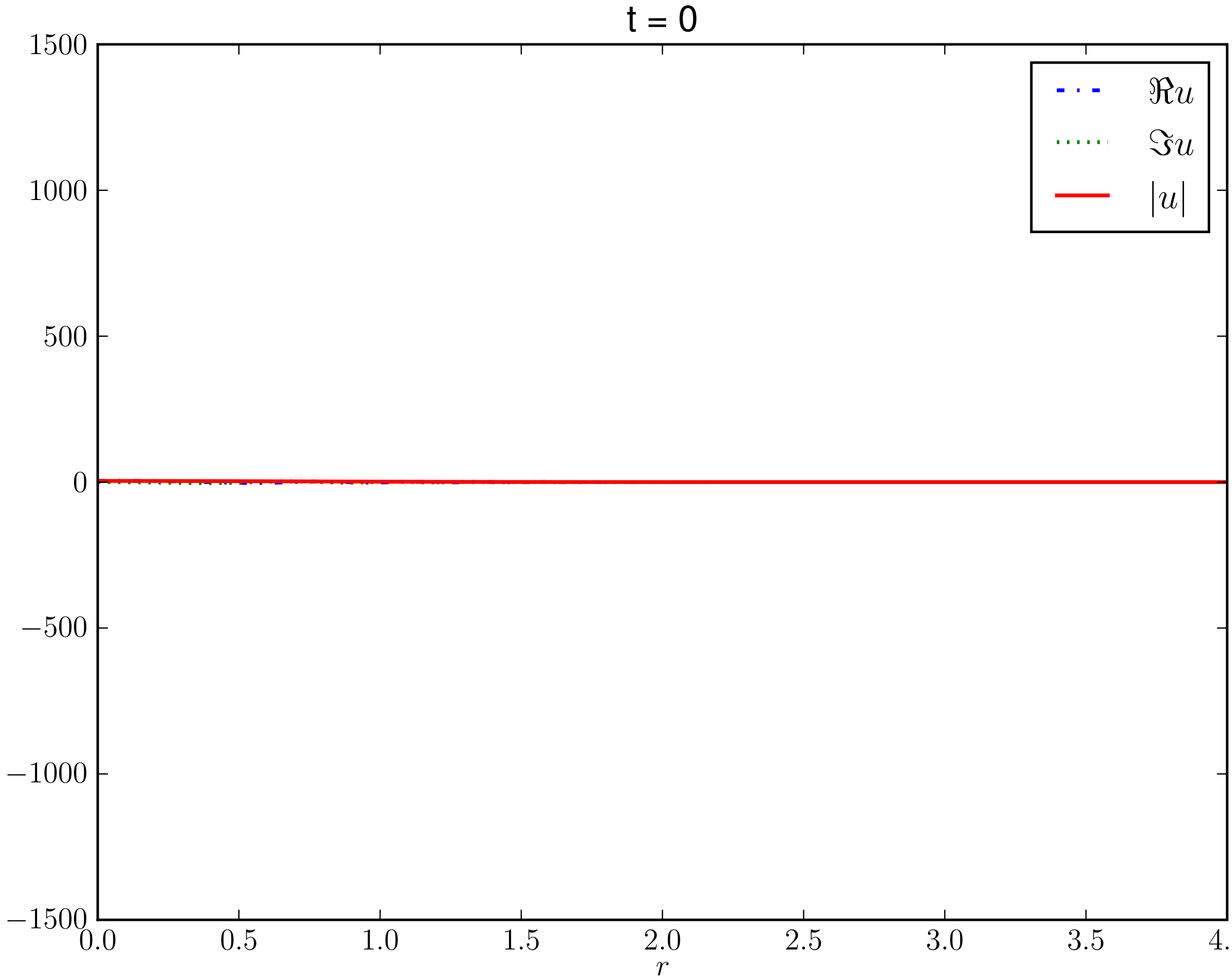
t = 0.095
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

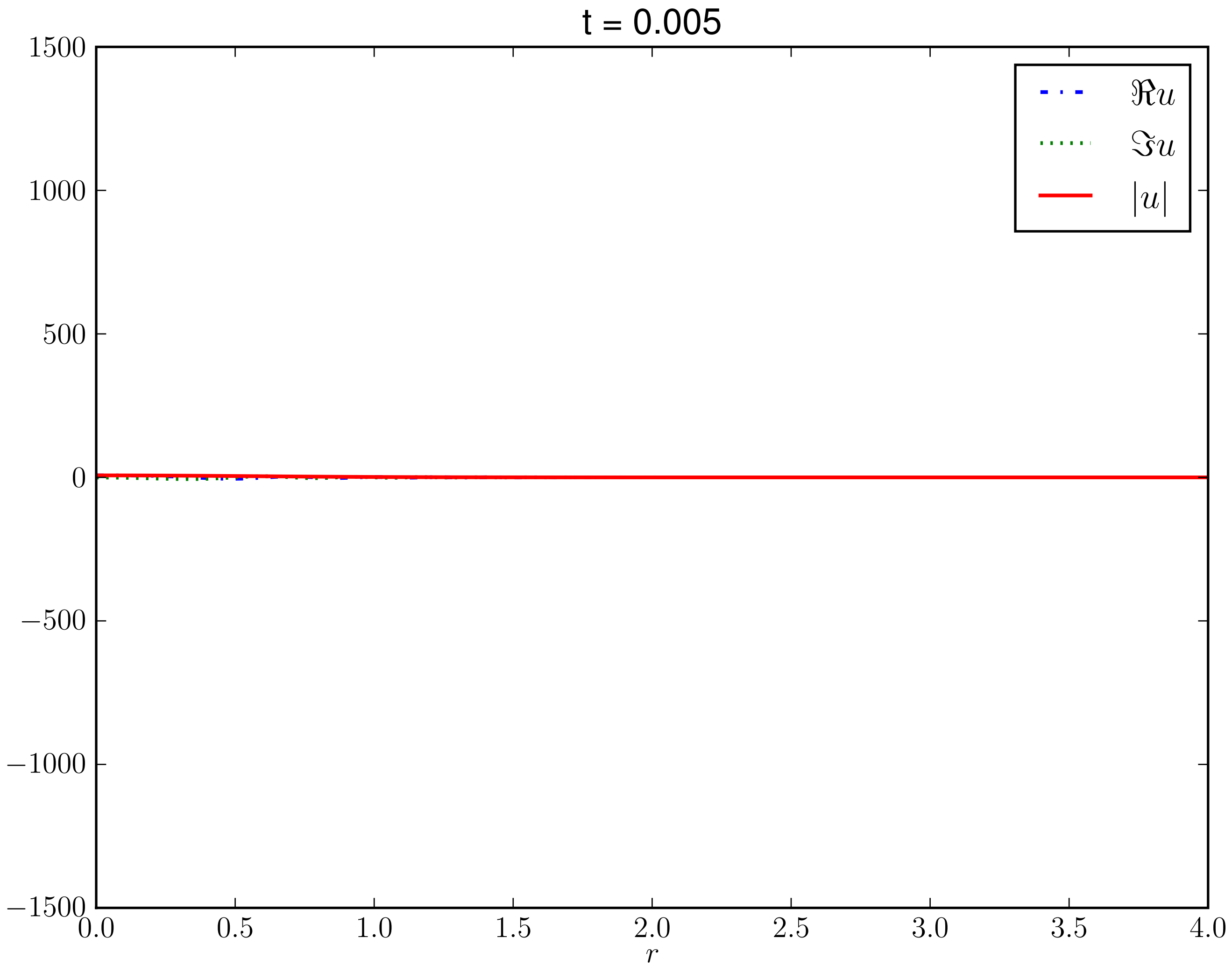

t = 0.1
8
6
4
2
0
−2
−4
−6
−8
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


Phased Centered Gaussian Fourier transform snapshots along linear flow
$|\hat{u}(k)|(t=0)$
$|\hat{u}(k)|(t=0.05)$
$|\hat{u}(k)|(t=0.1)$
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
0
50
100
150
200
k

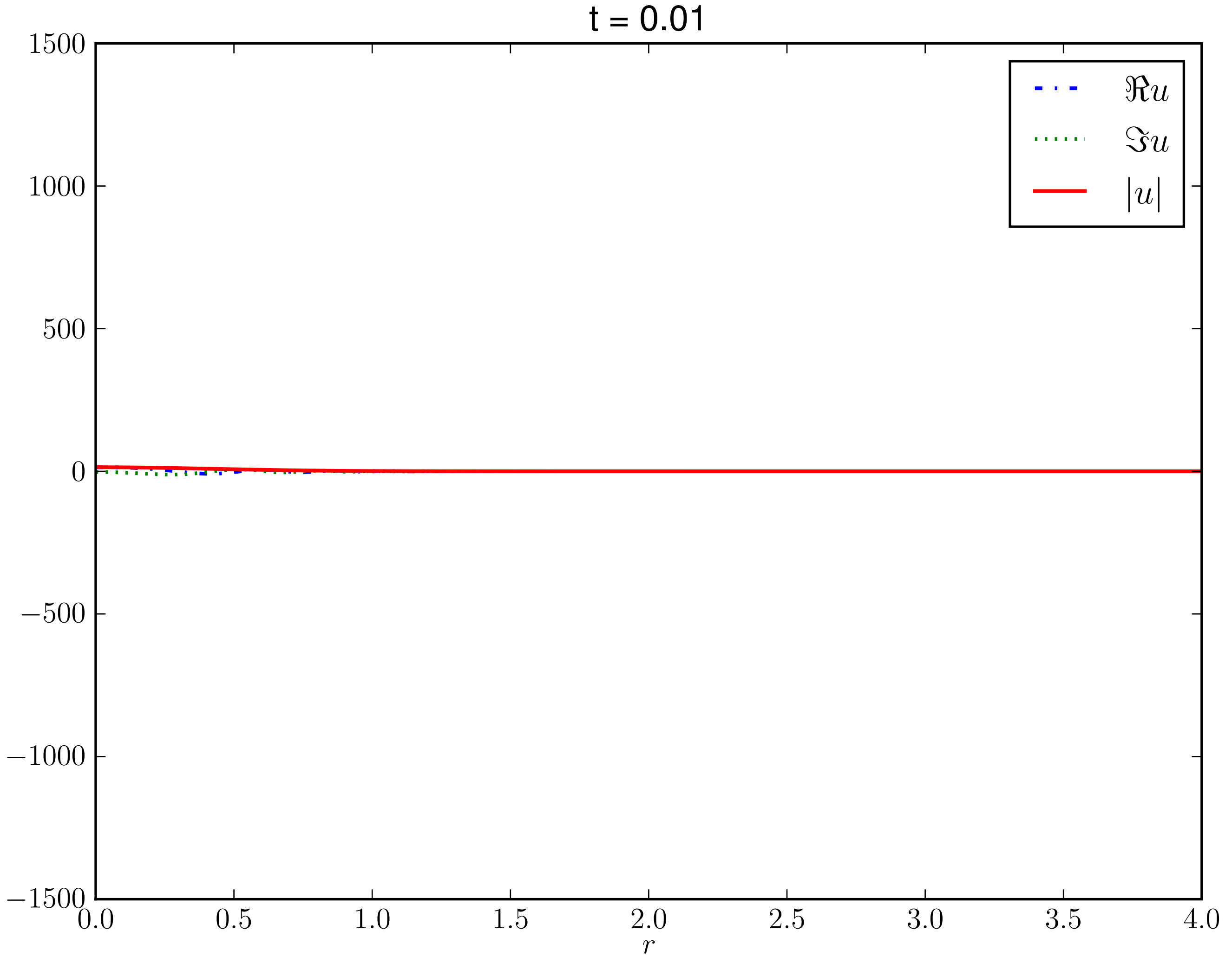

Potential Energy Norm under linear flow
90
80
70
60
50
40
30
20
10
0
0.00
0.02
0.04
0.06
0.08
0.10
t
$\|u\|_{L^6}$

Phased Centered Gaussian under linear flow; bigger vertical axis

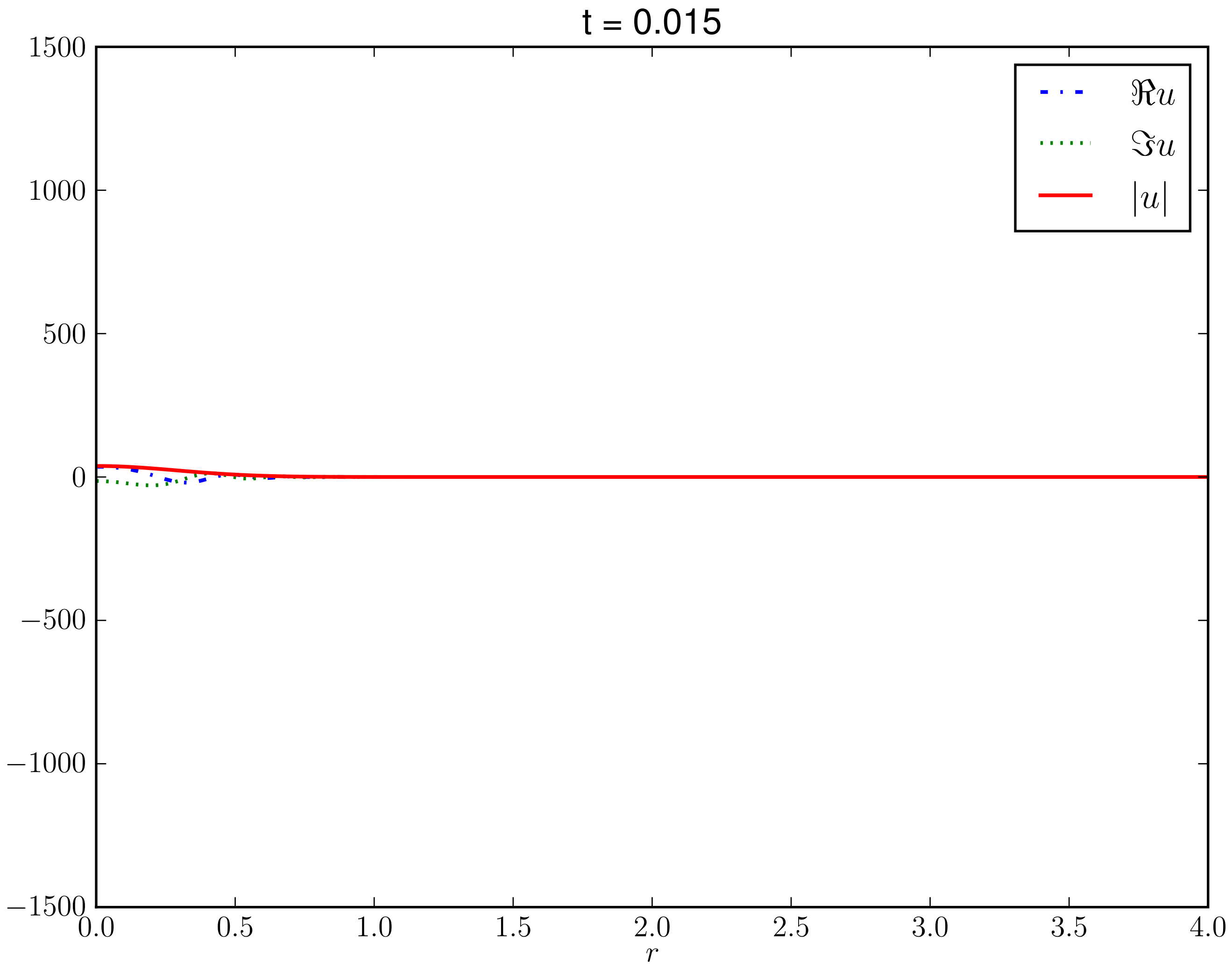




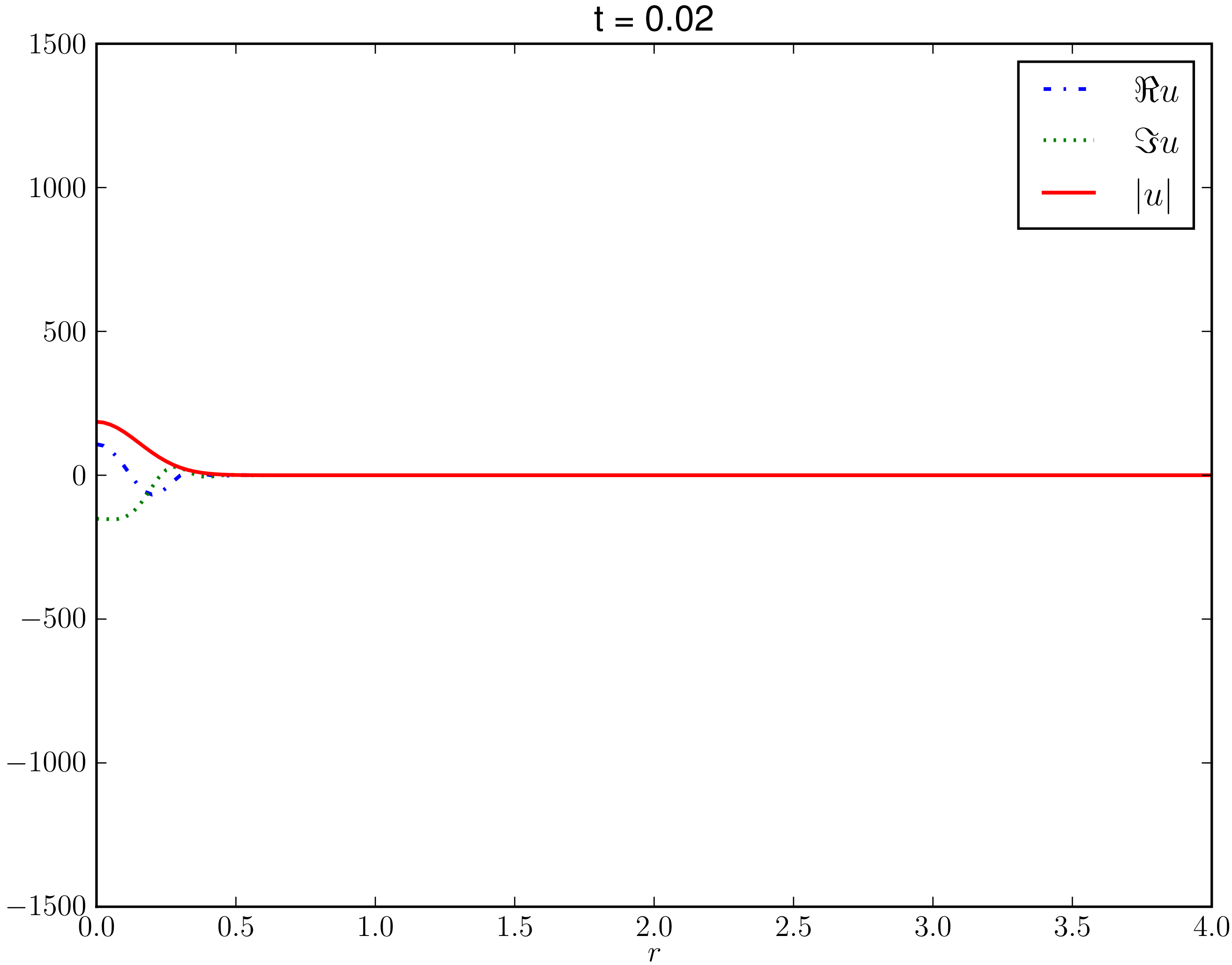
t = 0.005
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



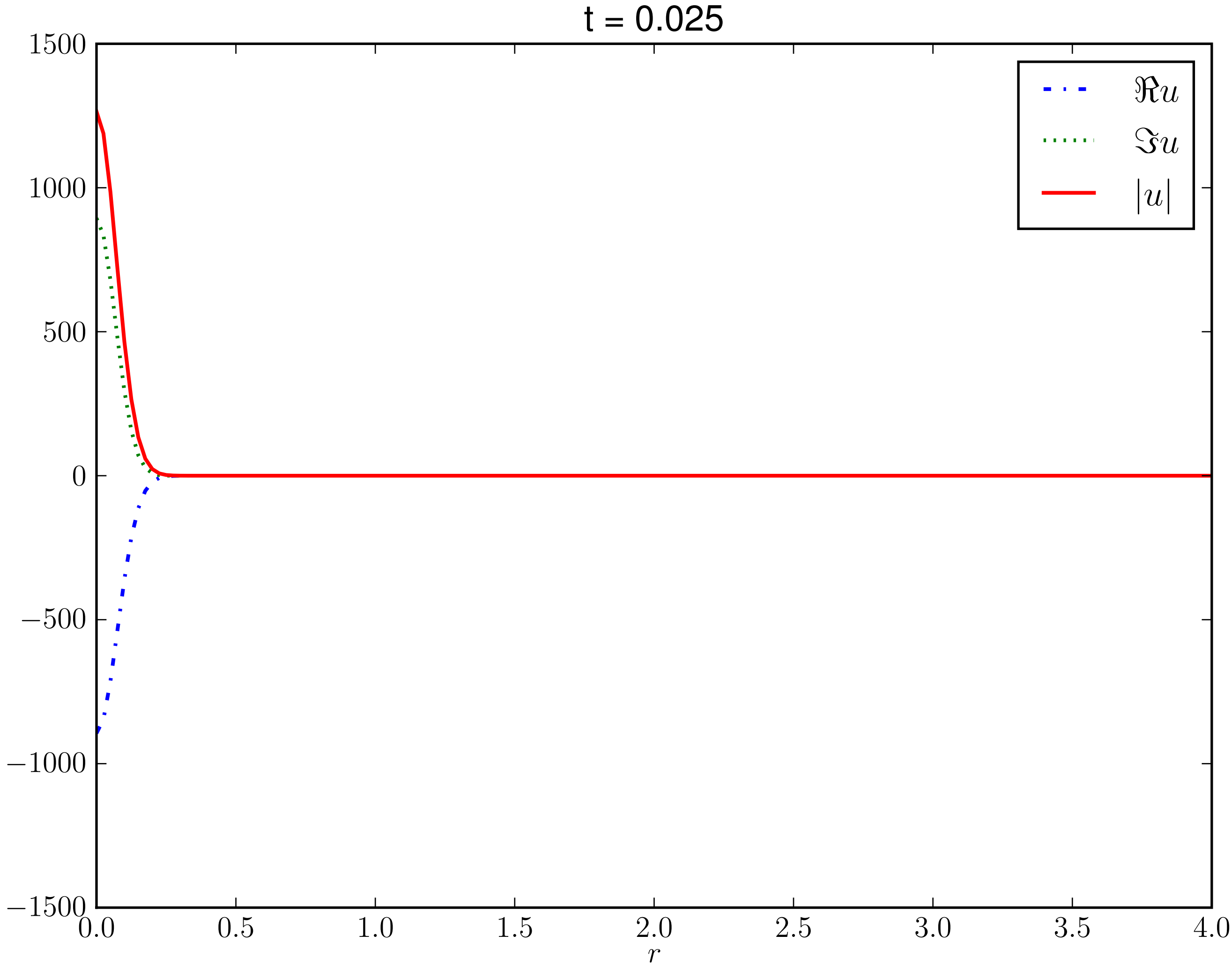
t = 0.01
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



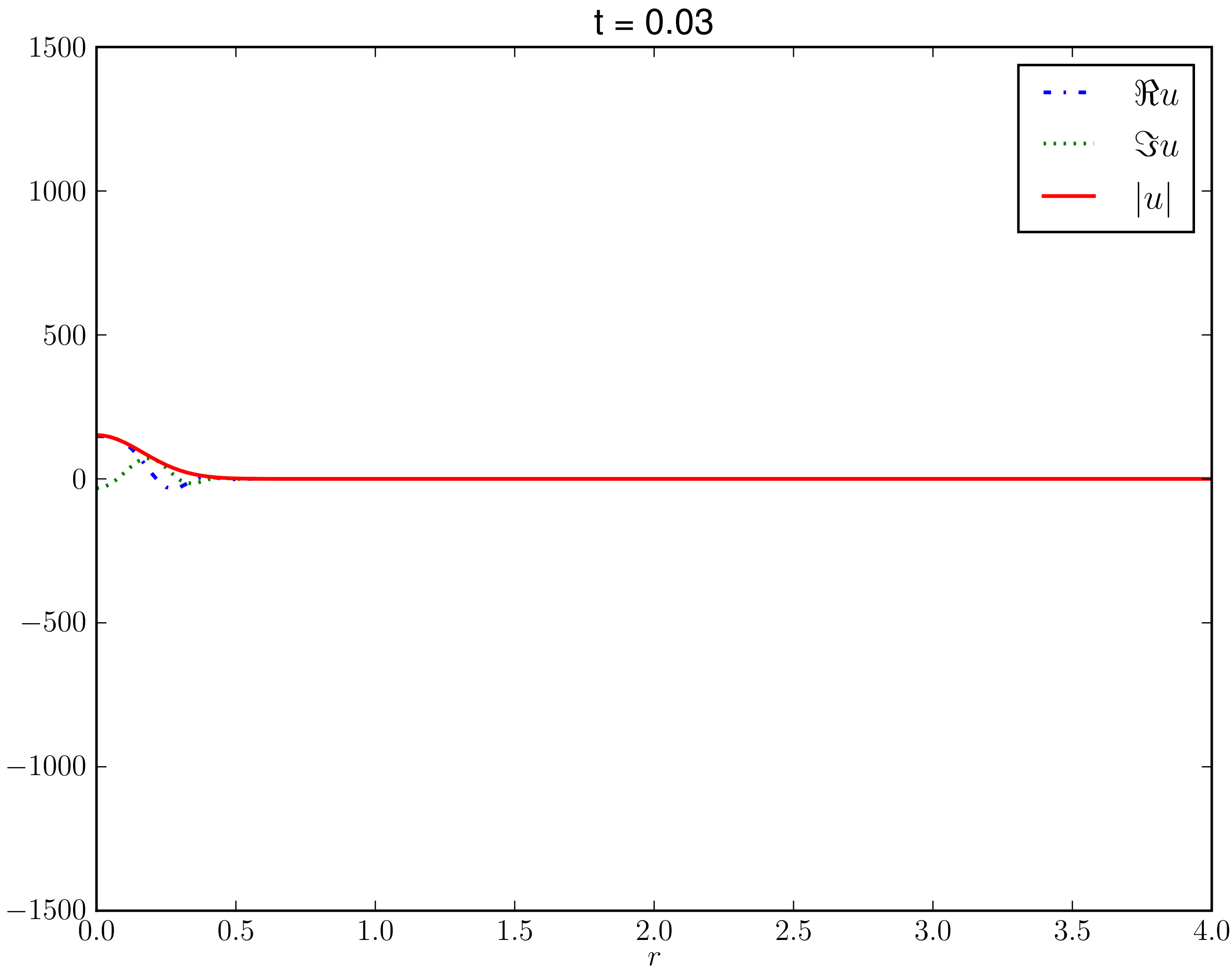
t = 0.015
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



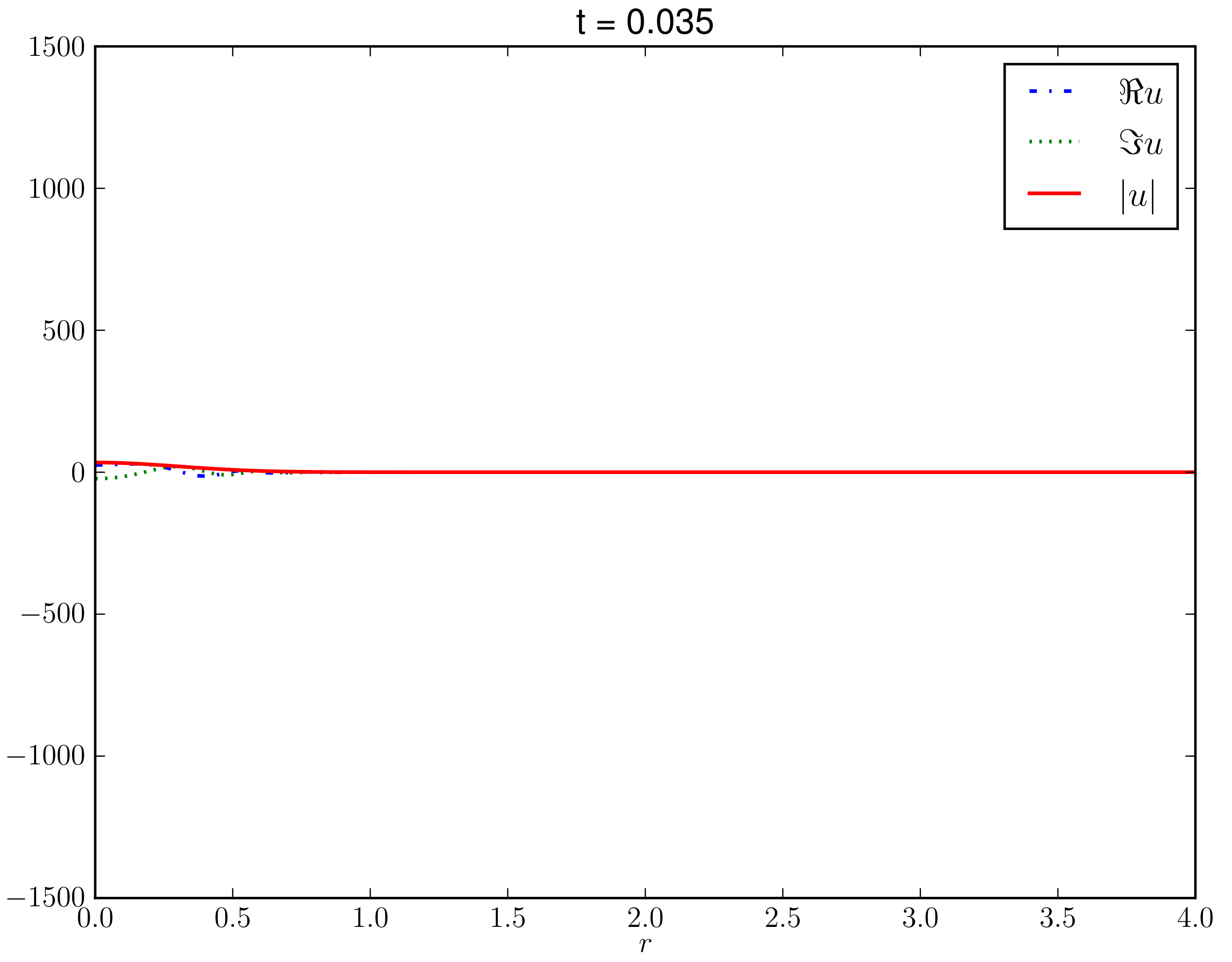
t = 0.02
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$



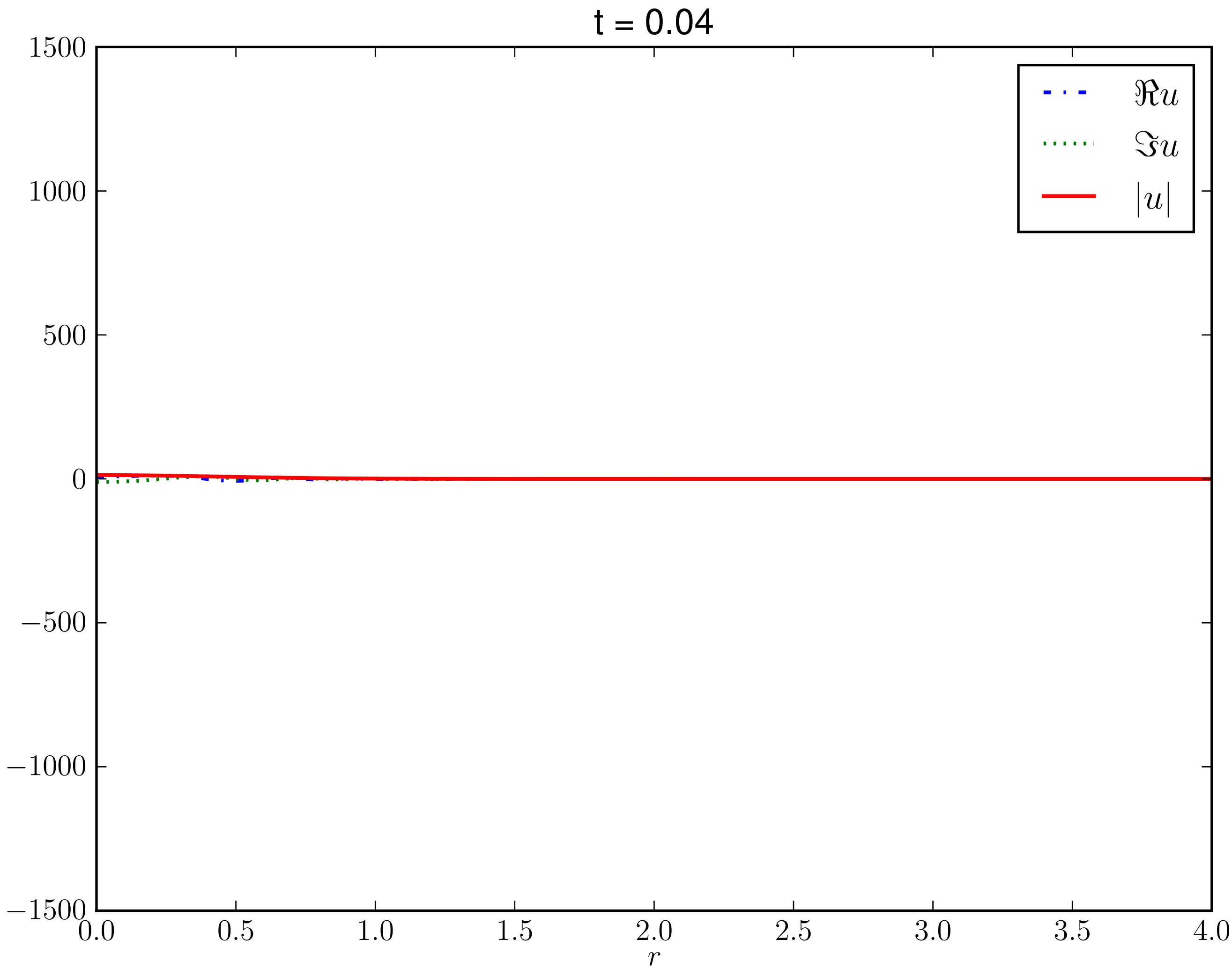
t = 0.025
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

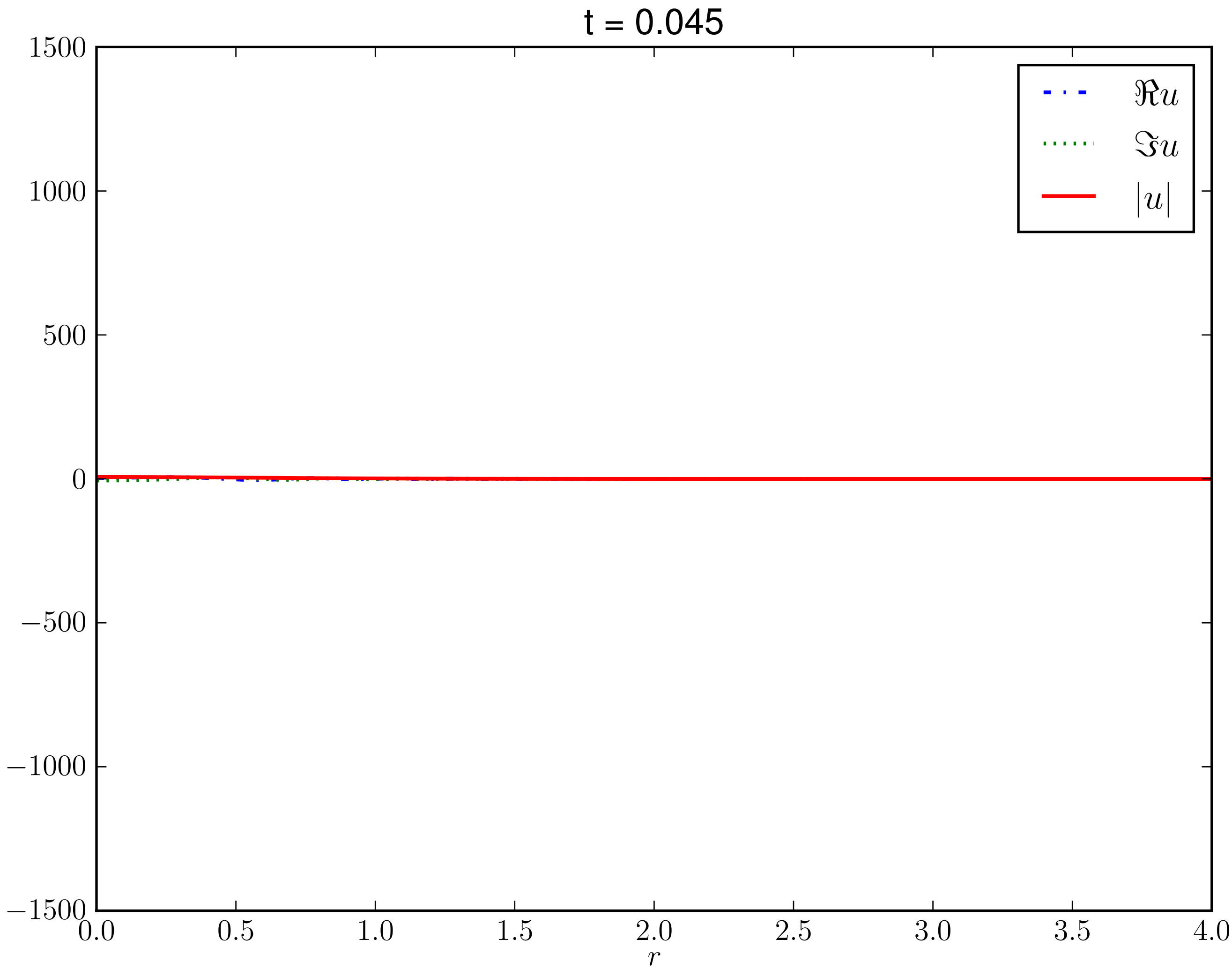


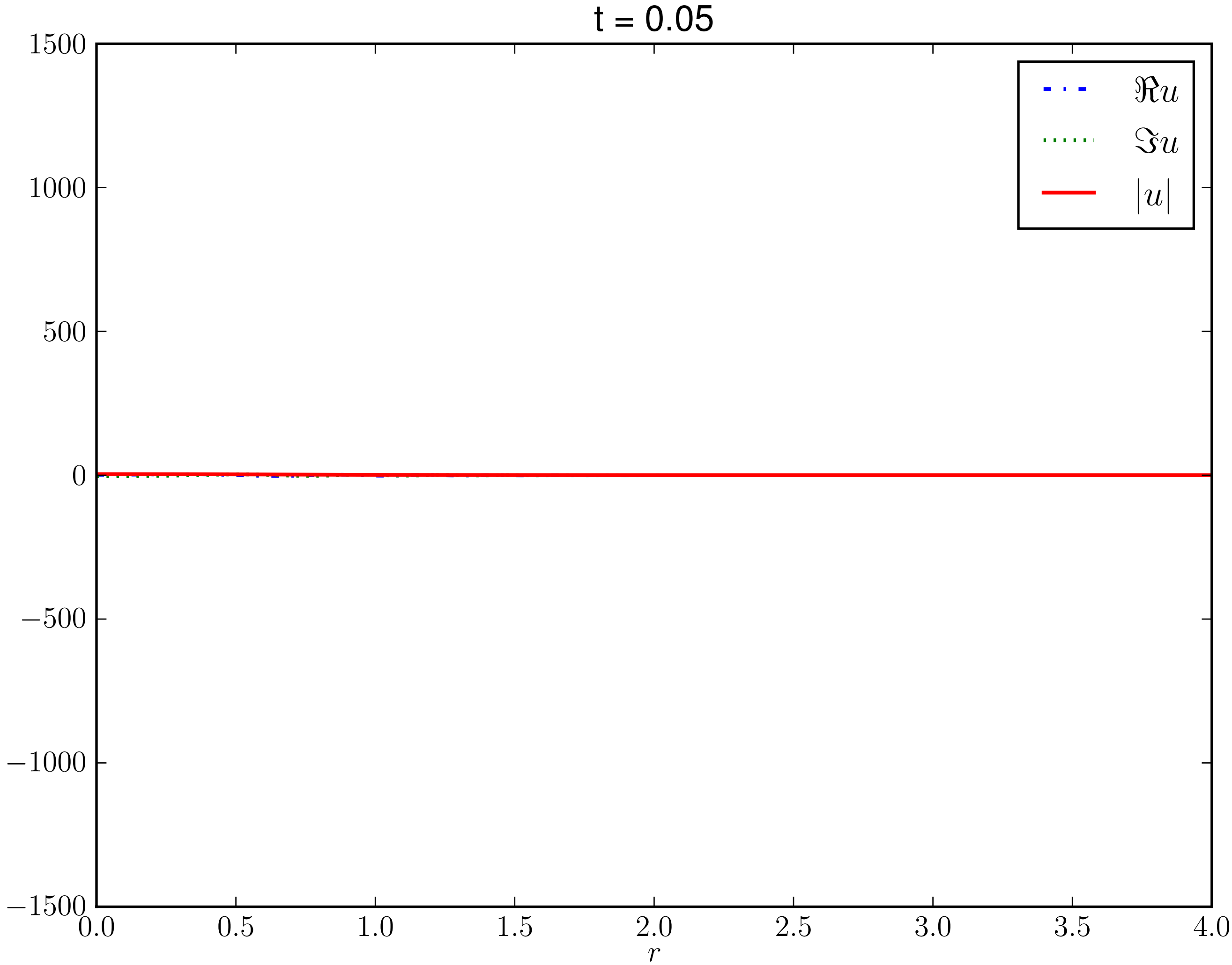
t = 0.03
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$



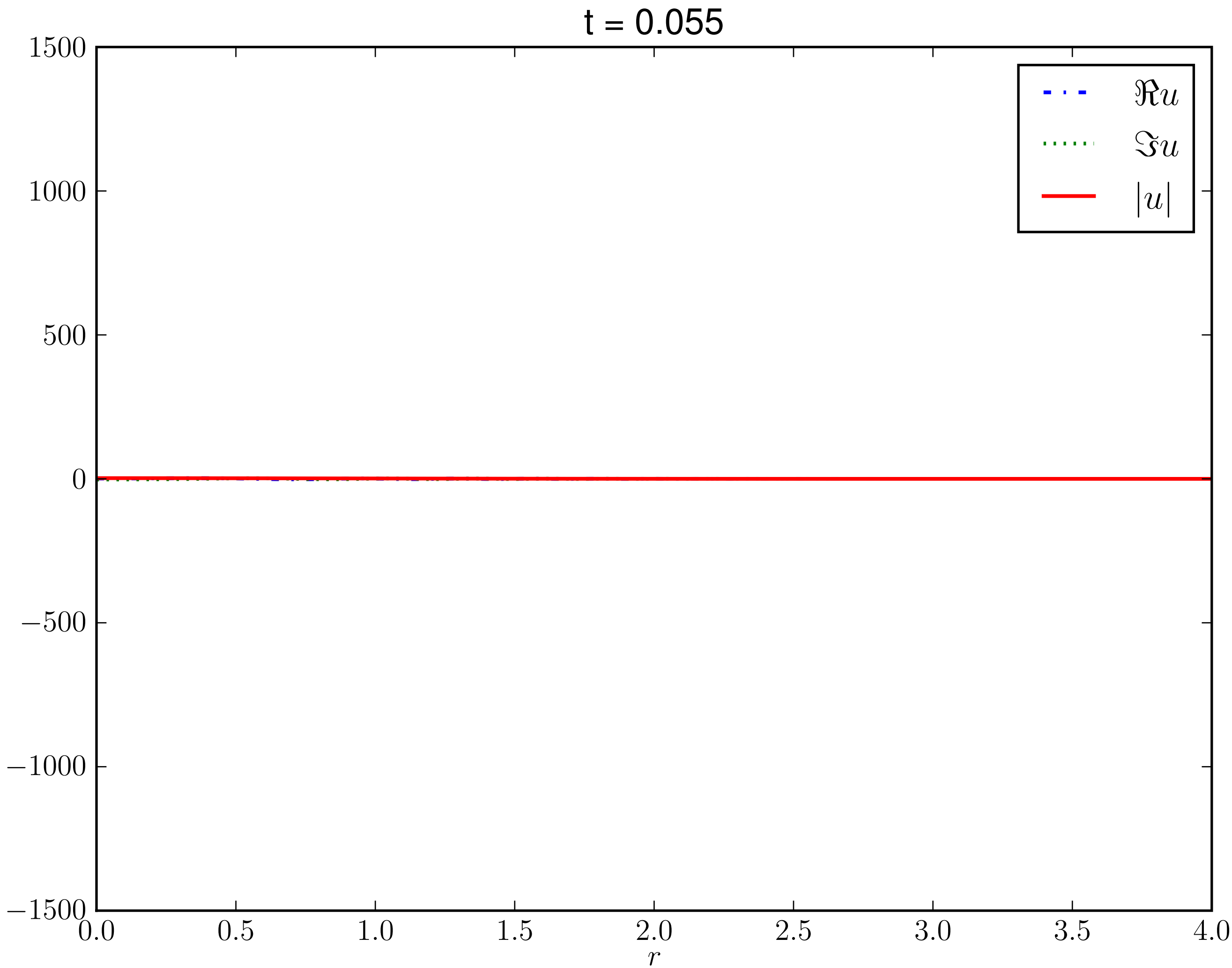
t = 0.035
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.04
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

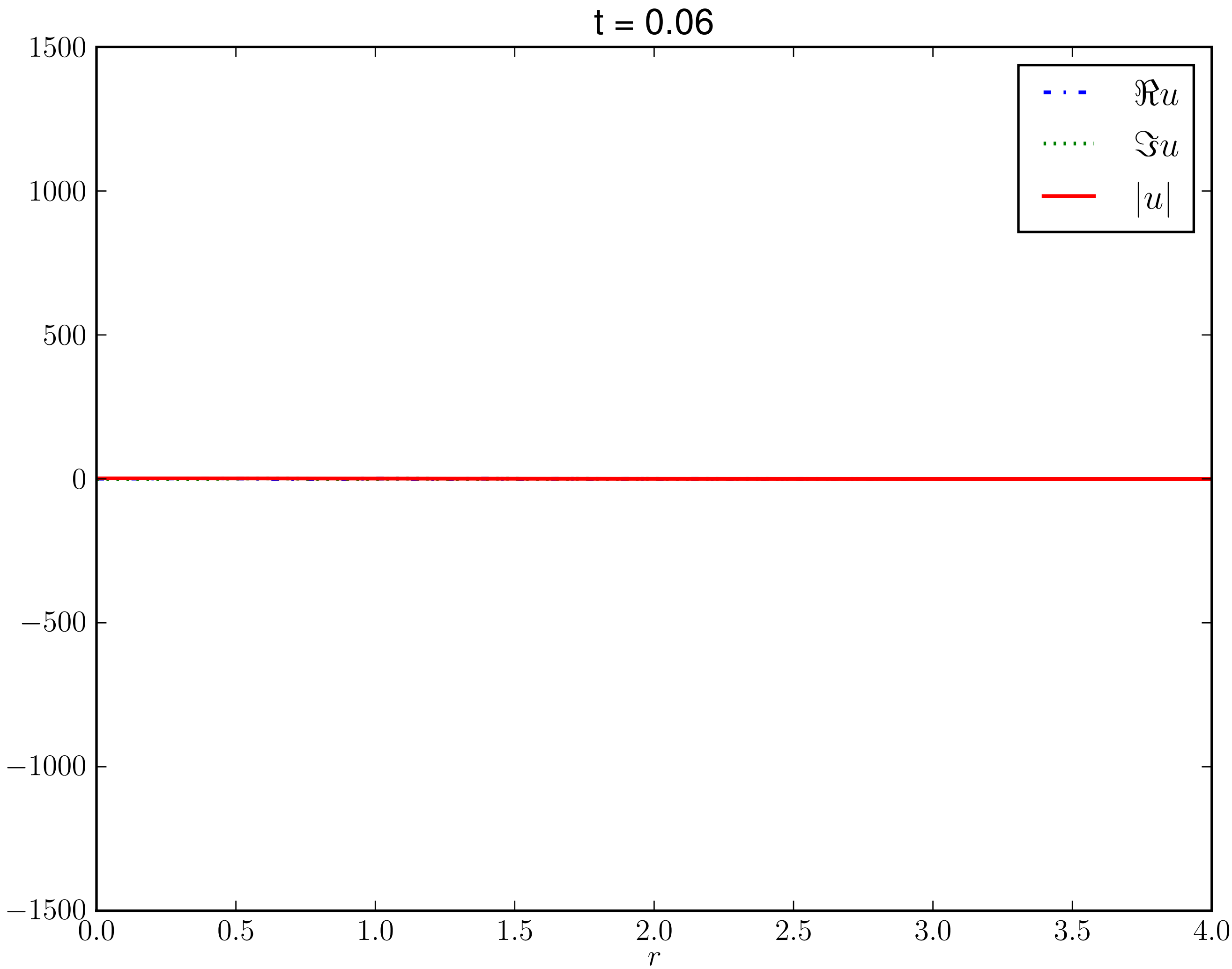

t = 0.045
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



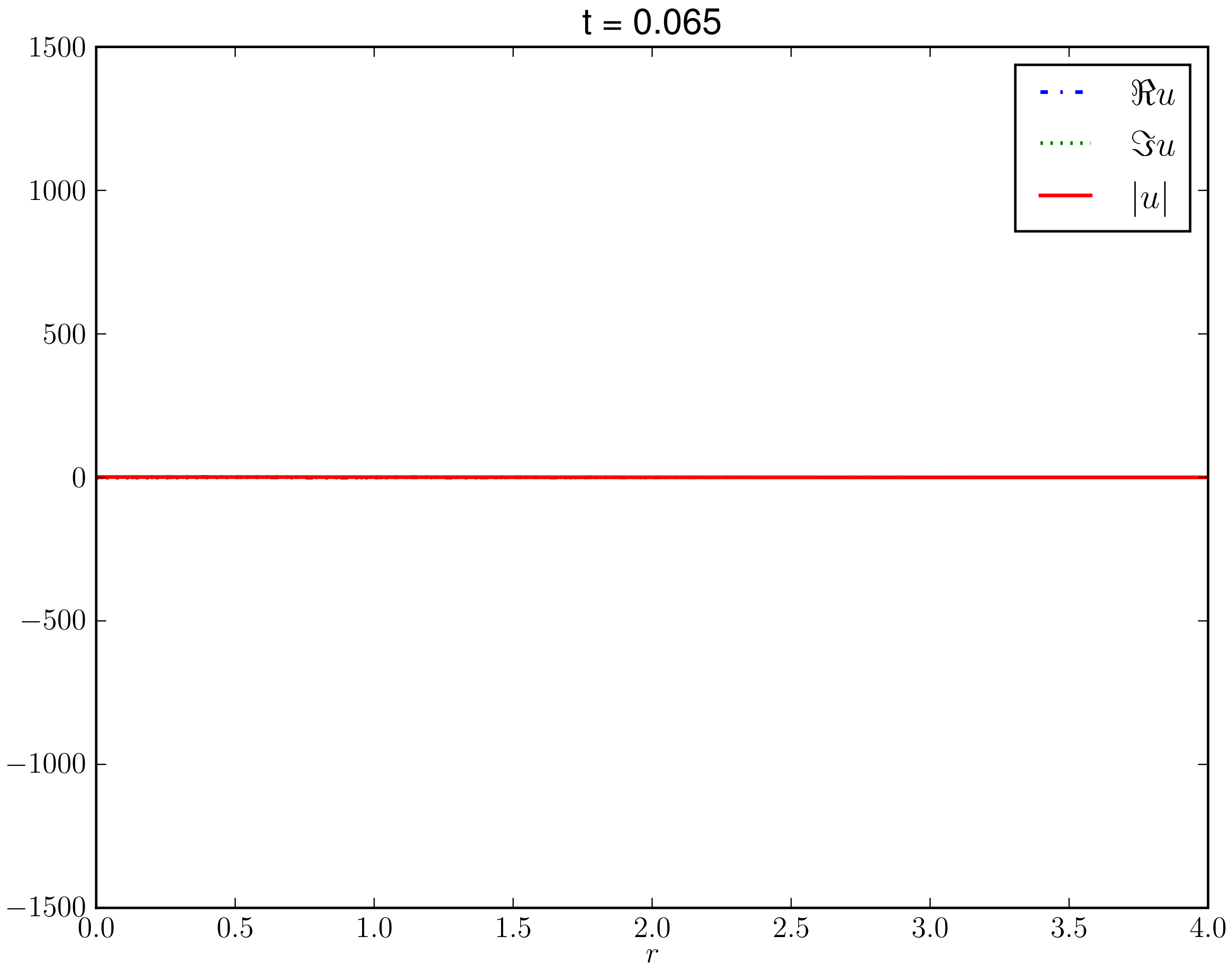
t = 0.05
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



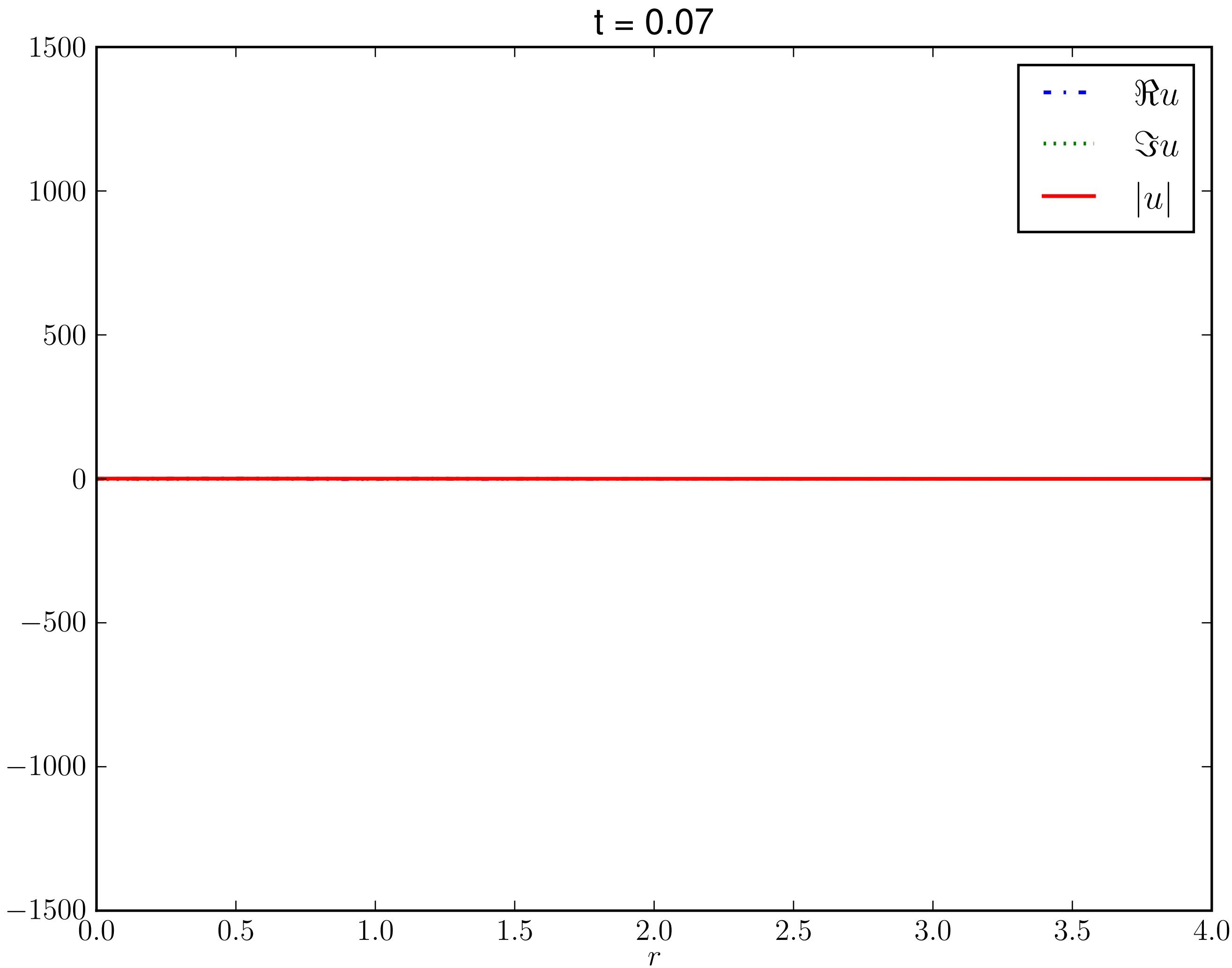
t = 0.055
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

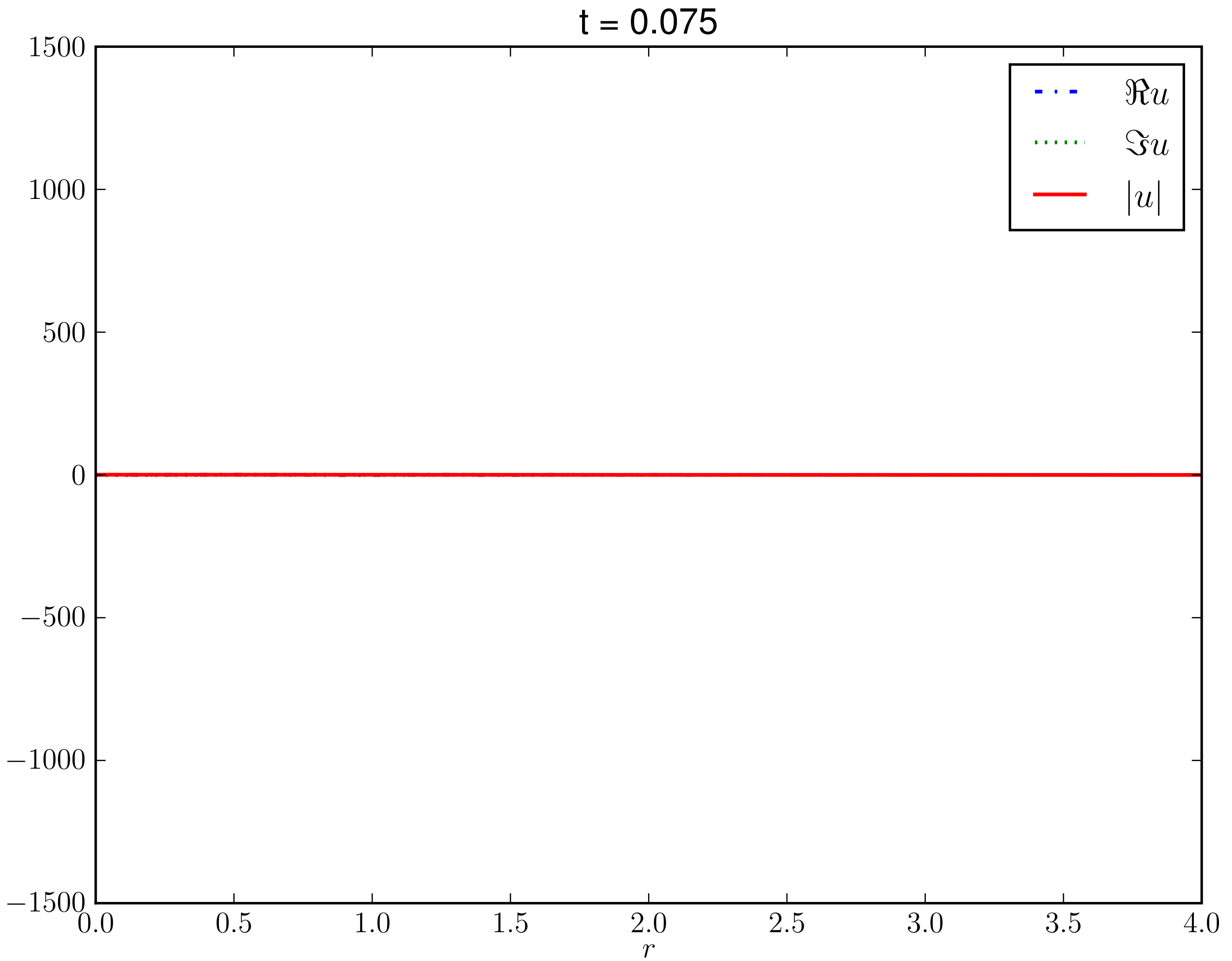


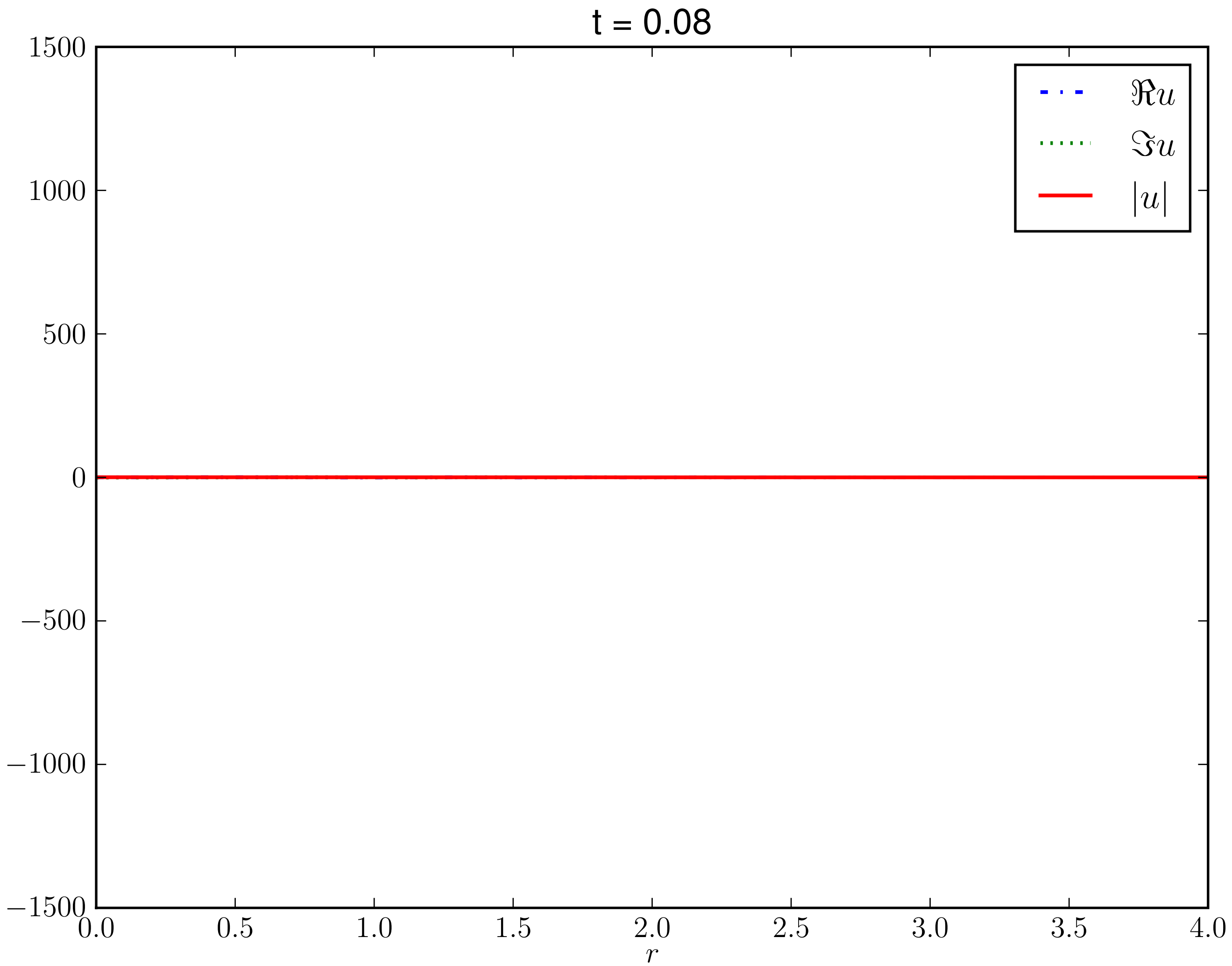
t = 0.06
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



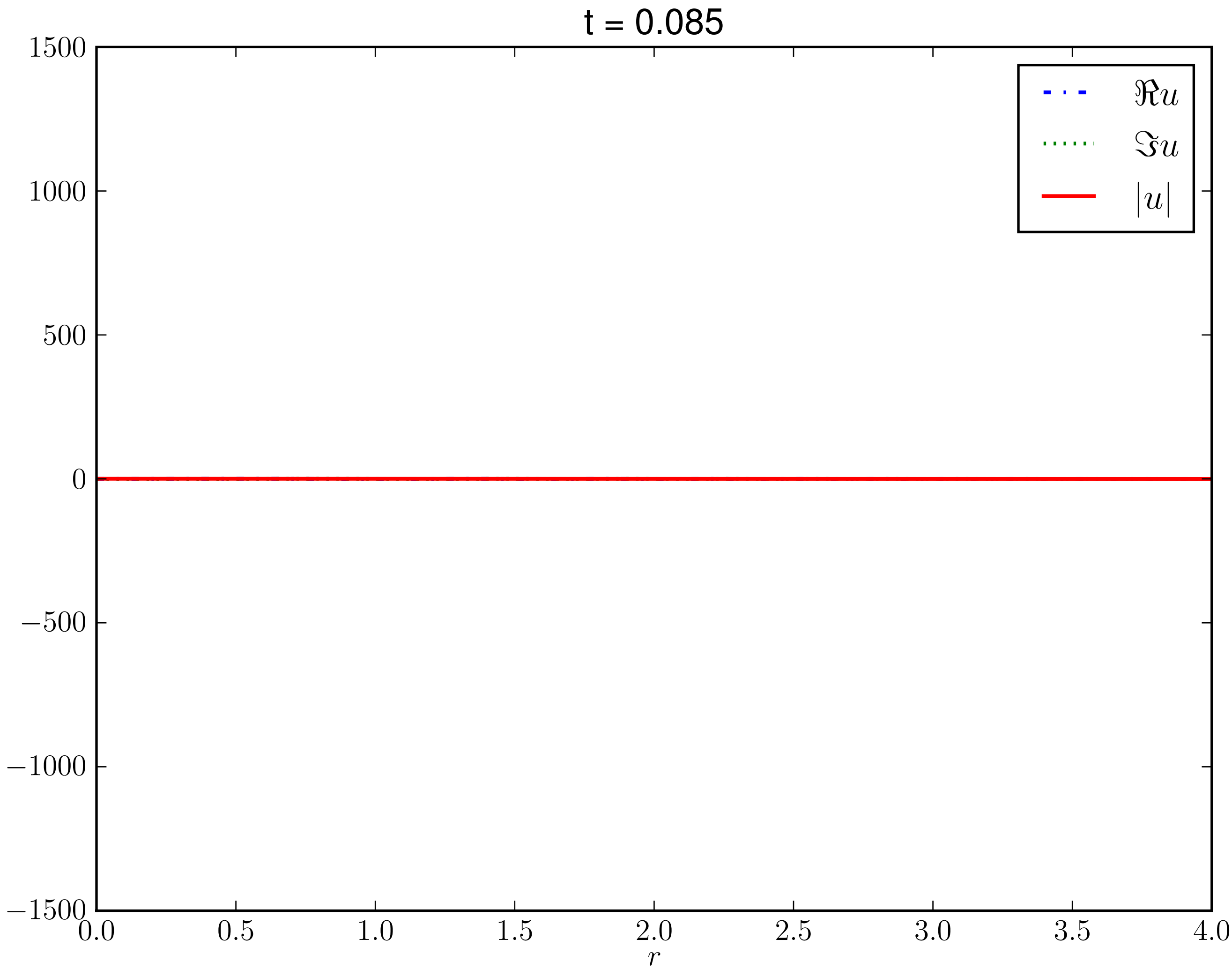
t = 0.065
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|


t = 0.07
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

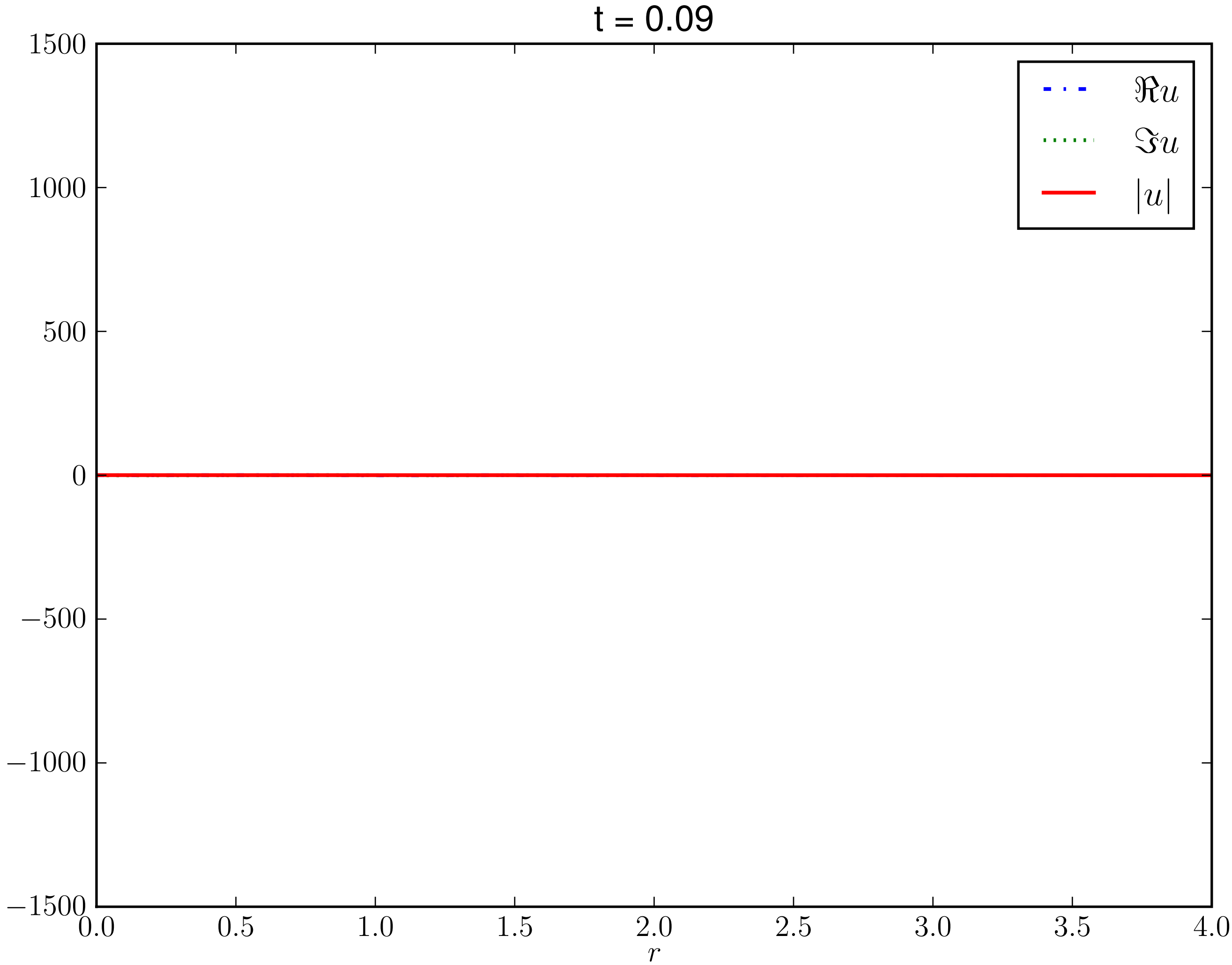

t = 0.075
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

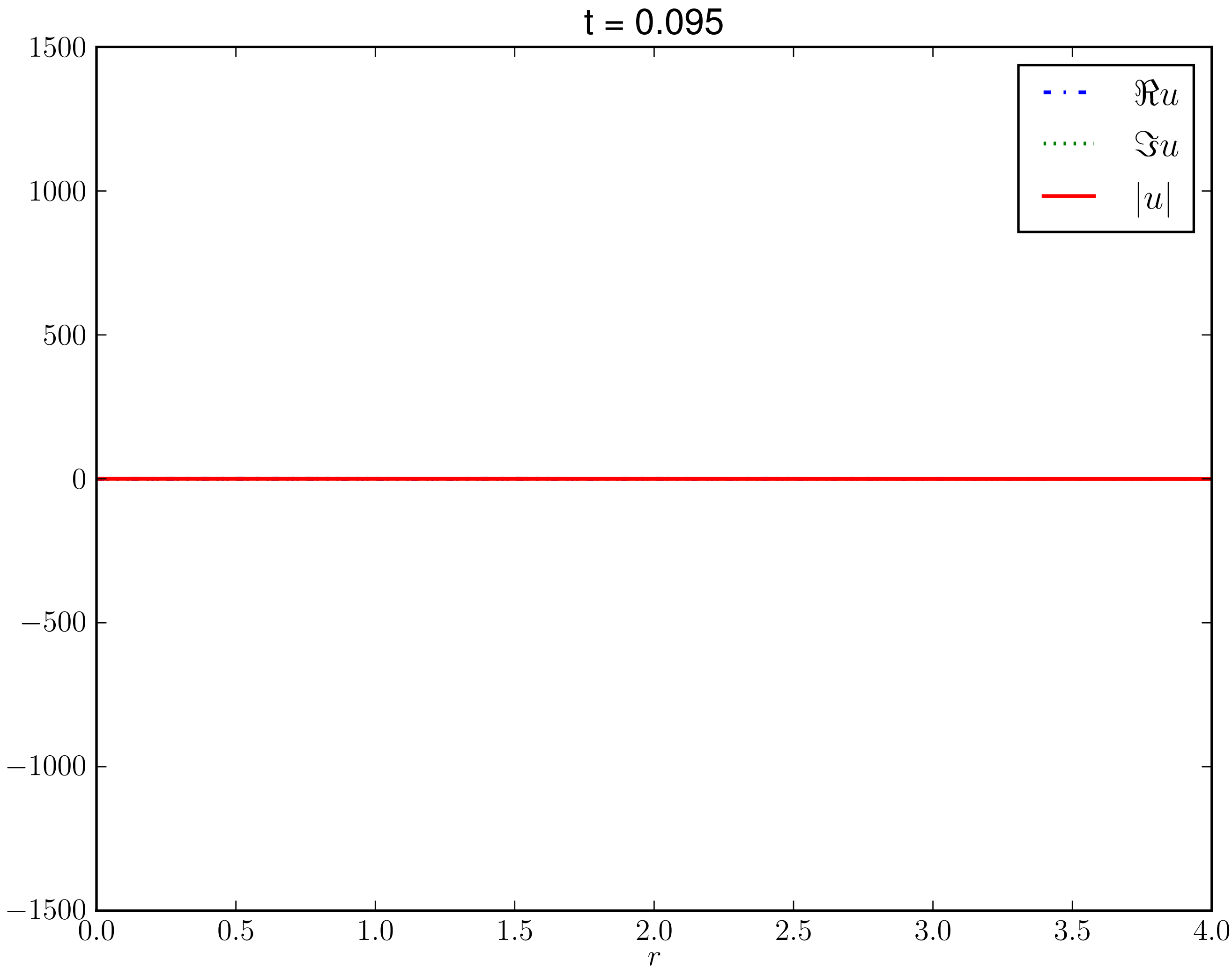


t = 0.08
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

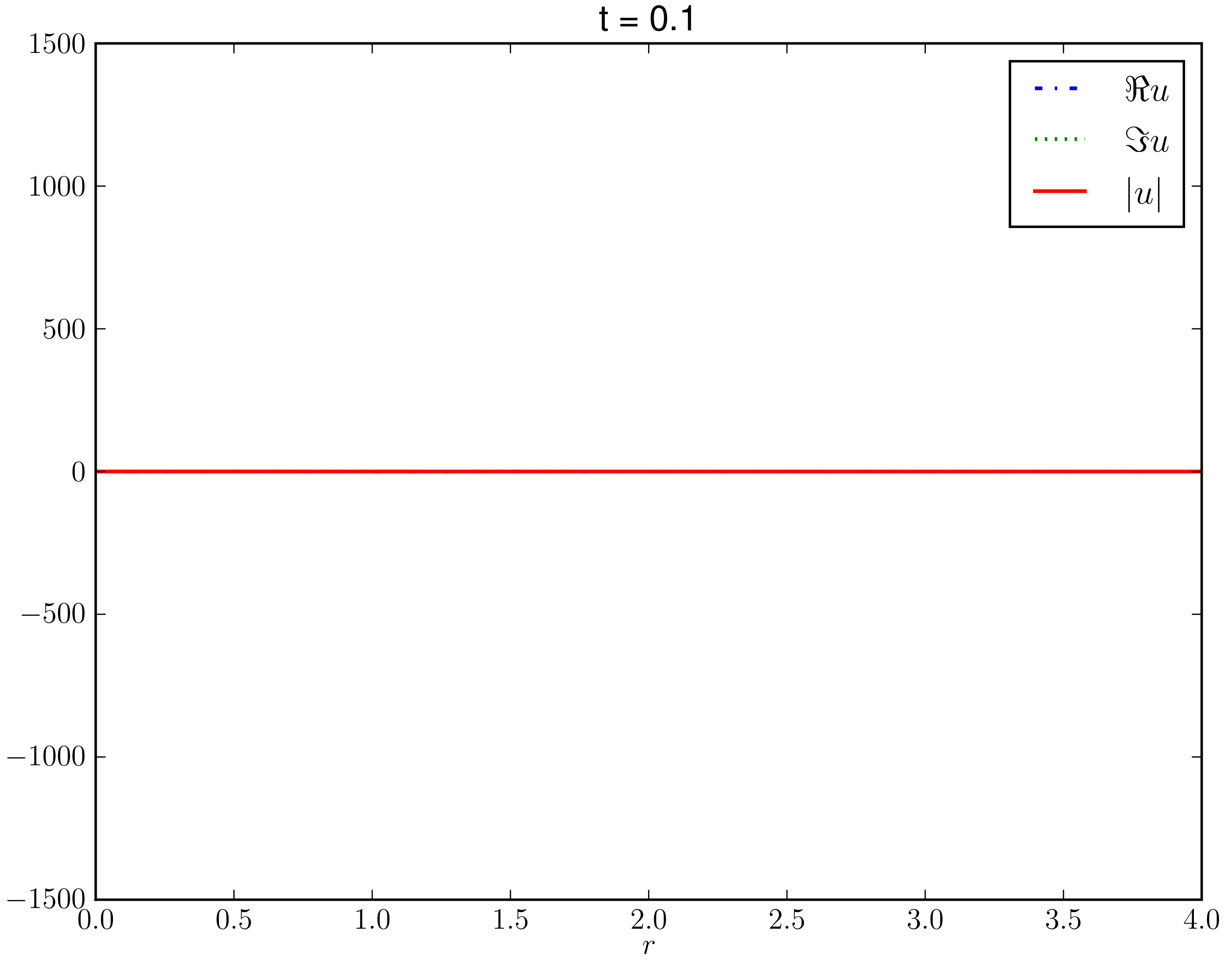


t = 0.085
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$

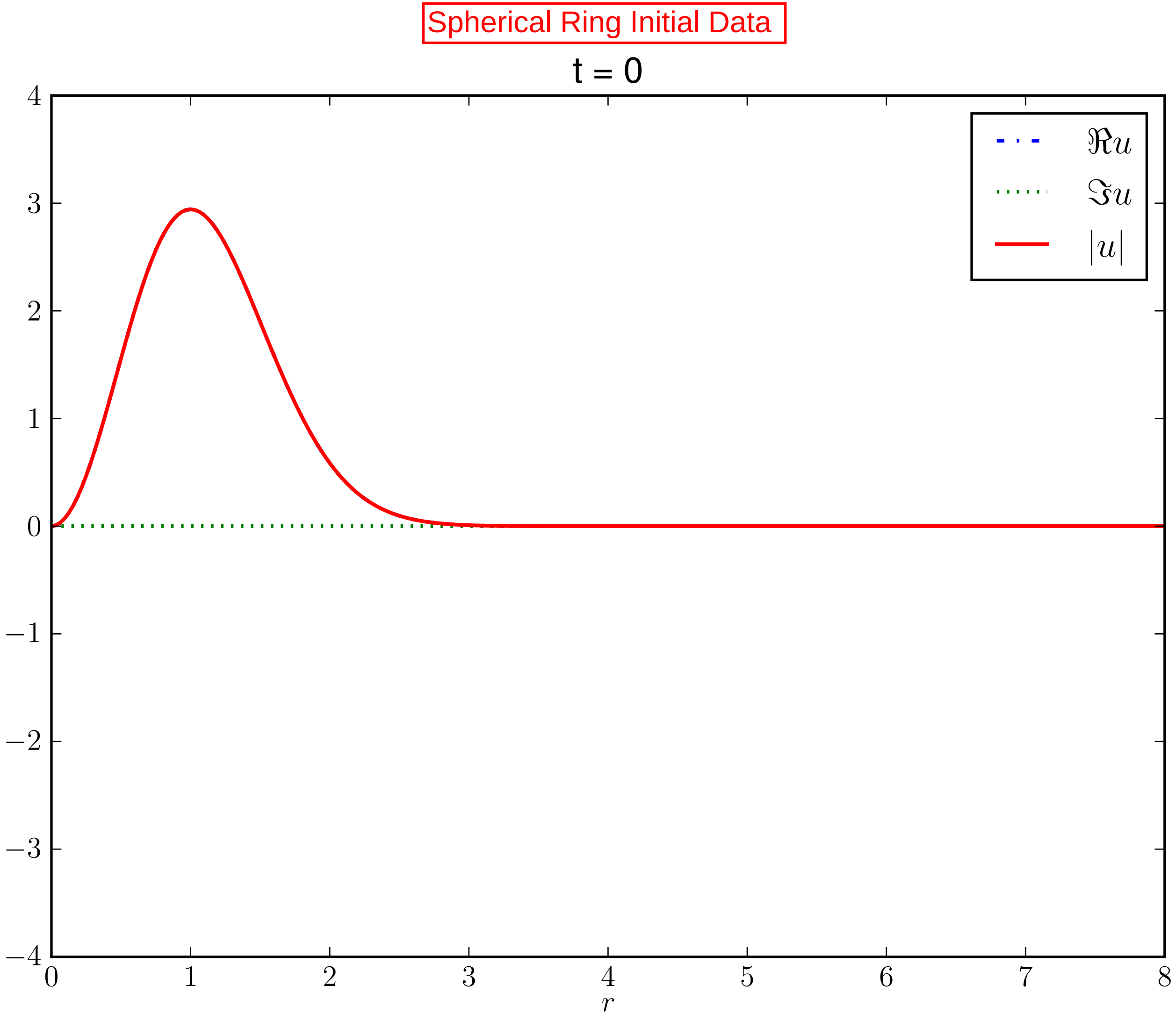

t = 0.09
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
$\Re u$
$\Im u$
$|u|$



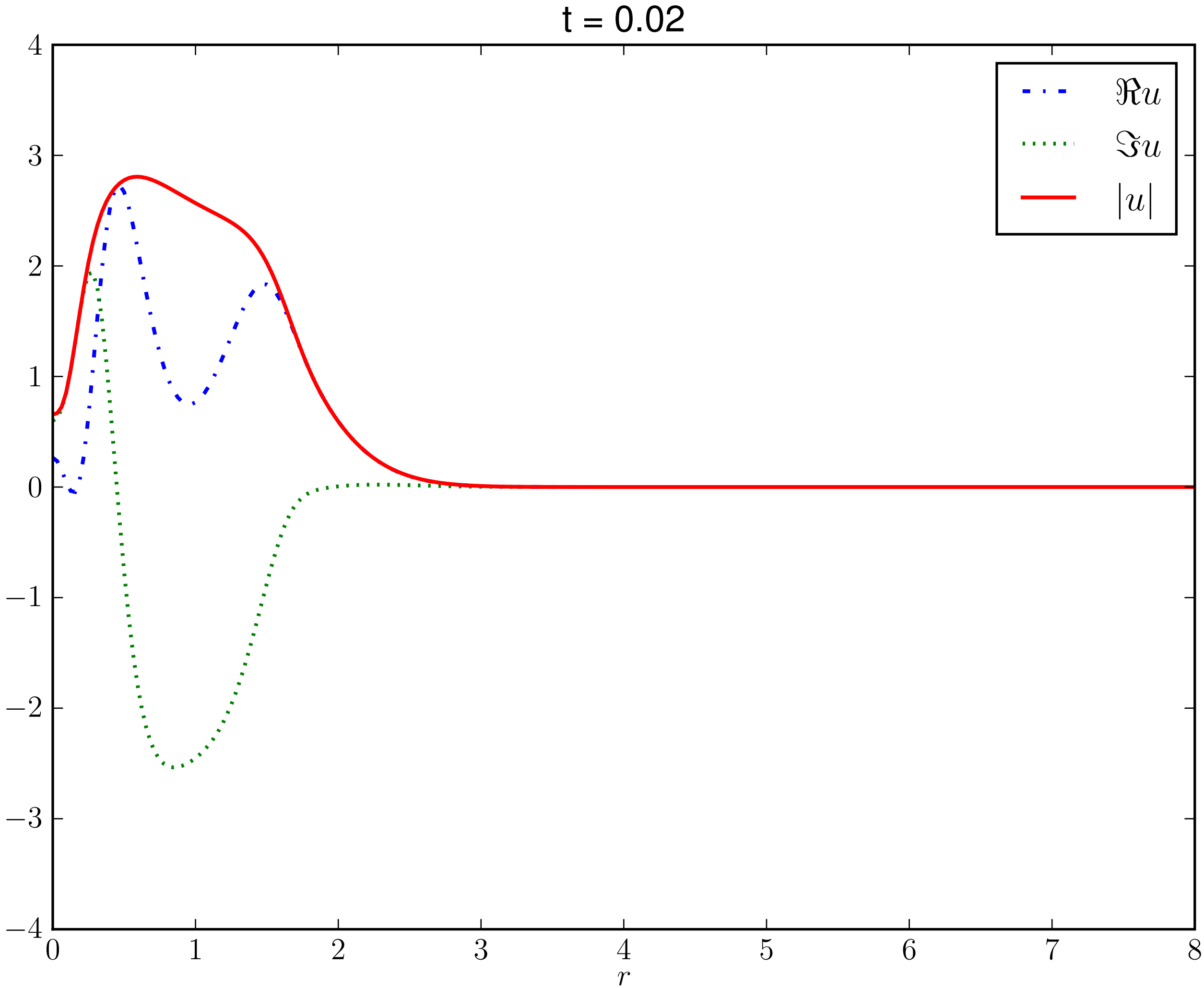
t = 0.095
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|



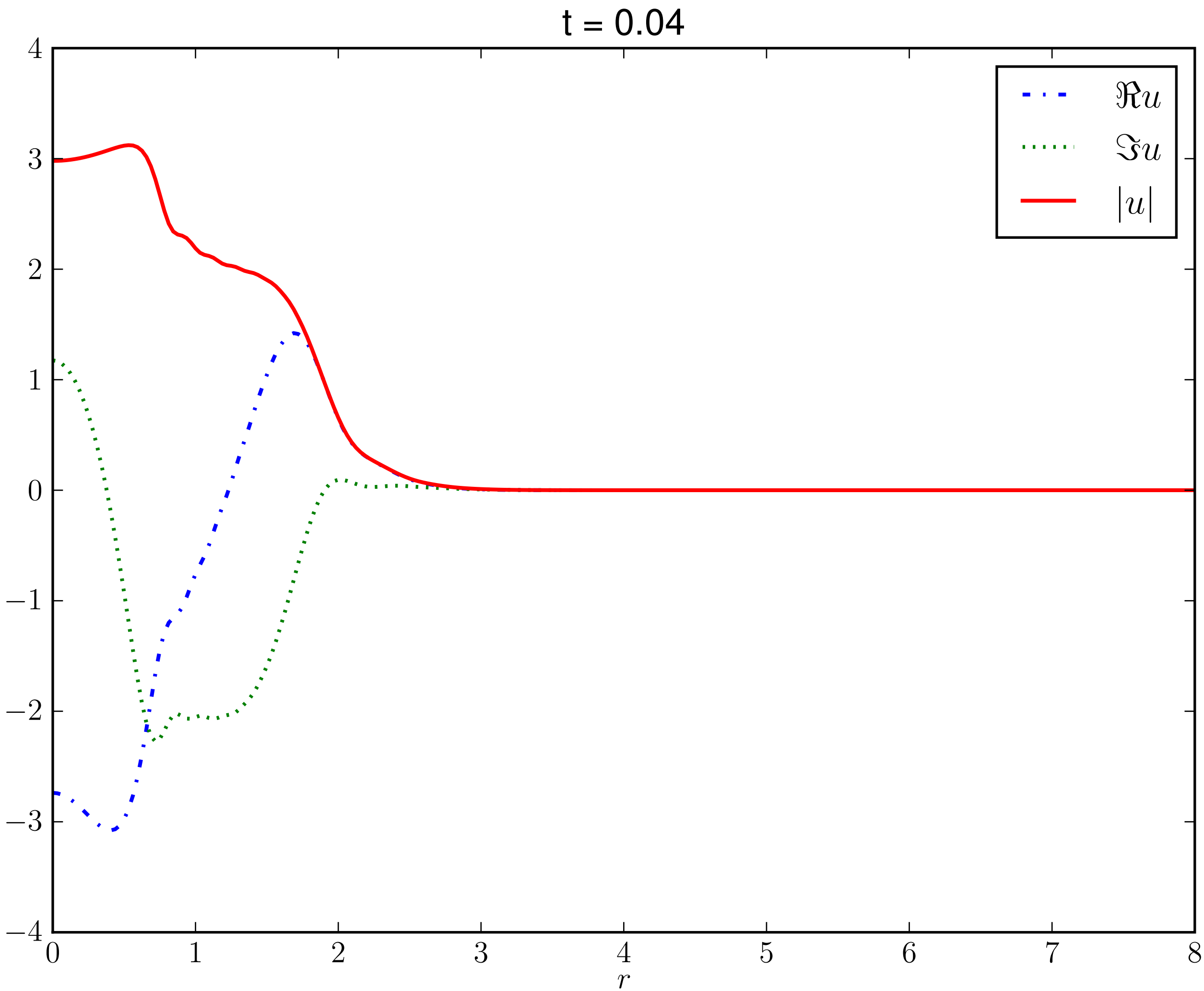
t = 0.1
1500
1000
500
0
−500
−1000
−1500
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
ℜu
ℑu
|u|

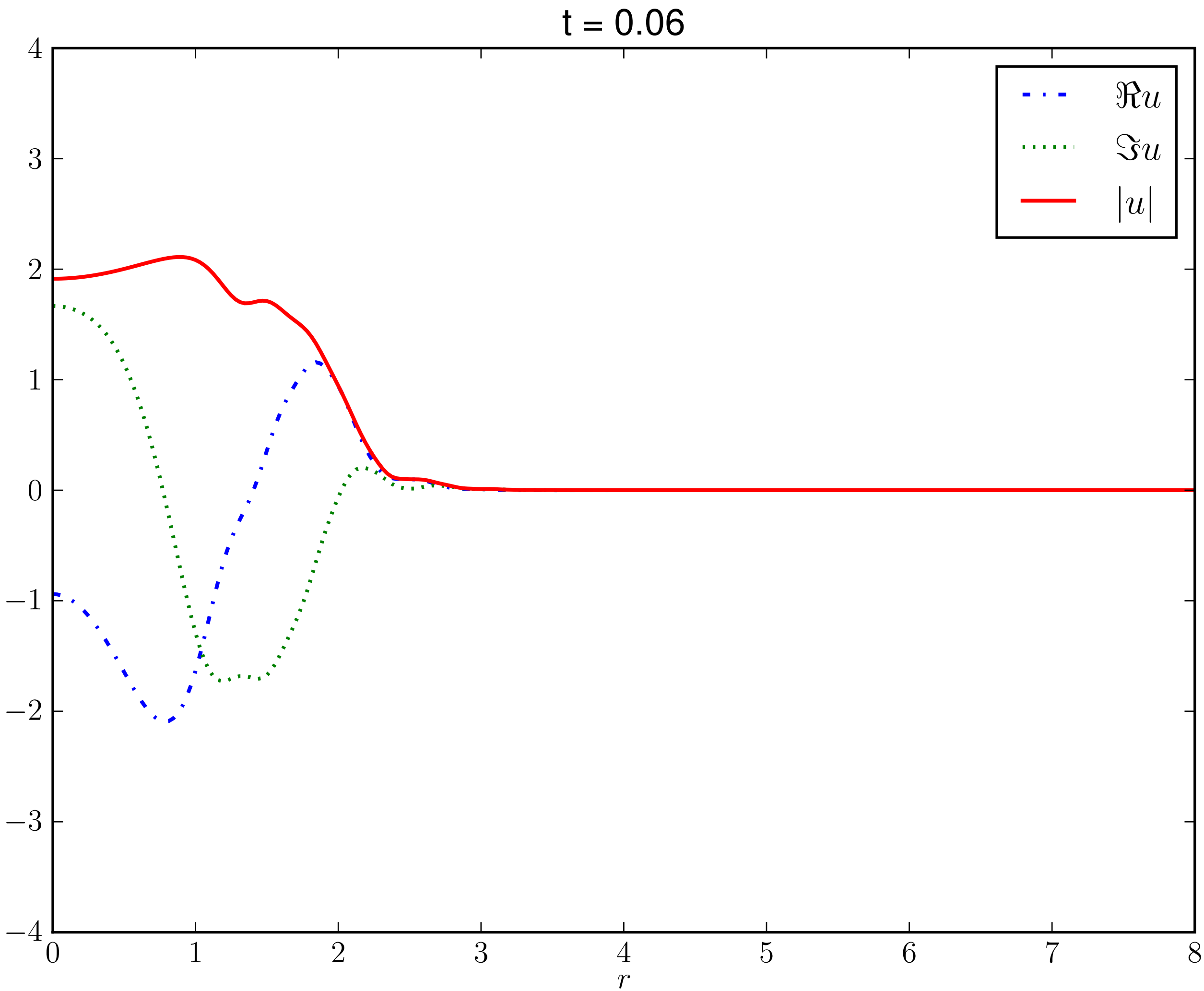


Spherical Ring Initial Data
t = 0
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r
ℜu
ℑu
|u|

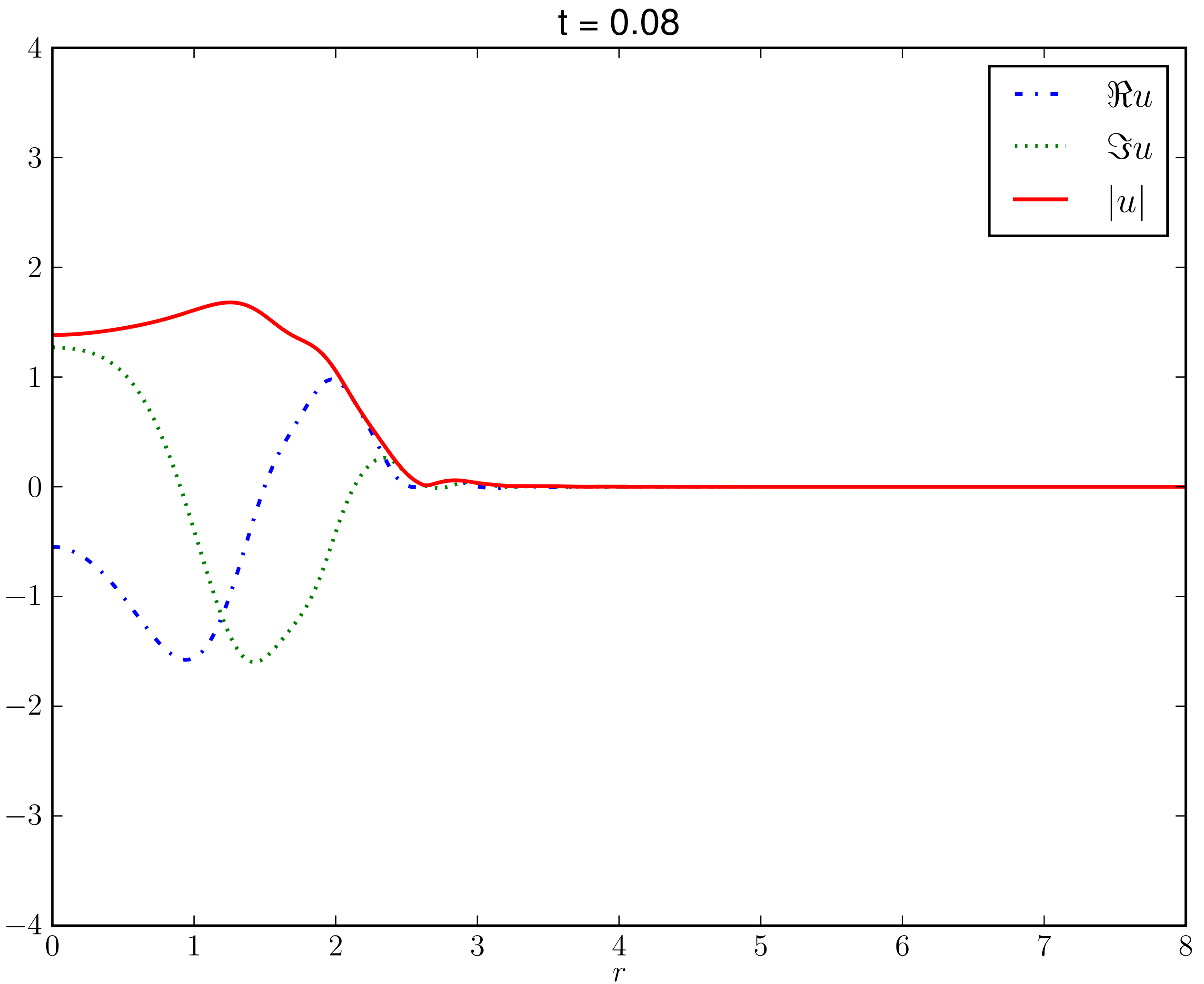


t = 0.02
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

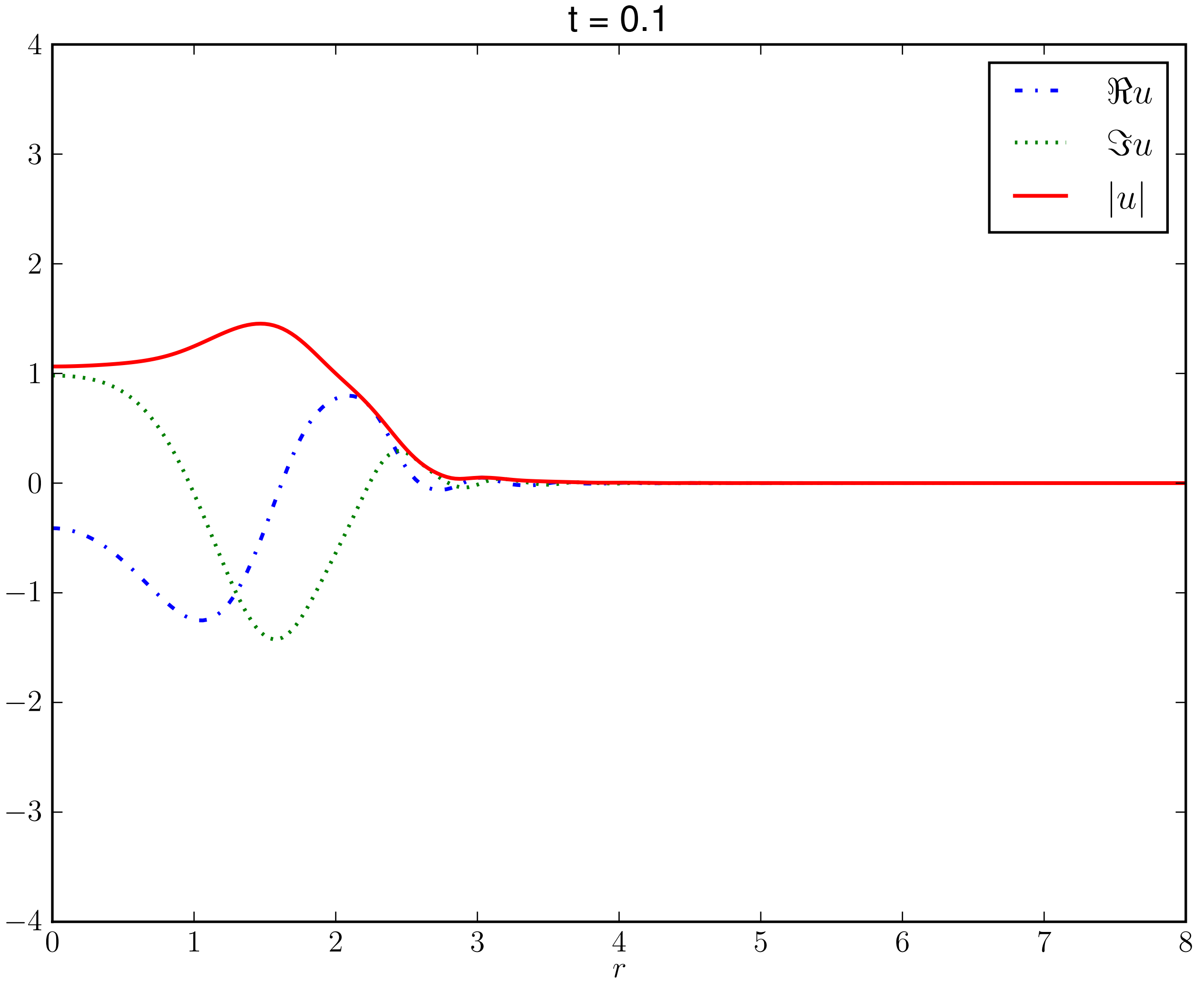

t = 0.04
$\Re u$
$\Im u$
$|u|$
r
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8



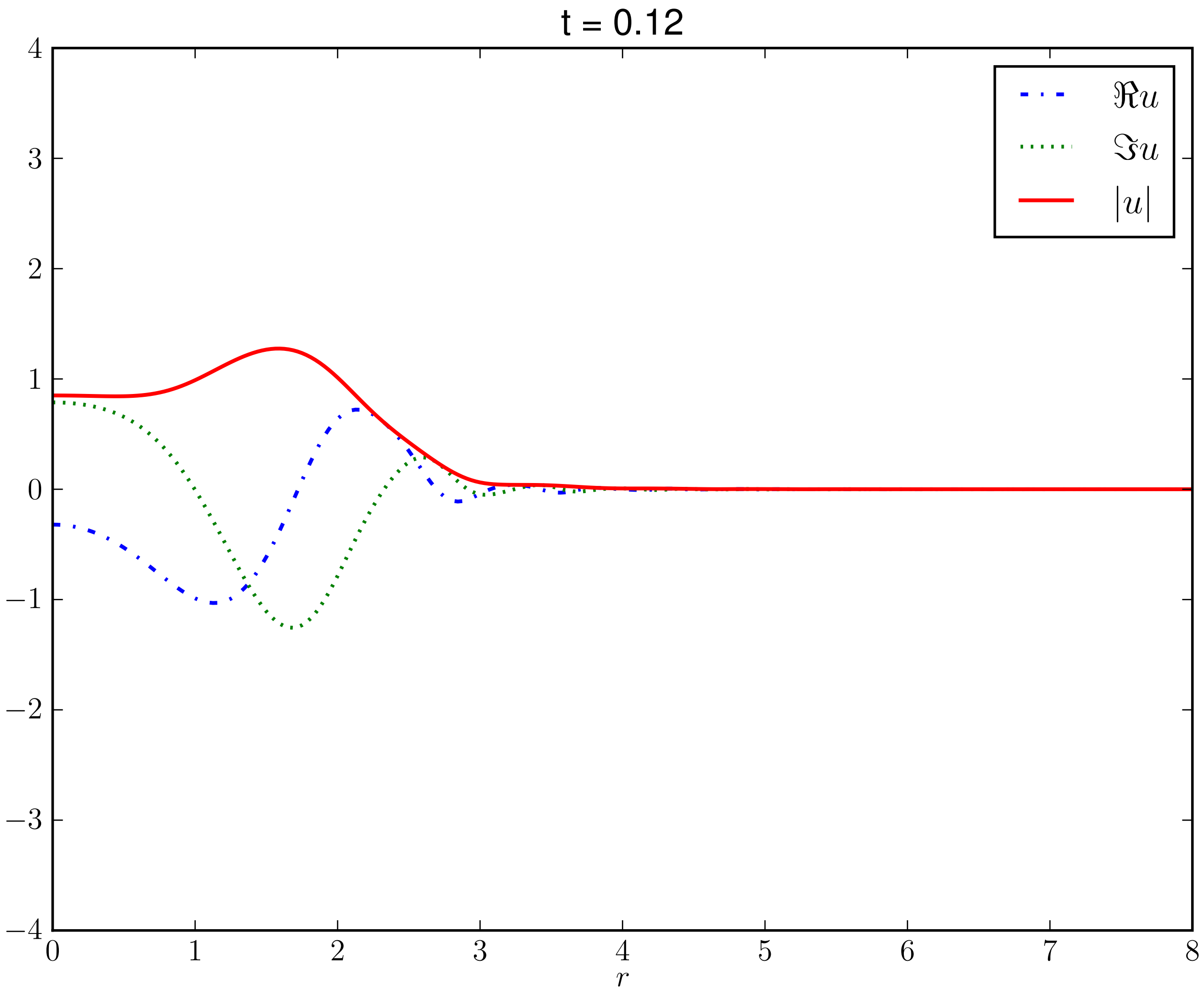
t = 0.06
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r
ℜu
ℑu
|u|

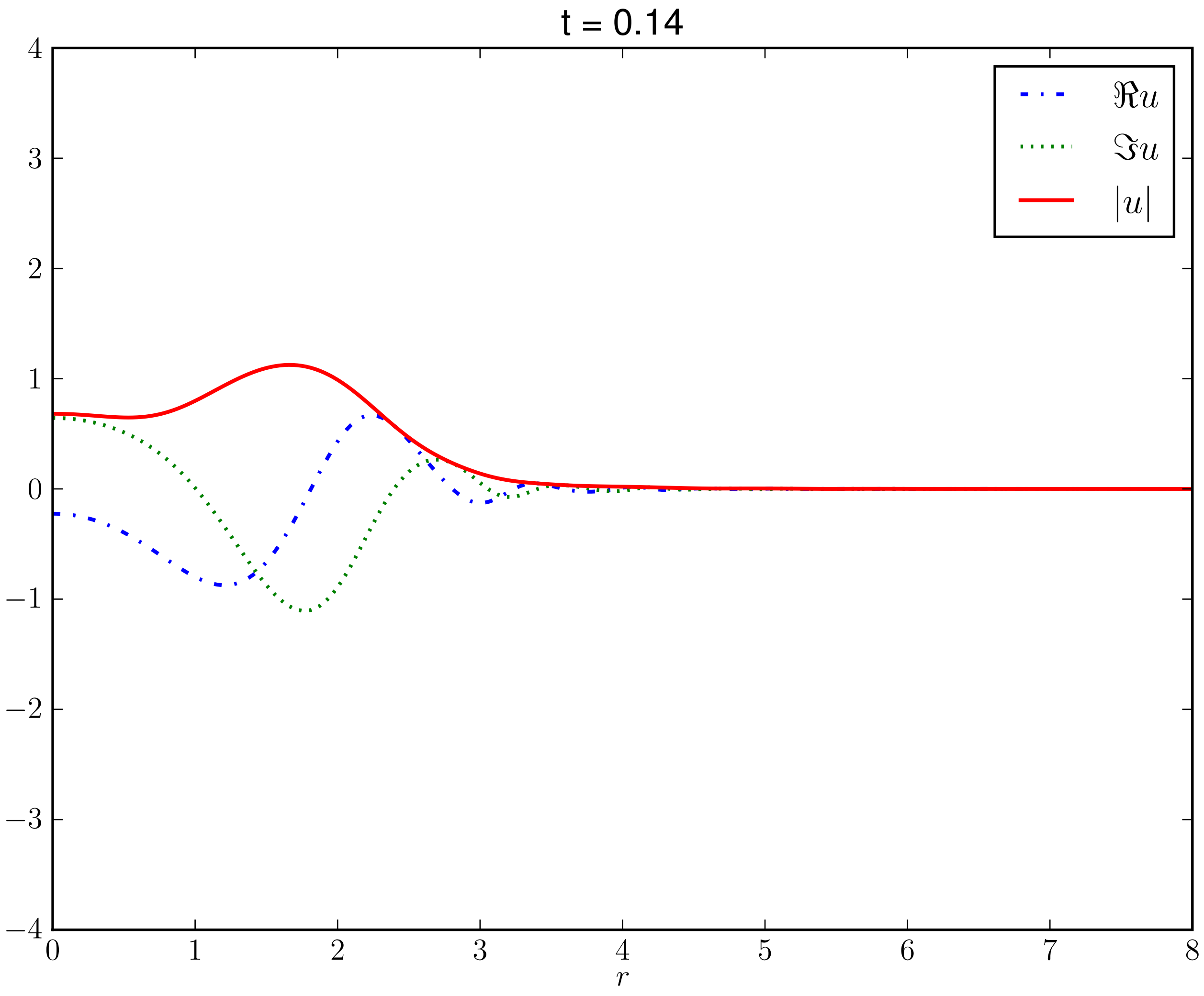


t = 0.08
ℜu
ℑu
|u|
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

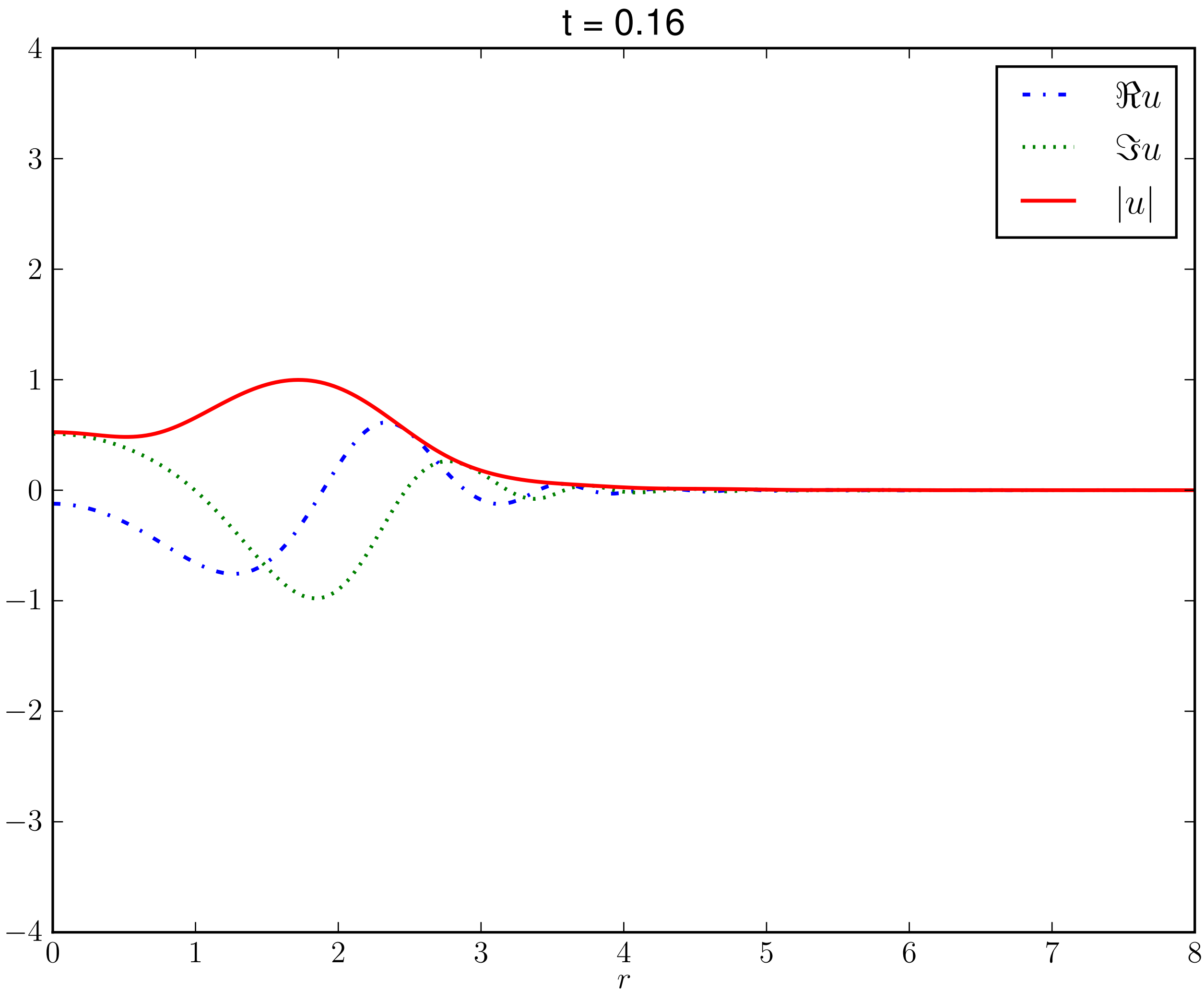


t = 0.1
ℜu
ℑu
|u|
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

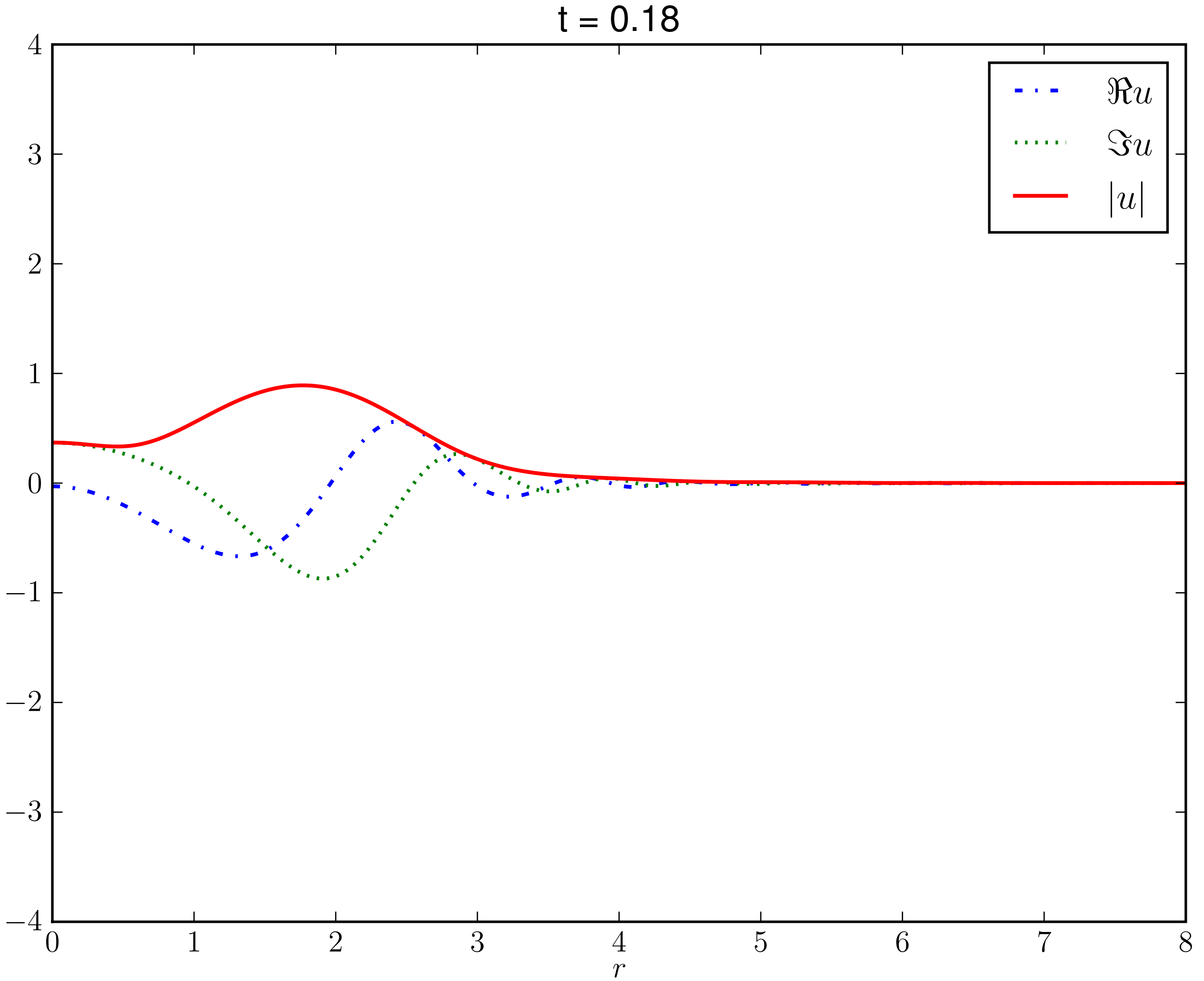

t = 0.12
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r

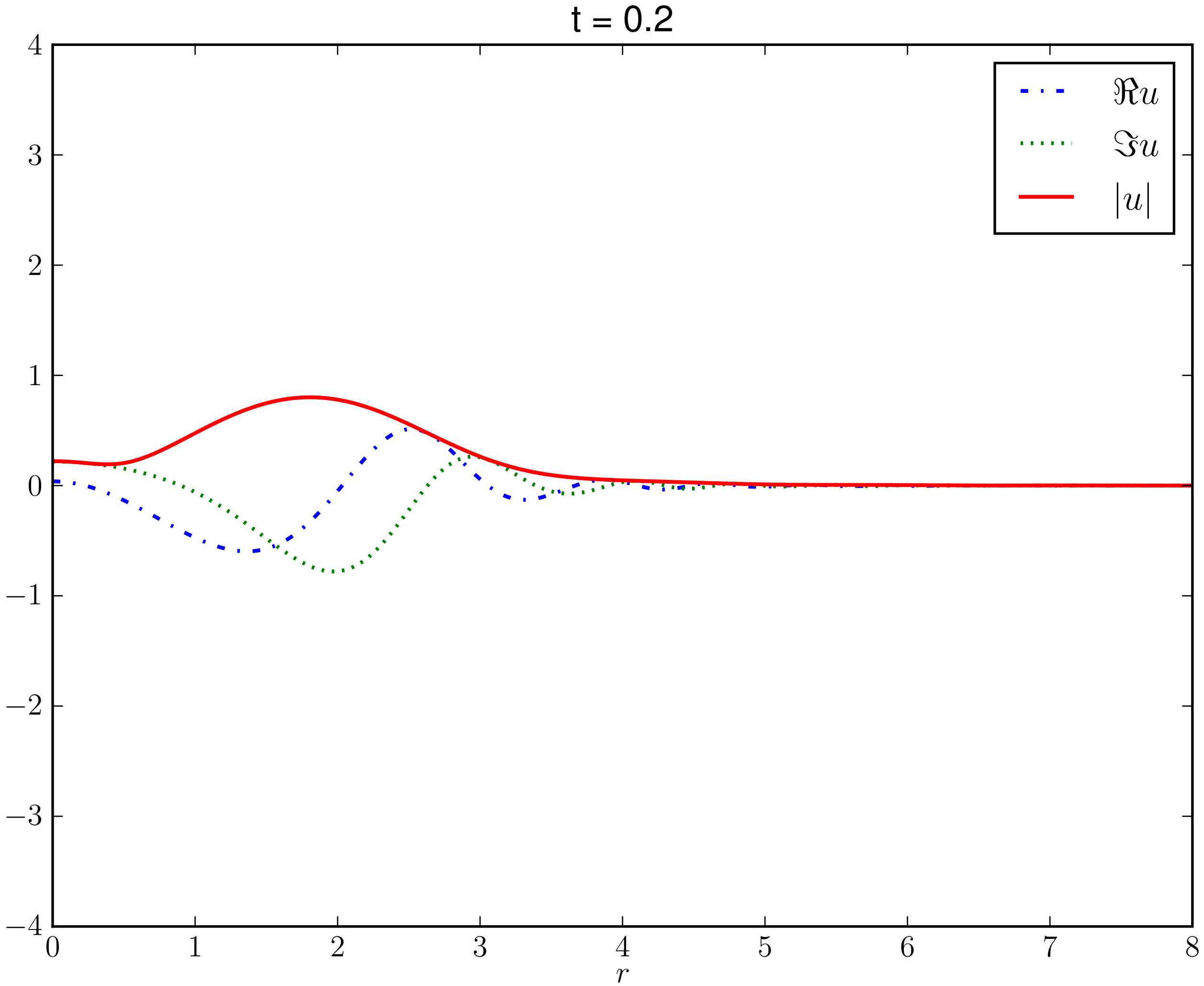


t = 0.14
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r
ℜu
ℑu
|u|

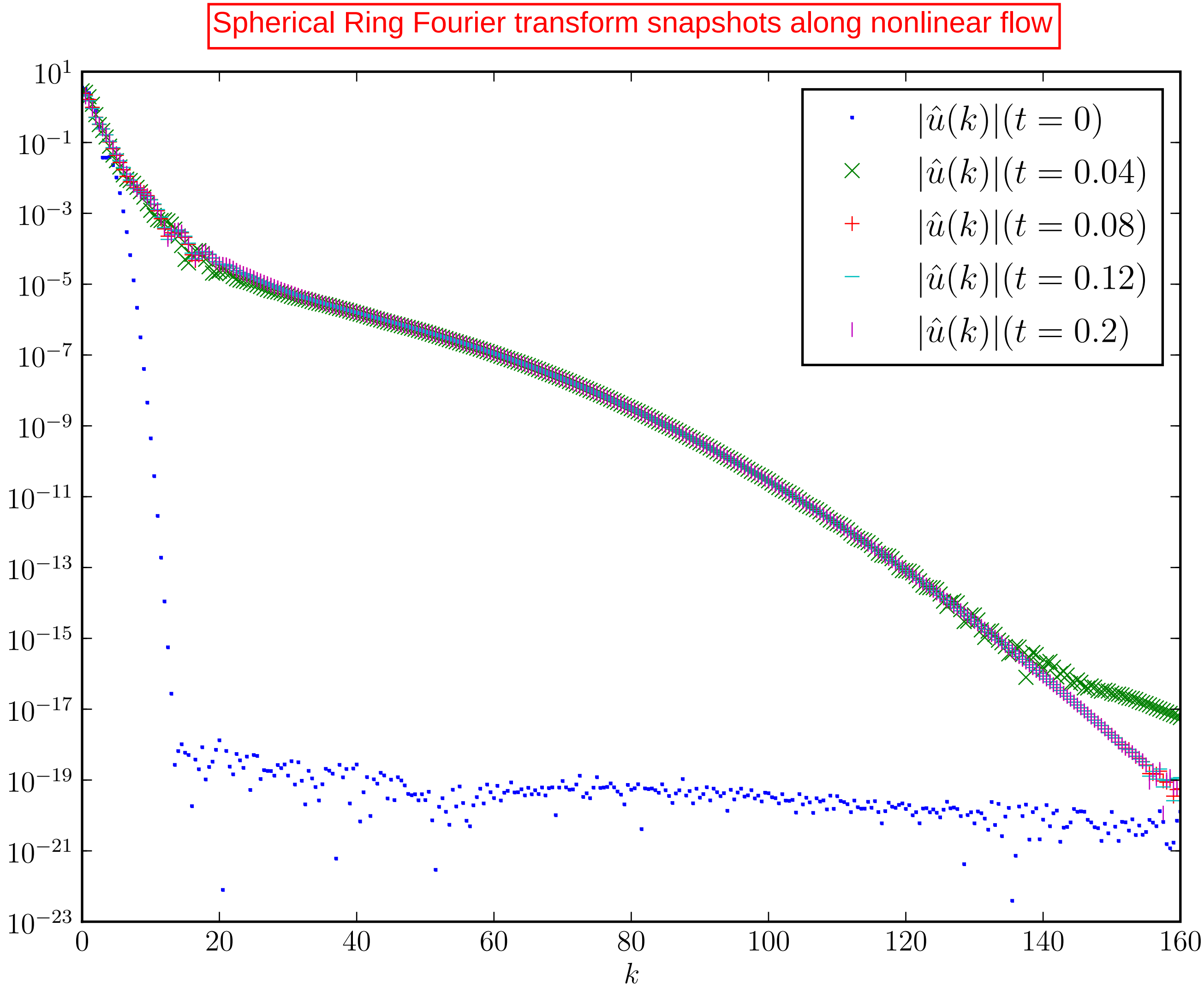


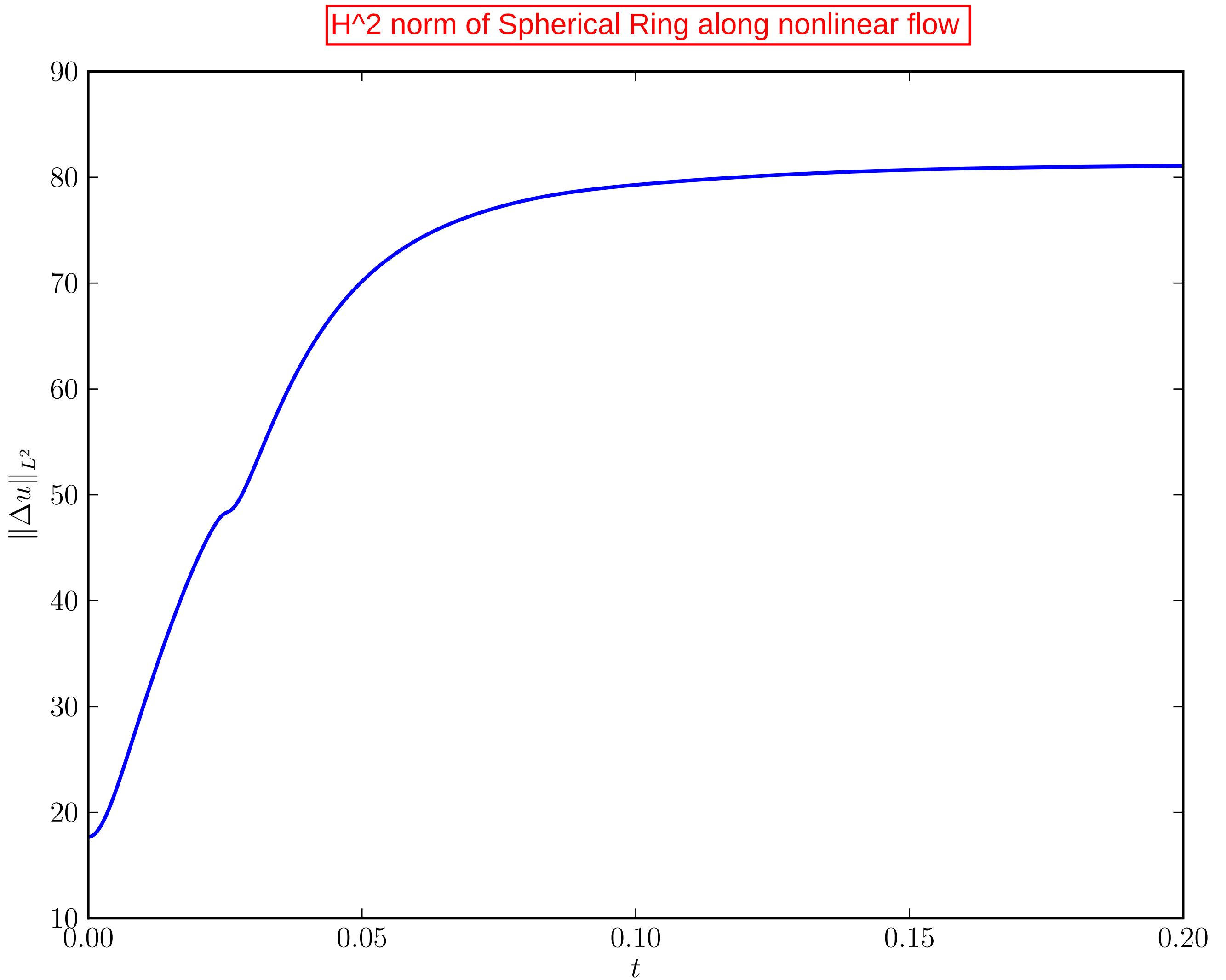
t = 0.16
$\Re u$
$\Im u$
$|u|$
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r


t = 0.18
4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
r
ℜu
ℑu
|u|

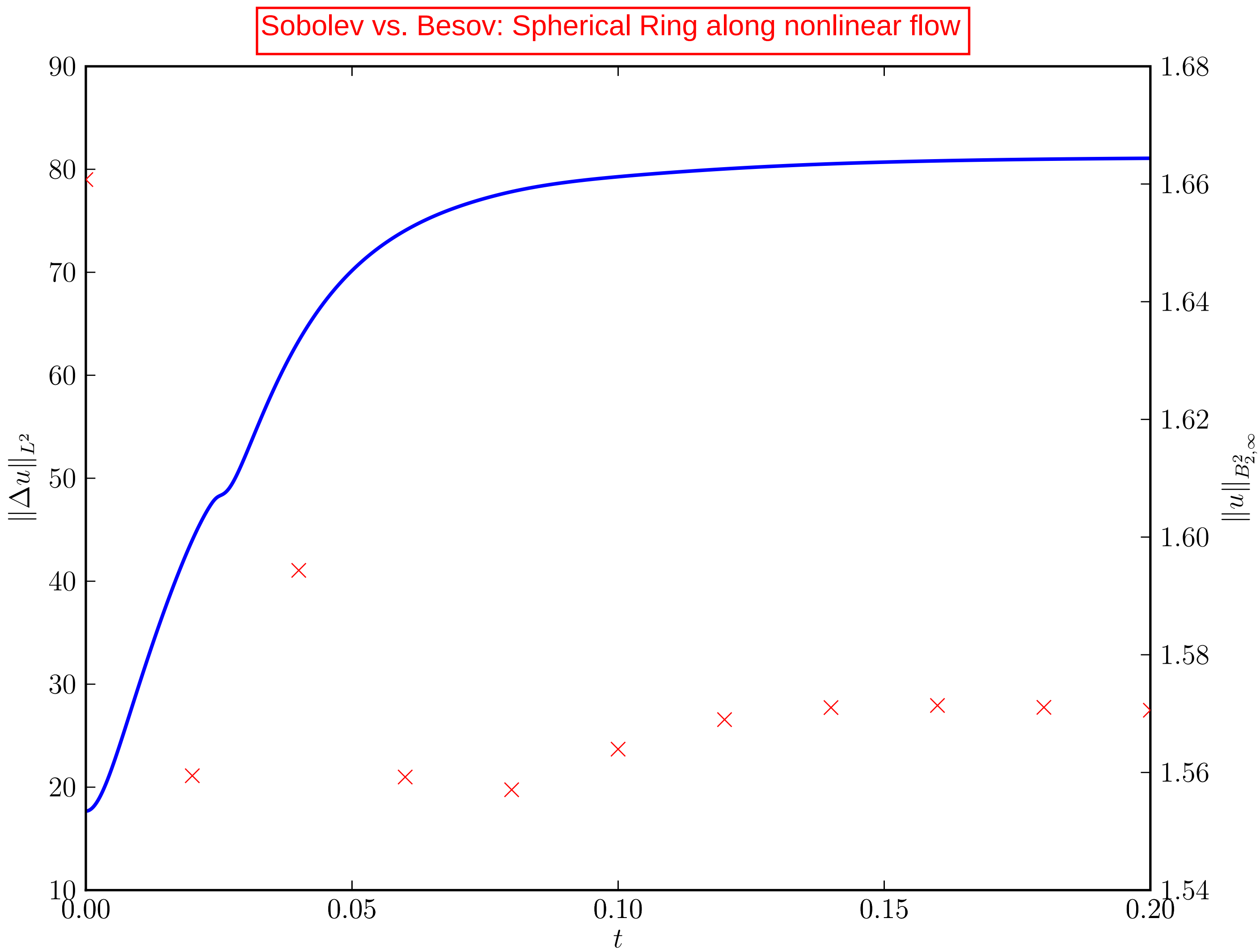


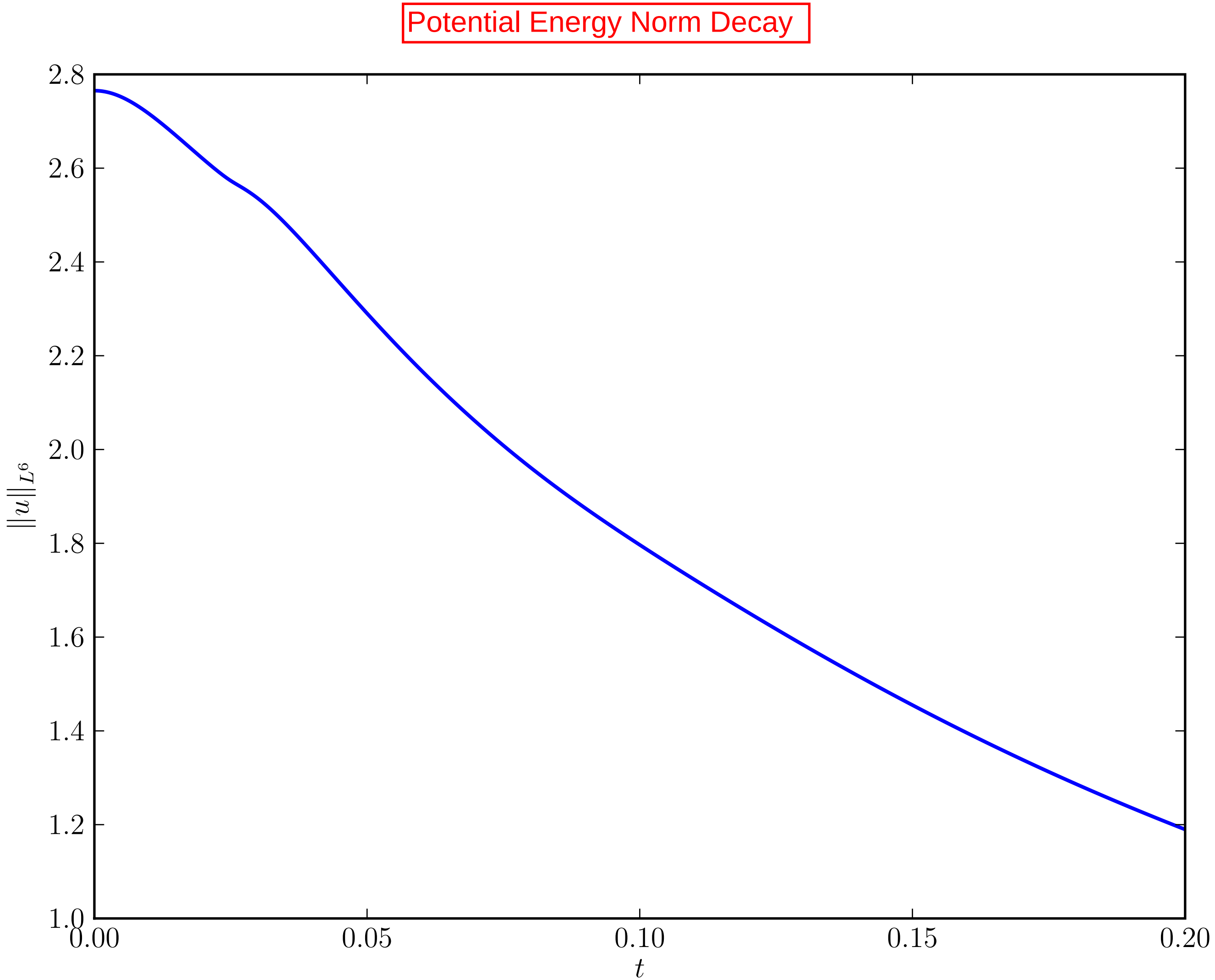
t = 0.2
ℜu
ℑu
|u|
r
0
1
2
3
4
5
6
7
8
4
3
2
1
0
−1
−2
−3
−4


Spherical Ring Fourier transform snapshots along nonlinear flow
10^{1}
10^{-1}
10^{-3}
10^{-5}
10^{-7}
10^{-9}
10^{-11}
10^{-13}
10^{-15}
10^{-17}
10^{-19}
10^{-21}
10^{-23}
0
20
40
60
80
100
120
140
160
k
$|\hat{u}(k)|(t=0)$
$|\hat{u}(k)|(t=0.04)$
$|\hat{u}(k)|(t=0.08)$
$|\hat{u}(k)|(t=0.12)$
$|\hat{u}(k)|(t=0.2)$



H^2 norm of Spherical Ring along nonlinear flow
90
80
70
60
50
40
30
20
10
0.00
0.05
0.10
0.15
0.20
t
$\|\Delta u\|_{L^2}$

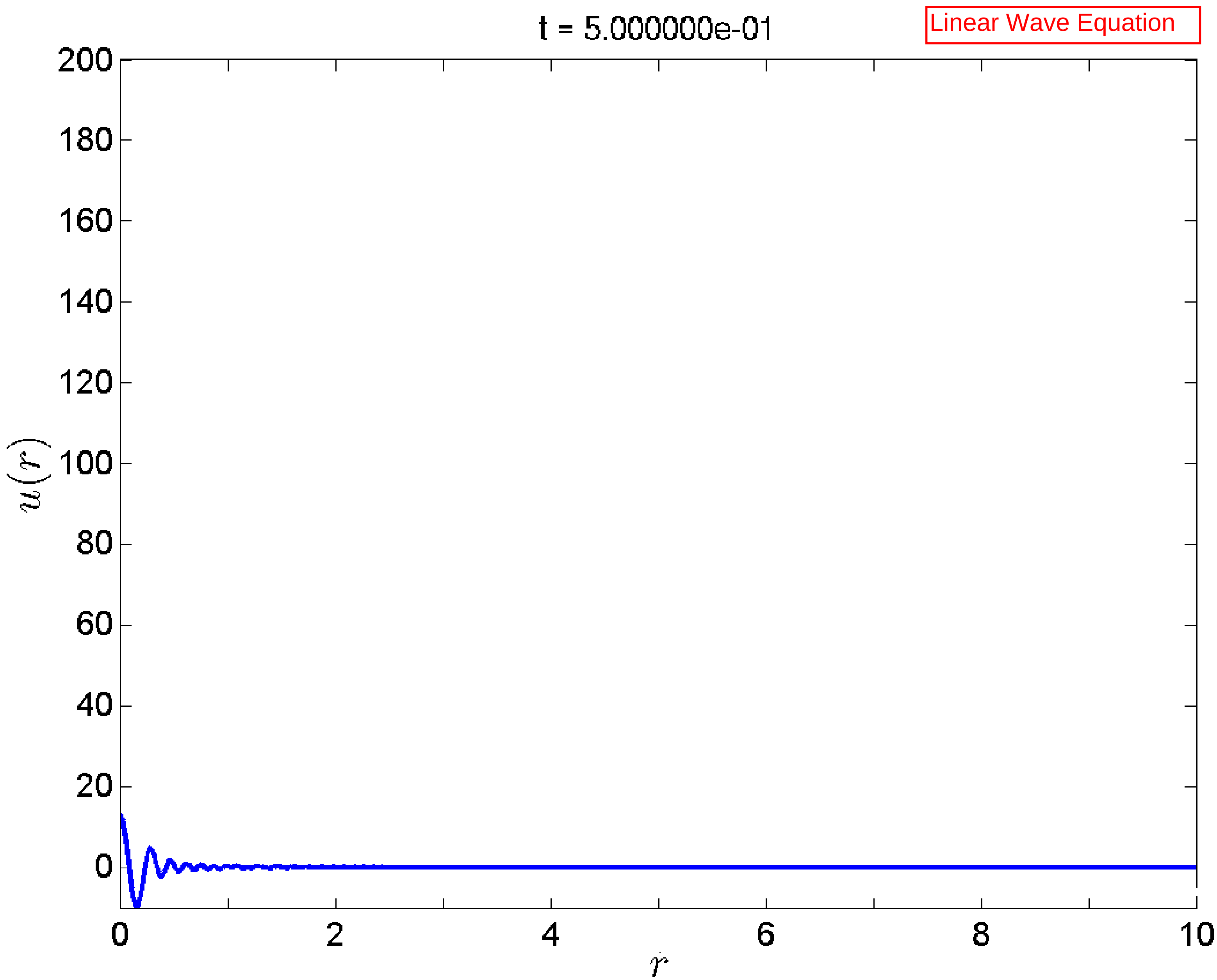

Sobolev vs. Besov: Spherical Ring along nonlinear flow
$\|\Delta u\|_{L^2}$
$\|u\|_{B^2_{2,\infty}}$
t
10
20
30
40
50
60
70
80
90
1.54
1.56
1.58
1.60
1.62
1.64
1.66
1.68
0.00
0.05
0.10
0.15
0.20



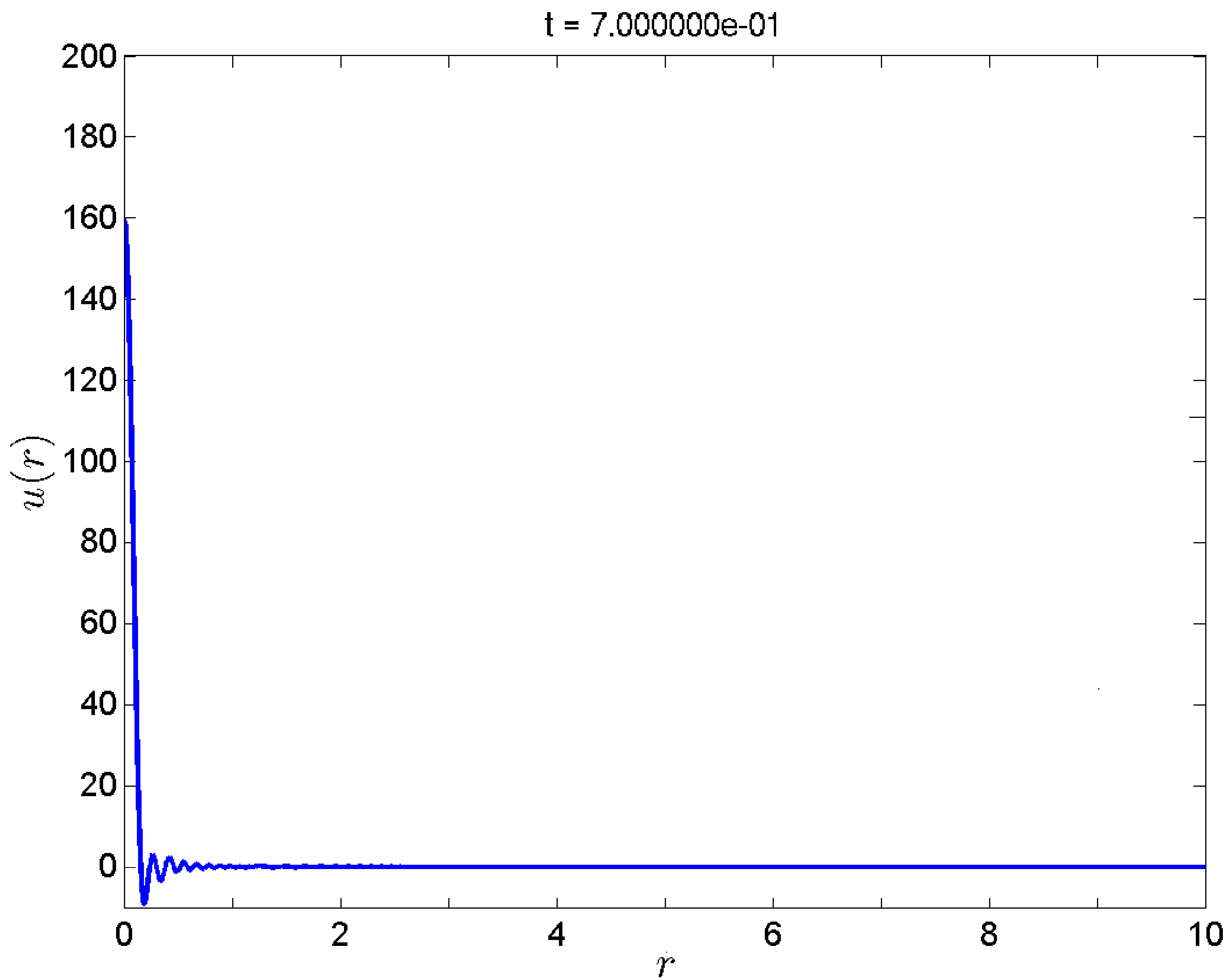
Potential Energy Norm Decay
$\|u\|_{L^6}$
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.00
0.05
0.10
0.15
0.20
t

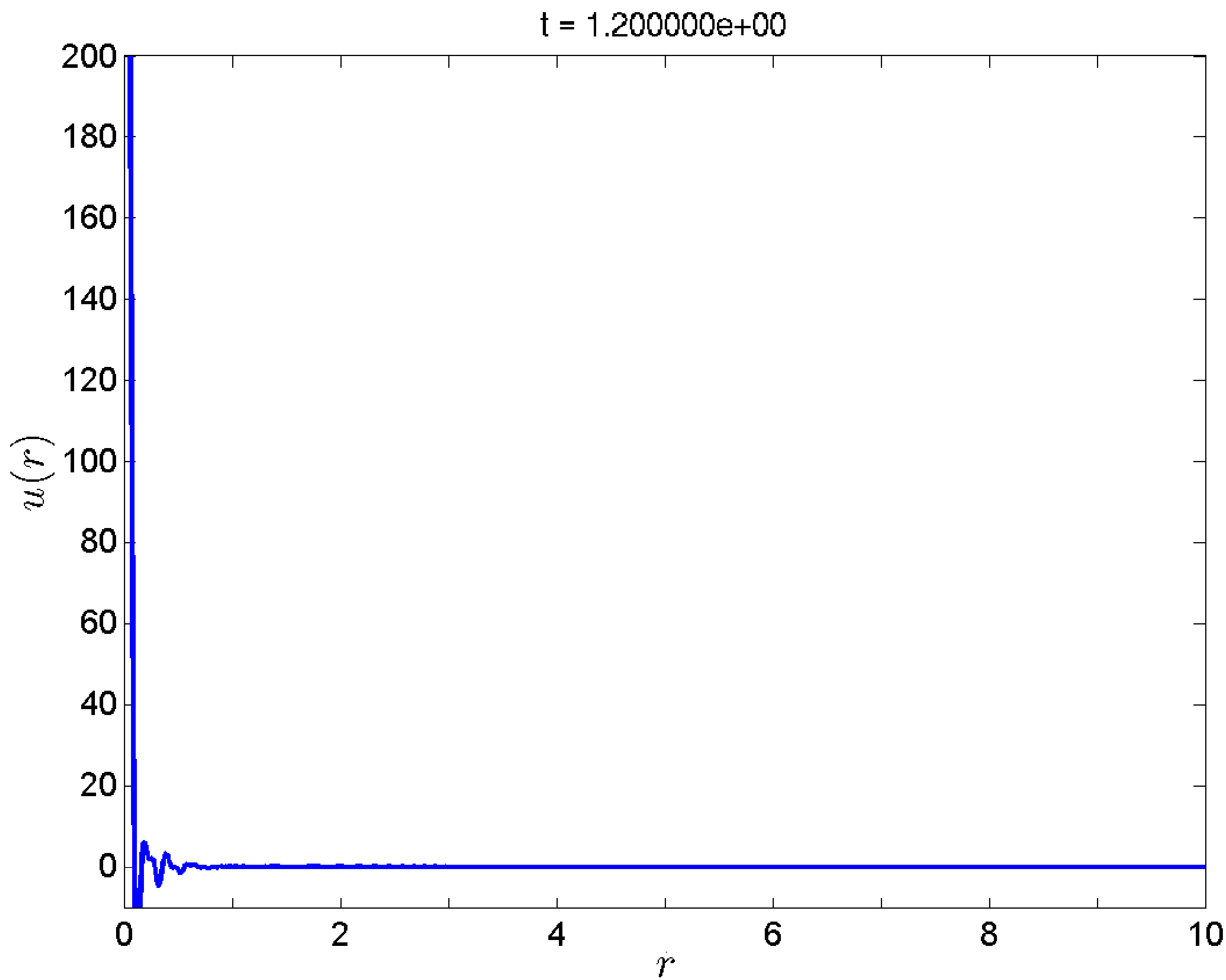
C-Pang-Simpson-Sulem simulations

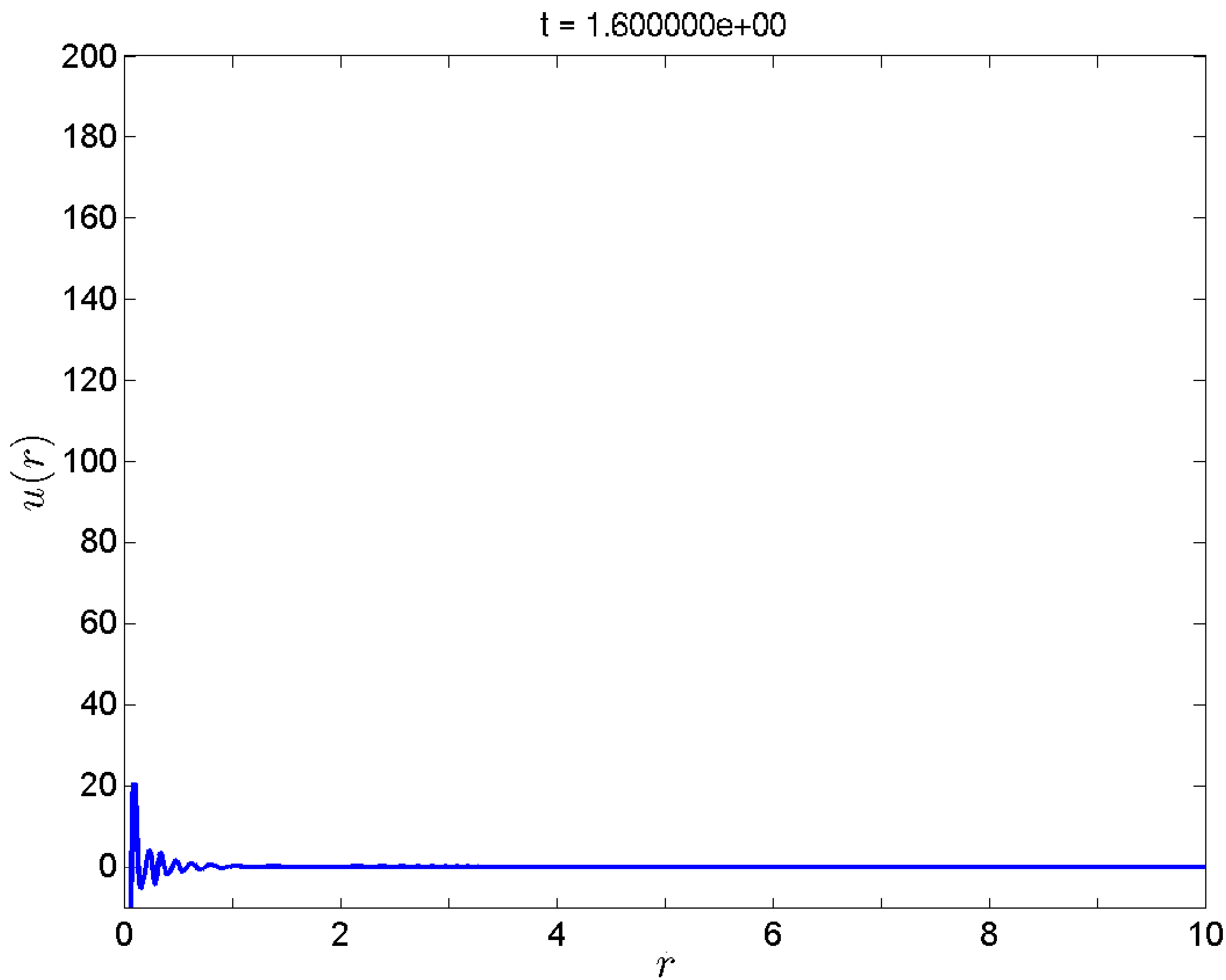
NLW_5 (R^5)



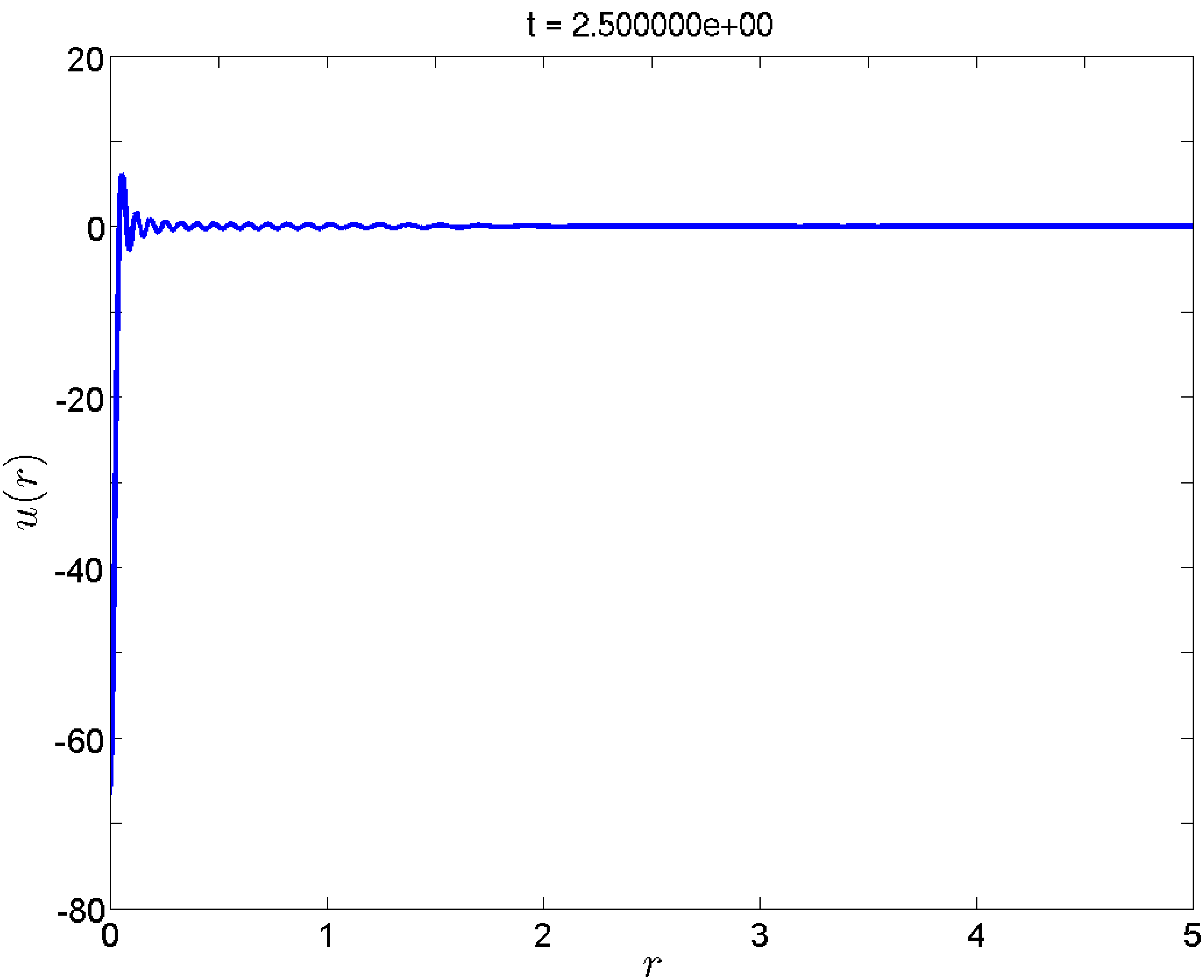
t = 5.000000e-01
Linear Wave Equation
200
180
160
140
120
100
80
60
40
20
0
0
2
4
6
8
10
$u(r)$
r

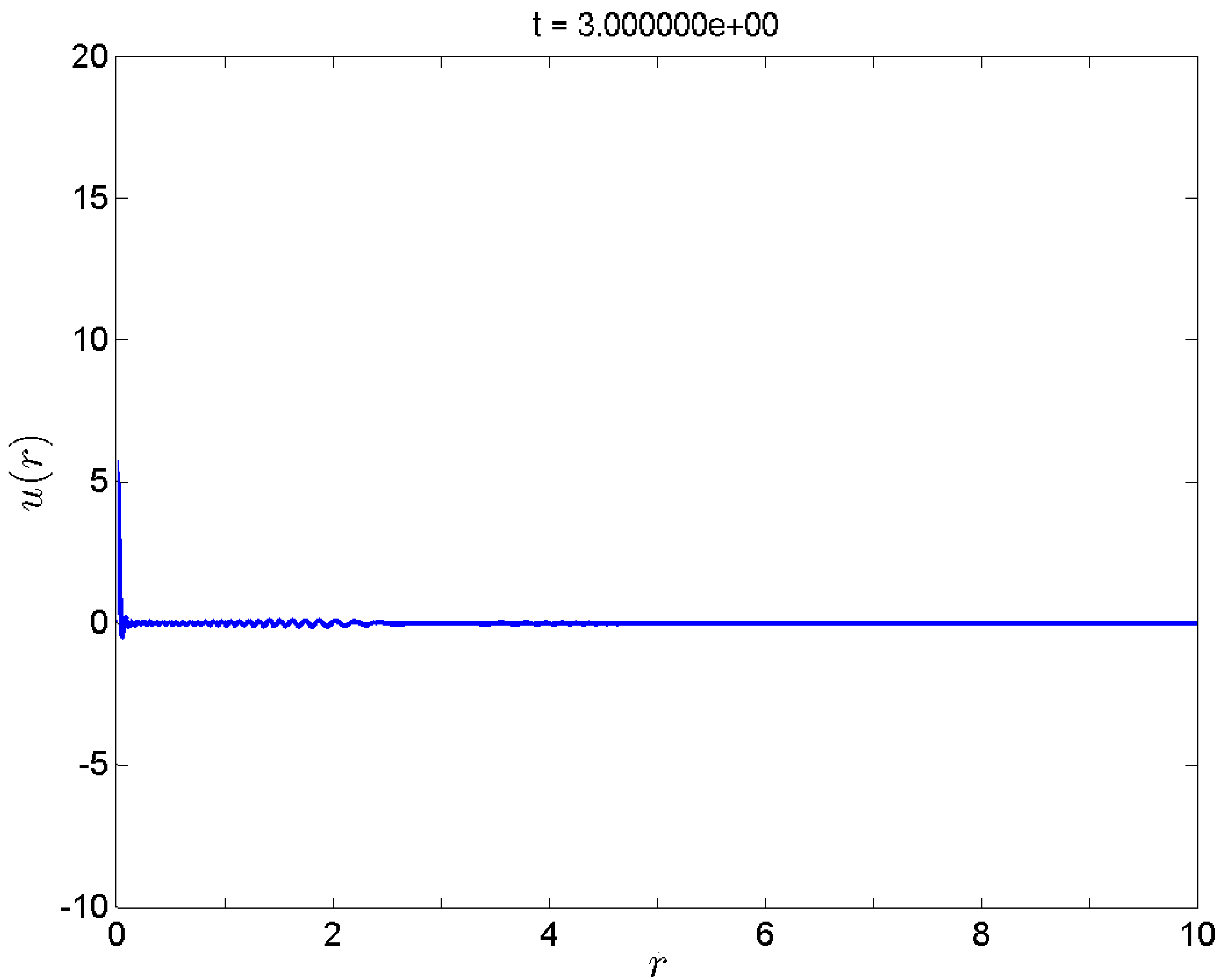

t = 7.000000e-01
200
180
160
140
120
100
80
60
40
20
0
$u(r)$
0
2
4
6
8
10
r


t = 1.200000e+00
200
180
160
140
120
100
80
60
40
20
0
$u(r)$
0
2
4
6
8
10
r

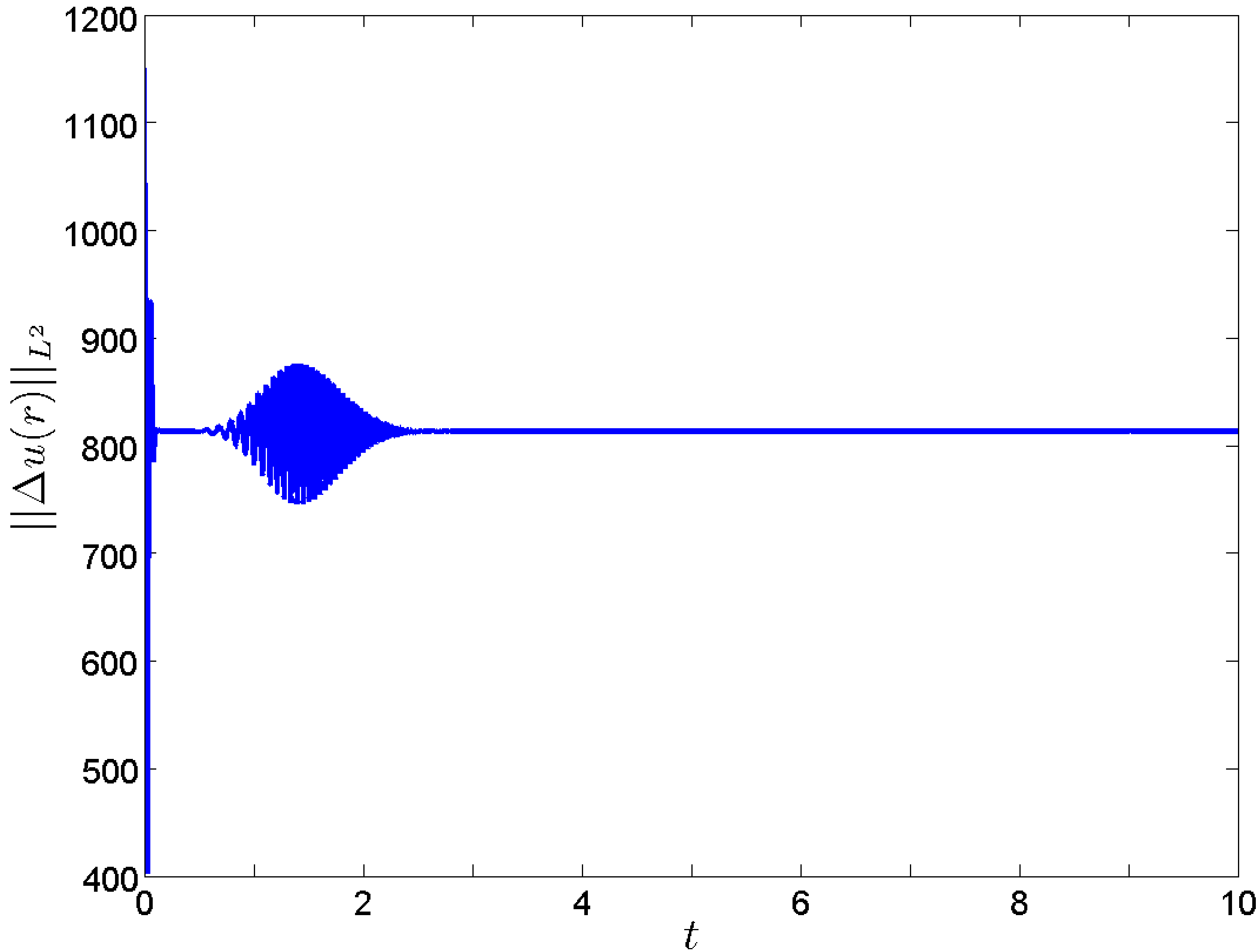


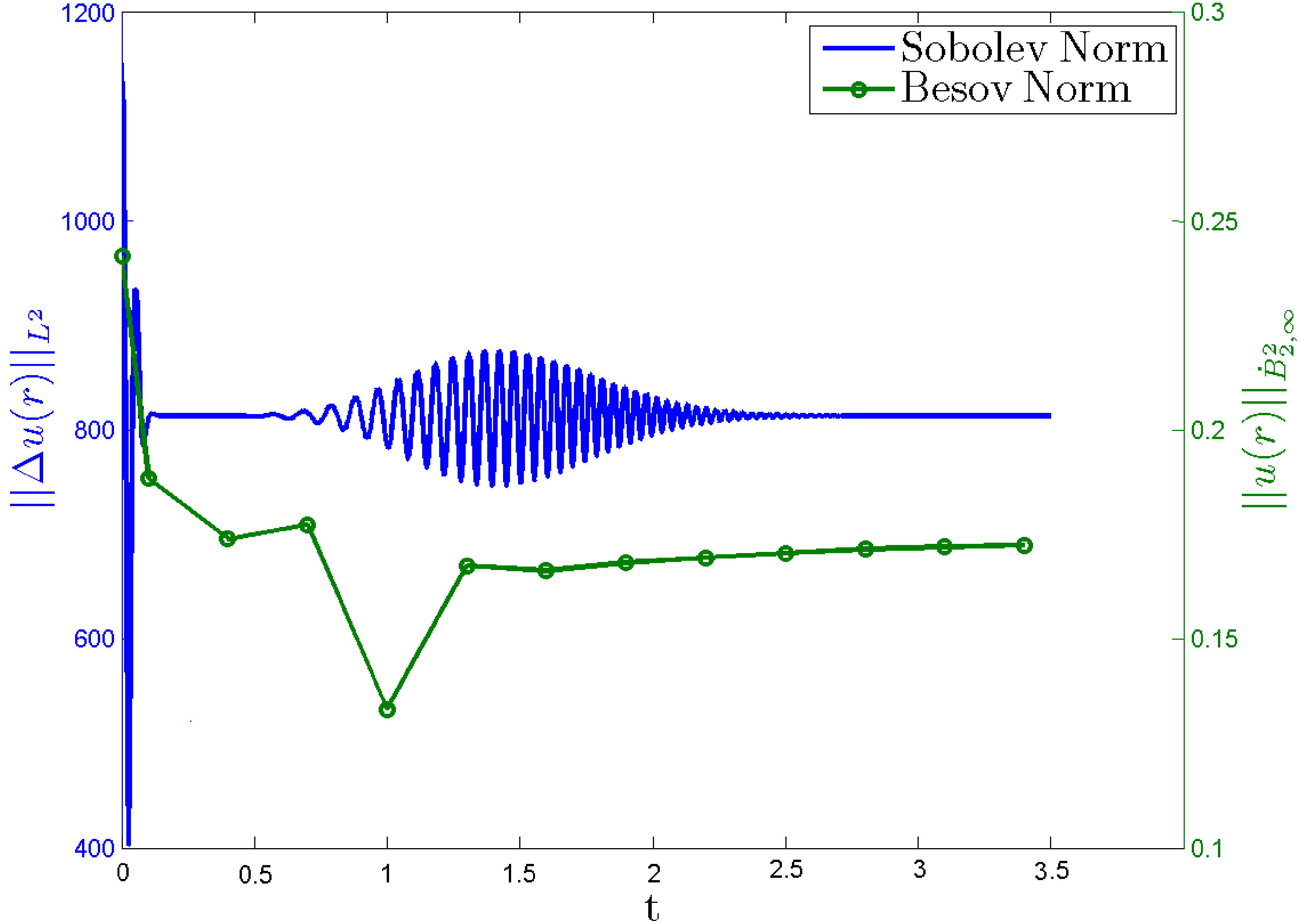
t = 1.600000e+00
200
180
160
140
120
100
80
60
40
20
0
u(r)
0
2
4
6
8
10
r


t = 2.500000e+00
20
0
-20
-40
-60
-80
0
1
2
3
4
5
$u(r)$
r

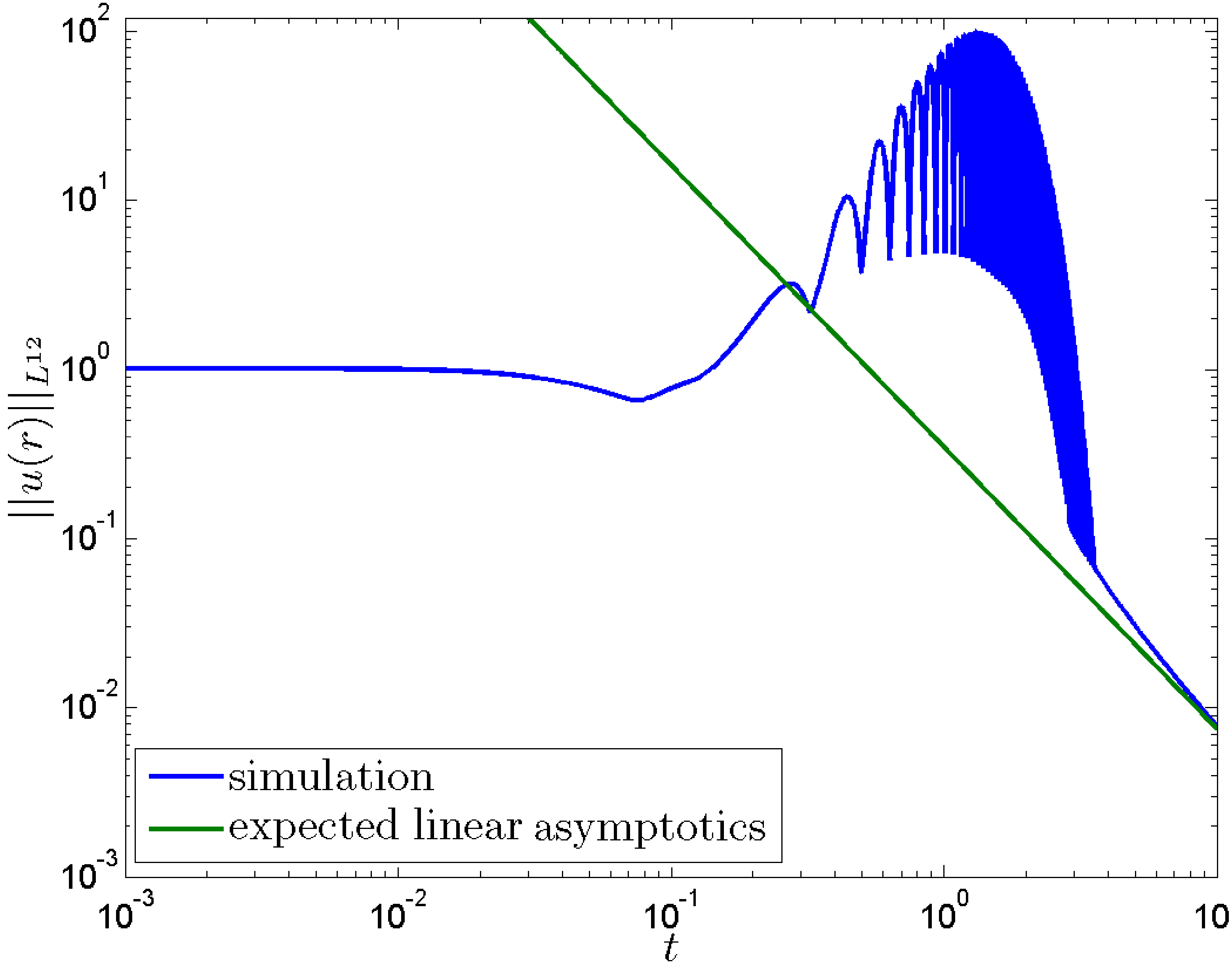


t = 3.000000e+00
20
15
10
5
0
-5
-10
0
2
4
6
8
10
$u(r)$
r

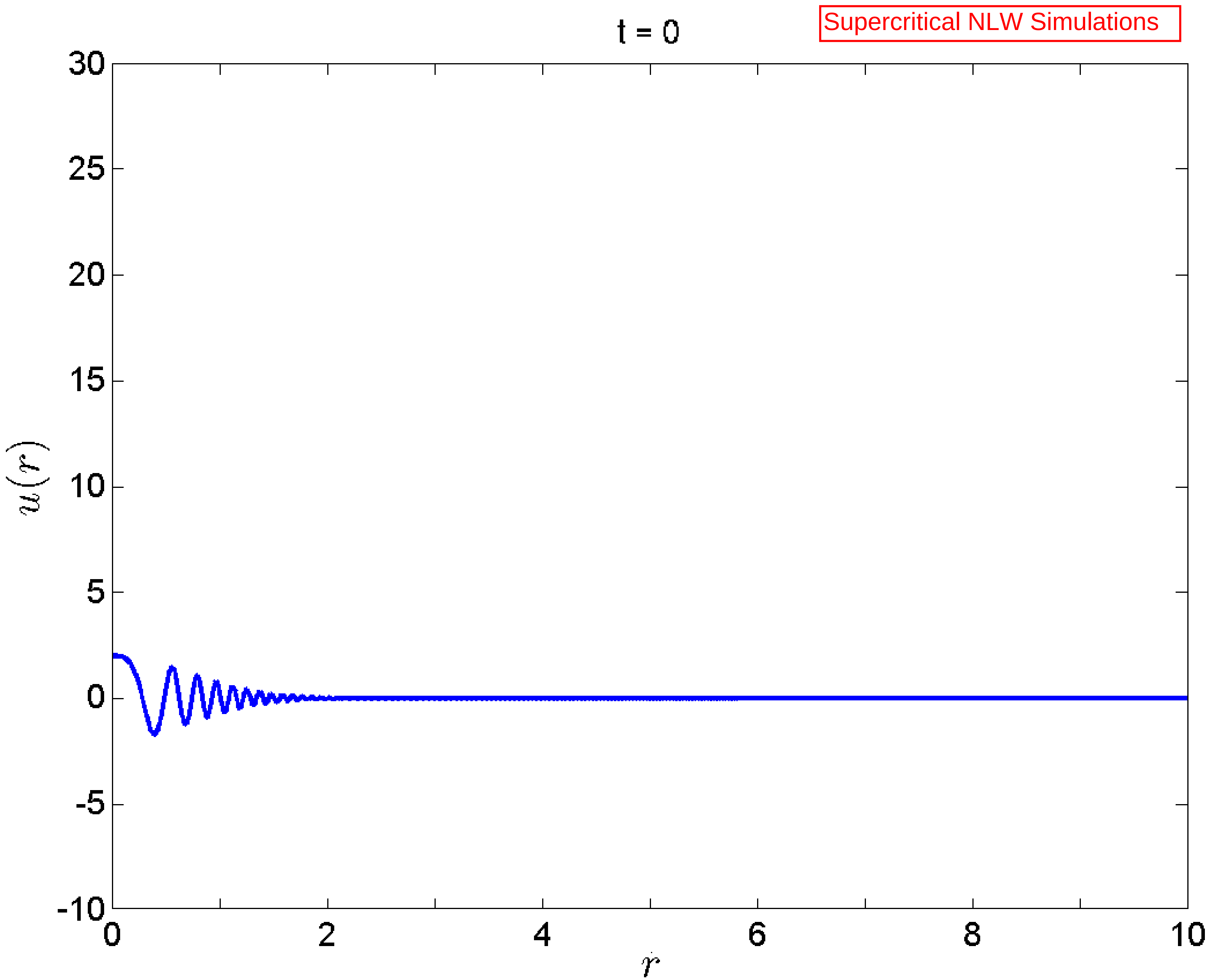

1200
1100
1000
900
800
700
600
500
400
0
2
4
6
8
10
t
$||\Delta u(r)||_{L^2}$



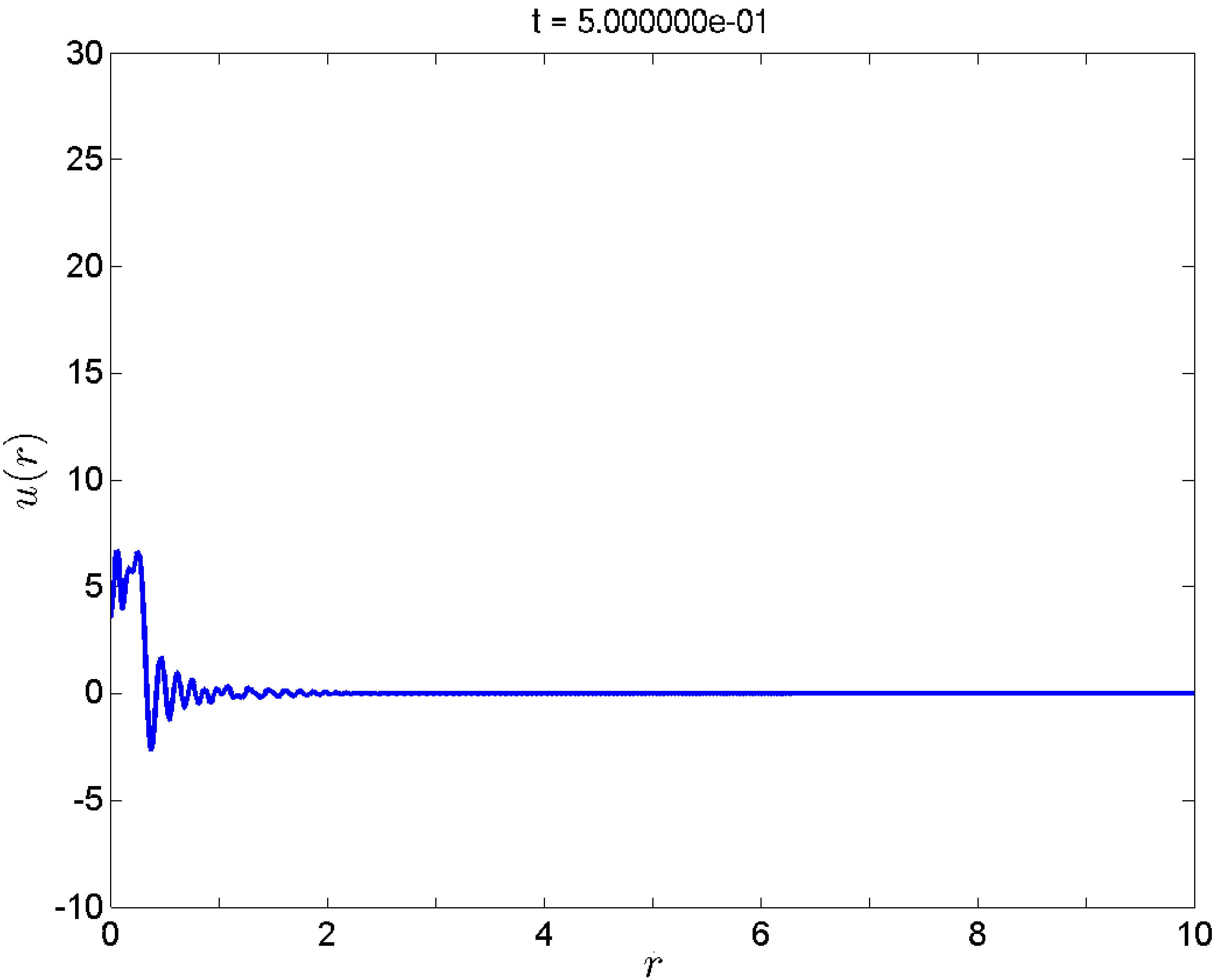
Sobolev Norm
Besov Norm
$||\Delta u(r)||_{L^2}$
$||u(r)||_{\dot{B}^2_{2,\infty}}$
t
1200
1000
800
600
400
0.3
0.25
0.2
0.15
0.1
0
0.5
1
1.5
2
2.5
3
3.5

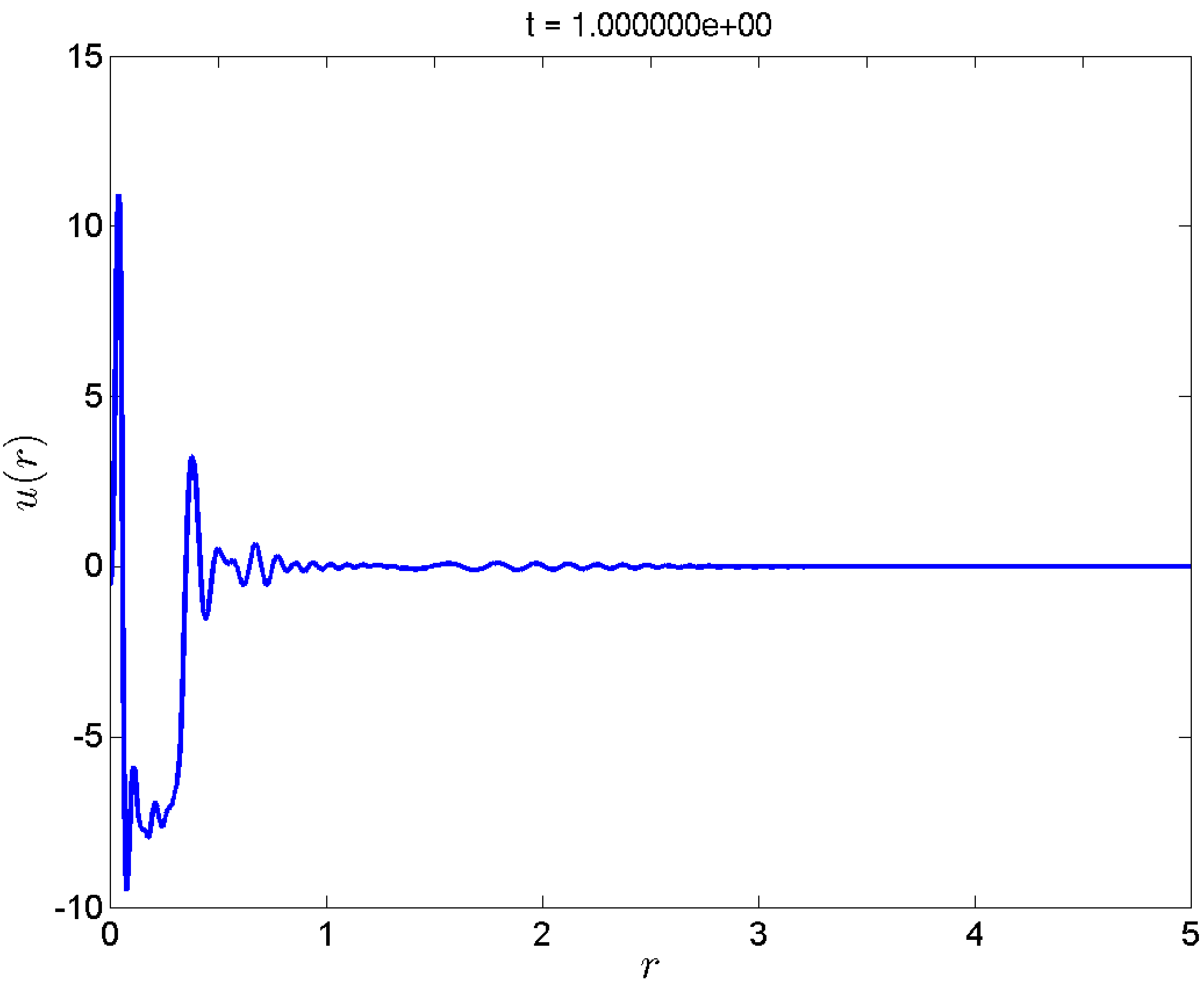


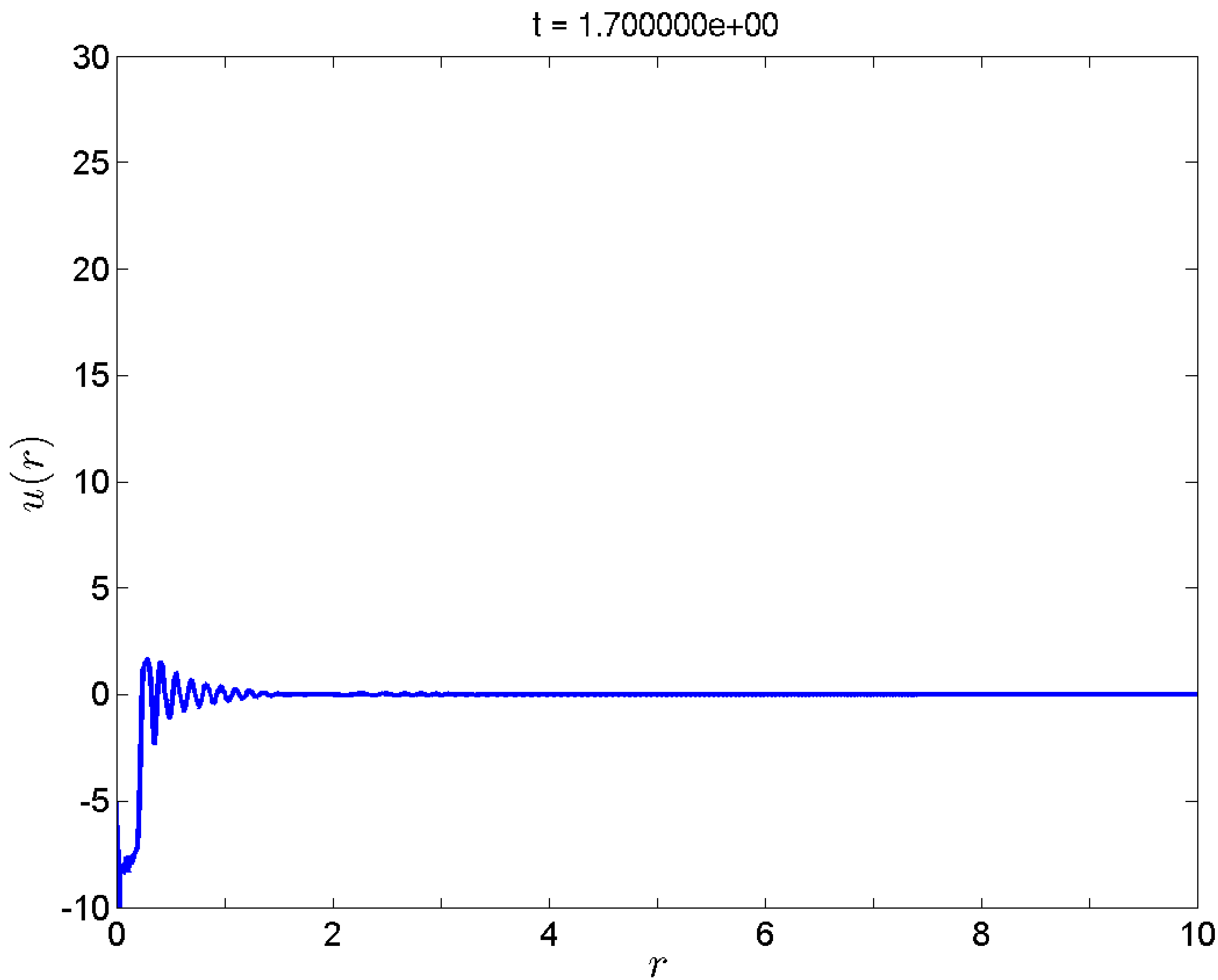
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
$||u(r)||_{L^{12}}$
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
t
simulation
expected linear asymptotics

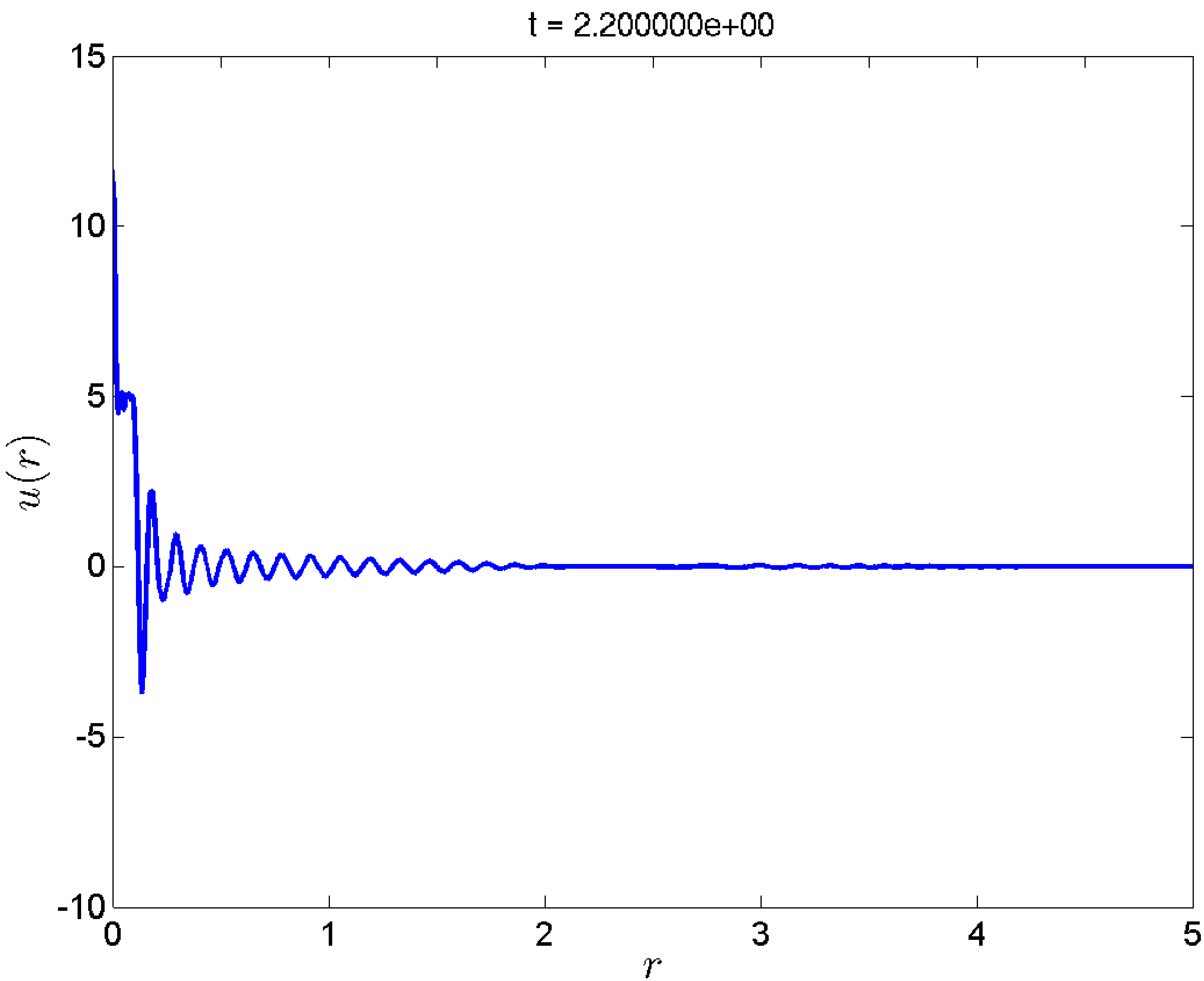


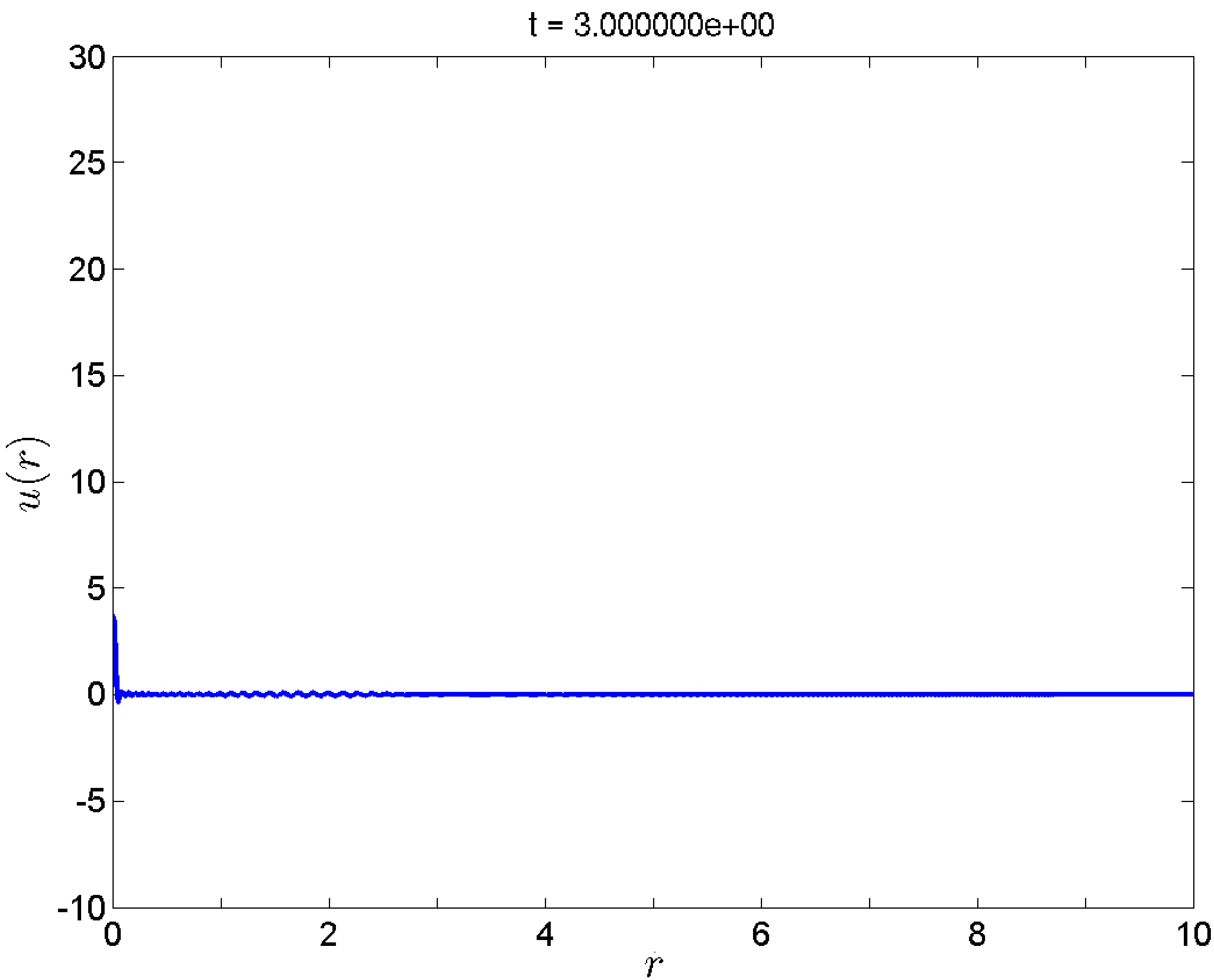
t = 0
Supercritical NLW Simulations
30
25
20
15
10
5
0
-5
-10
u(r)
0
2
4
6
8
10
r


t = 5.000000e-01
30
25
20
15
10
5
0
-5
-10
$u(r)$
0
2
4
6
8
10
r

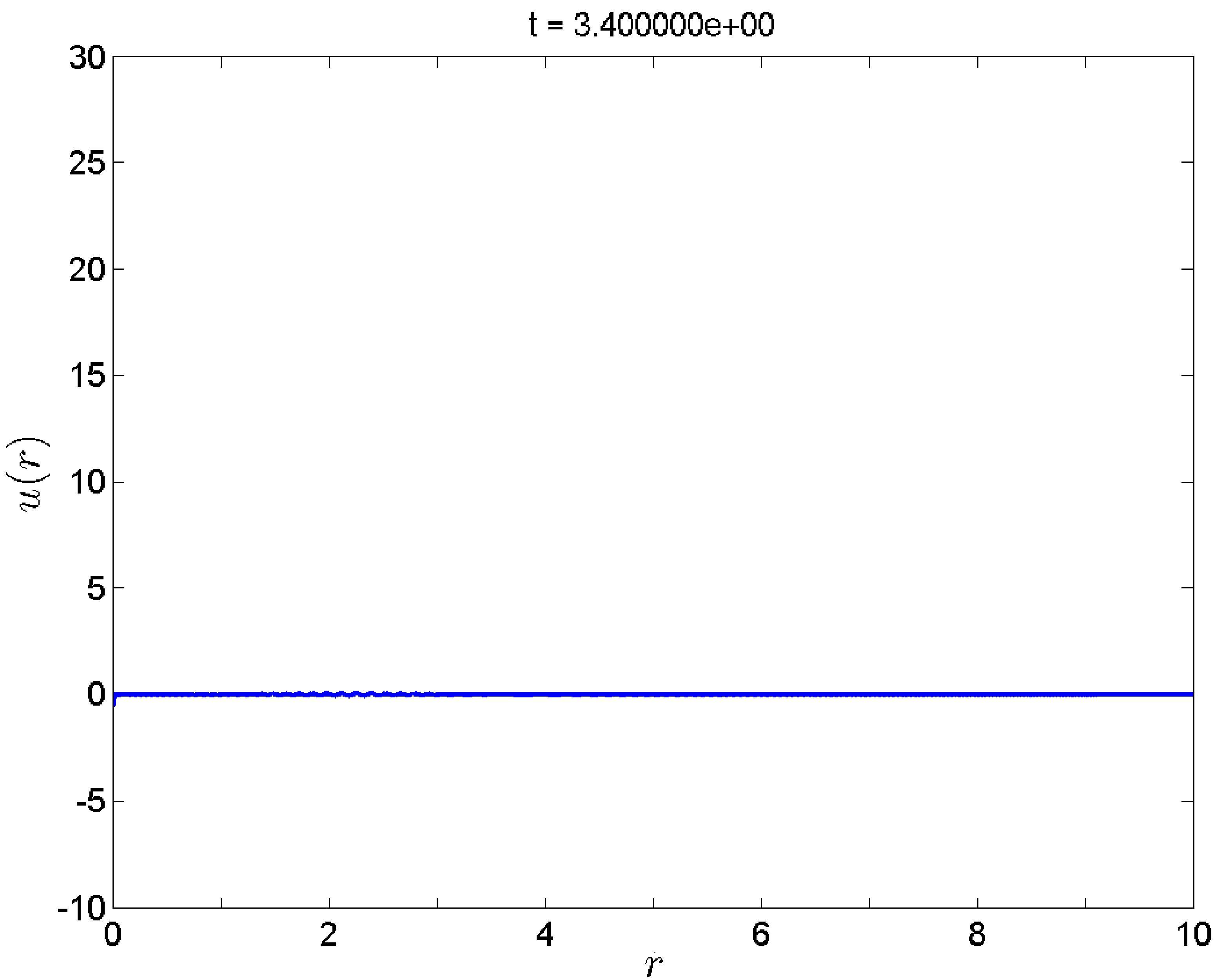

t = 1.000000e+00
15
10
5
0
-5
-10
$u(r)$
0
1
2
3
4
5
r

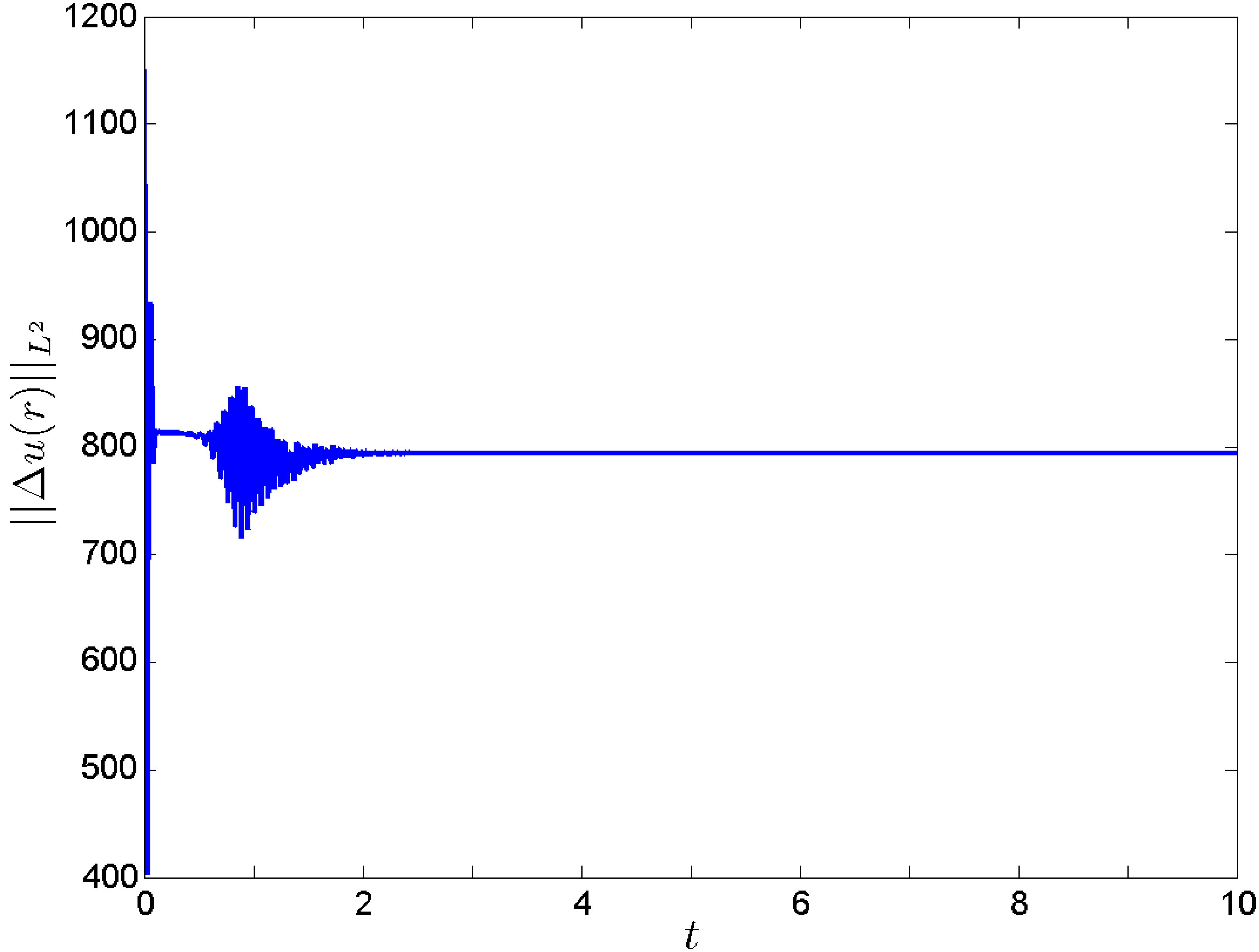

t = 1.700000e+00
30
25
20
15
10
5
0
-5
-10
$u(r)$
0
2
4
6
8
10
r


t = 2.200000e+00
15
10
5
0
-5
-10
0
1
2
3
4
5
u(r)
r

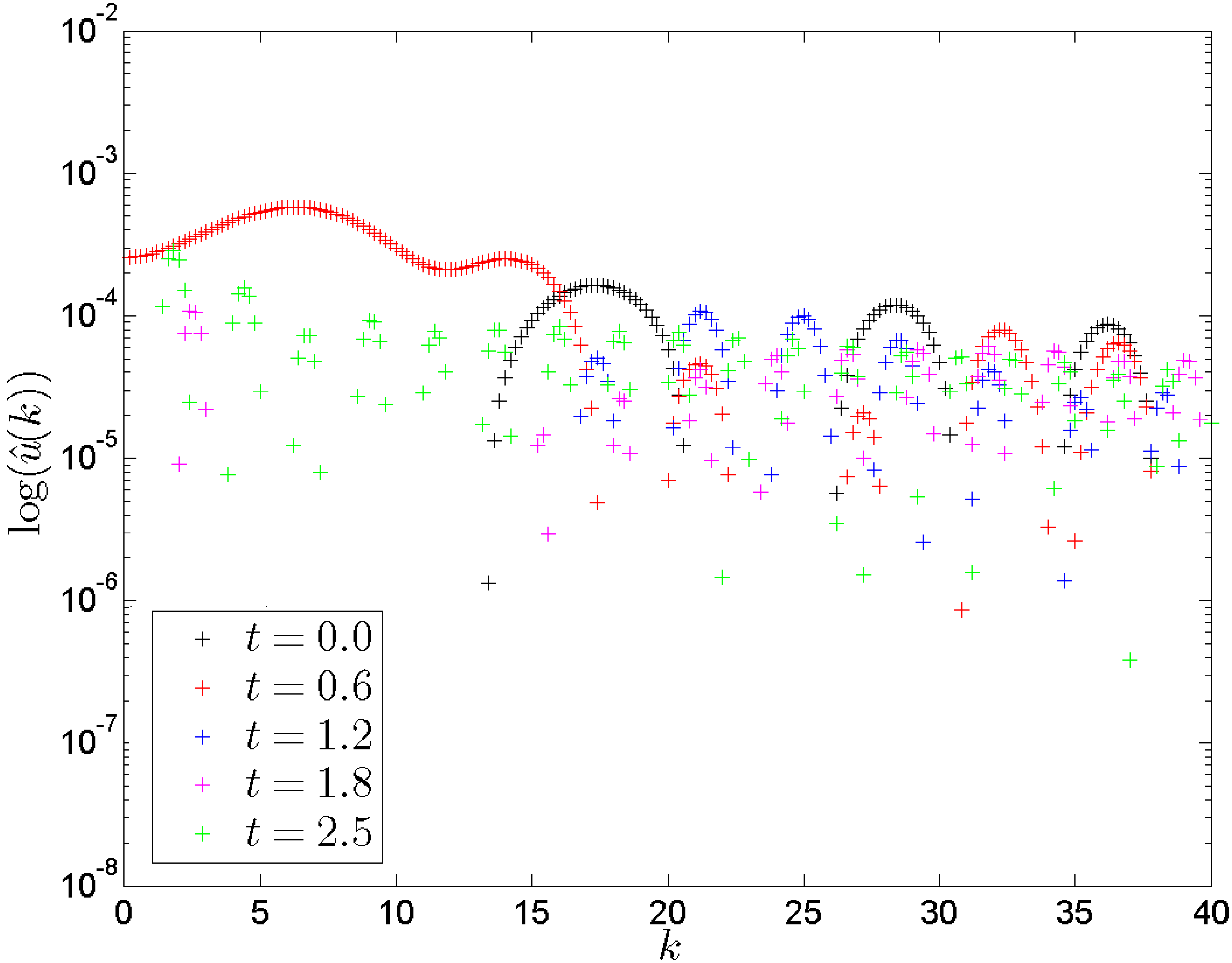


t = 3.000000e+00
30
25
20
15
10
5
0
-5
-10
$u(r)$
0
2
4
6
8
10
r

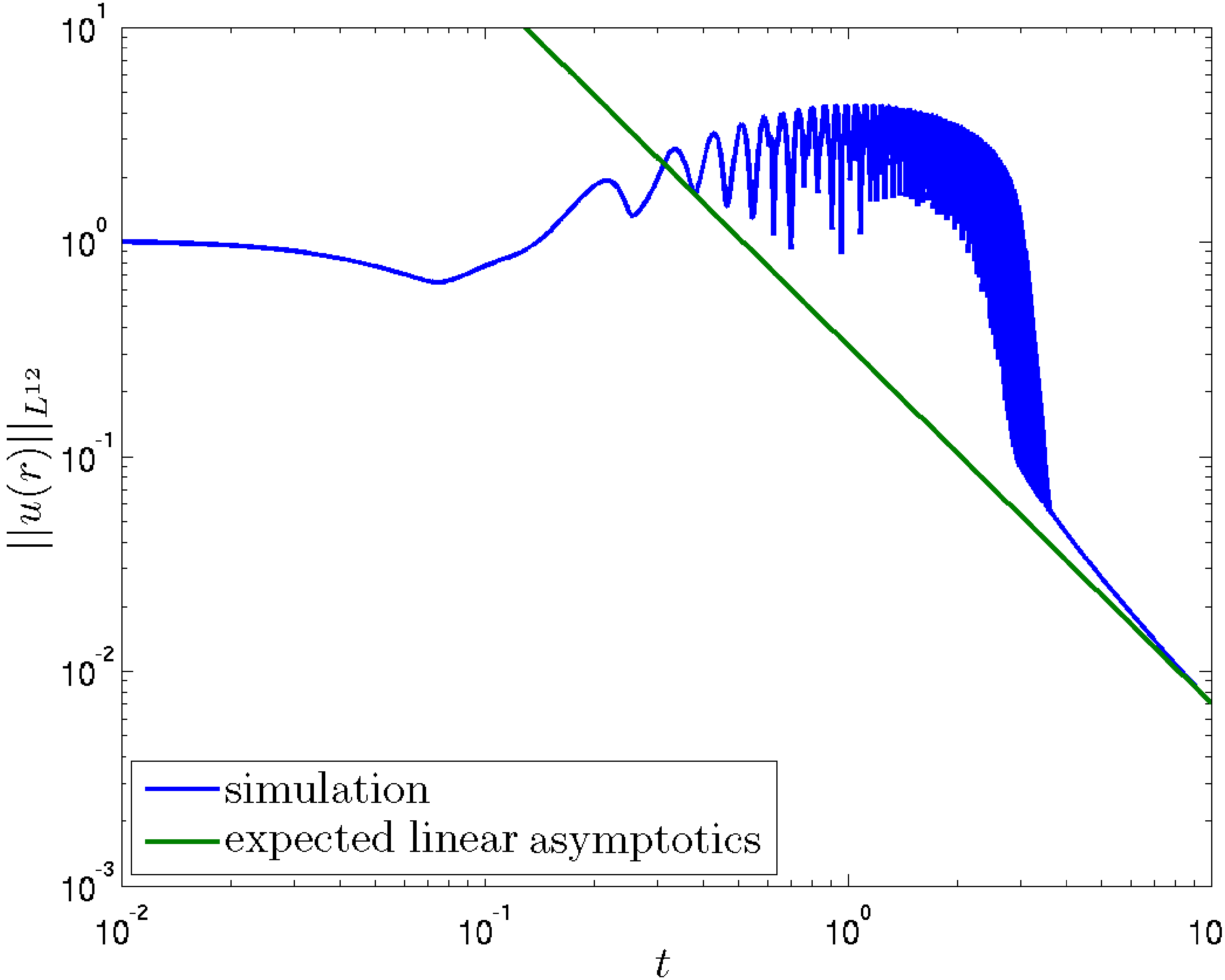

t = 3.400000e+00
30
25
20
15
10
5
0
-5
-10
$u(r)$
0
2
4
6
8
10
r



1200
1100
1000
900
800
700
600
500
400
0
2
4
6
8
10
t
$||\Delta u(r)||_{L^2}$



10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-8
log(û(k))
0
5
10
15
20
25
30
35
40
k
t = 0.0
t = 0.6
t = 1.2
t = 1.8
t = 2.5



10^1
10^0
10^{-1}
10^{-2}
10^{-3}
$||u(r)||_{L^{12}}$
10^{-2}
10^{-1}
10^0
10^1
t
simulation
expected linear asymptotics

4. Ideas Towards a Supercritical Theory?

Ideas Towards a Supercritical Theory?

GAFA, Geom. funct. anal.
Special Volume – GAFA2000, 32 – 56
1016-443X/00/S1032-25 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

PROBLEMS IN HAMILTONIAN PDE'S

J. Bourgain

1 Introduction

The purpose of this exposé is to describe a line of research and problems, which I believe, will not be by any means completed in the near future. As such, we certainly hope to encourage further investigations. The list of topics in this field is fairly extensive and only a few will be commented on here. Their choice was mainly dictated by personal research involvement. It should also be mentioned that the different groups of researchers may have very different styles and aims. As a science, claims and results range from pure experimentation to rigorous mathematical proofs. Although my primary interest is this last aspect, I have no doubt that numerics or heuristic argumentation may be equally interesting and important. The history of the Korteweg-de-Vries equation for instance is a striking example of how a problem may evolve through these different interacting stages to eventually create a beautiful theory. As a mathematician, I feel however that

Ideas Towards a Supercritical Theory?

PROBLEM. *Is there global scattering in the energy space for* $p = 2 + \frac{4}{d}$*?*

(See also [C1,2] for other results on scattering).

(iv) We like to sketch the theoretical possibility for computer assisted proofs of global existence and scattering, for a given data ϕ. Consider for instance the 3D supercritical problem

$$\begin{cases} iu_t + \Delta u - u|u|^6 = 0 \\ u(0) = \phi \end{cases} \tag{3.22}$$

where ϕ is a given smooth function. We do expect a global smooth solution + scattering. For this to hold, it is sufficient to show that for some time, $0 < T < \infty$,

(a) (3.22) has a smooth solution on [0, T]. Equivalently, $T^* > T$, where T^* refers to Theorem 3.7

(b) The norm $\|e^{i(t-T)\Delta} u(T)\|_{L^{15}_{t \geq T} L^{15}_x} < \delta$

where $\delta > 0$ is some numerical constant (we do not explain the role of the L^{15}-norm here). About step (a). If we fix a time T, one may establish the result numerically. To do this, one first gathers sufficiently many discrete data and interpolates them with a (smooth) function $v = v(x, t)$, $t < T$. Assuming (3.22) has indeed a smooth solution, the function v will

Ideas Towards a Supercritical Theory?

Numerical Observations:

- The critical Sobolev norm $\|u(t)\|_{H}^{s_c}$ stays bounded but is not monotone.
- The H^{s_c} density spreads out in Fourier space.
- The Besov $B^2_{2,\infty}$ norm shrinks to small values as time advances.

Ideas Towards a Supercritical Theory?

From [Bou98], the *small data* theory is relaxed:

$$\|(e^{it\Delta}\phi)(x)\|_{L^4(dxdt)} \le C\|\phi\|_2^{1-\theta}\left\{\sup_{A,\tau\in\mathcal{C}_A}\left\|\frac{1}{A}\widehat{\phi}|_\tau\right\|_1\right\}^{\theta}. \qquad (57)$$

Inequality (57) permits in particular to generate initial data $\phi \in L^2(\mathbb{R}^2)$ with arbitrarily large $\|\phi\|_2$-norm such that $\|e^{it\Delta}\phi\|_4$ is small; hence the 2D-Cauchy problem

$$\begin{cases} iu_t + \Delta u + \lambda|u|^2 u = 0 \\ u(0) = \phi \end{cases} \qquad (58)$$

is globally well posed with a solution u in $\mathcal{C}(]-\infty,\infty[; L^2(\mathbb{R}^2)) \cap L^4(]-\infty,\infty[, L^4(\mathbb{R}^2))$. Observe that in the small data theory for (58), the smallness of the L^2–norm $\|\phi\|_2$ is only used to ensure smallness of $\|e^{it\Delta}\phi\|_4$.

Ideas Towards a Supercritical Theory?

The motif from [Pla02] sets the stage:

1 Existence and regularity theorems

Theorem 1. *Let $n = 2$, and $u_0 \in \dot{B}_2^{0,\infty}$ be such that*

(4) $$\|u_0\|_{\dot{B}_2^{0,\infty}} < \varepsilon_0.$$

Then there exists a global solution of (1) such that

(5) $$u(x,t) \in L_t^\infty(\dot{B}_2^{0,\infty}),$$

(6) $$u(x,t) \underset{t\to 0}{\longrightarrow} u_0(x) \quad \textit{weakly.}$$

Moreover, this solution is unique under the additional assumption

$$u \in \dot{B}_{4^+}^{0^-,\infty}(\mathcal{L}_t^{4^-}) \cap \dot{B}_{4^-}^{0^+,\infty}(\mathcal{L}_t^{4^+})$$

and

(7) $$\sup_j \left(2^{0^+ j}\|\Delta_j u\|_{L_x^{4^-}(L_t^{4^+})} + 2^{0^- j}\|\Delta_j u\|_{L_x^{4^+}(L_t^{4^-})}\right) < \varepsilon_1.$$

small Besov → GWP

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