

Mat 344F challenge set #2  
Solutions

1. Put two balls into box 1, one ball into box 2 and three balls into box 3. The remaining 4 balls can now be distributed in any way among the three remaining boxes. This can be done in  $C(4+3-1,4)=C(6,4)=15$  ways. Please note that the question should have said that box 1 has at least two balls.
  
2. There are  $mn$  entries in the  $m \times n$  matrix, two choices for each, and the choices are independent of one another, hence by the multiplication principle, there are  $2^{mn}$  matrices that have 0,1 entries. If we allow the entries to be 0,1,2,3 then each of the  $mn$  entries has four choices which are independent of one another. The multiplication principle tells us that there are  $4^{mn}$  such matrices.
  
3. Let A be the event, " exactly one pair of shoes is selected". Let B be the event "exactly two pairs of shoes are selected. Let S be the event " four shoes are selected from 20 shoes with order being irrelevant". We want to calculate  $P(\text{at least one pair of shoes is selected}) = P(A \text{ or } B) = P(A) + P(B)$  since events A and B are disjoint. We will use the probability formula  $P(A) = \frac{n(A)}{n(B)}$ . The number of ways to select exactly one pair of shoes =  $n(A) = C(10,1) \times (C(18,2) - 9)$ .  $C(10,1)$  counts the number of ways to pick one pair of shoes. There are 18 shoes remaining, pick two shoes. However we don't want to pick another pair so we must subtract the 9 remaining pairs of shoes. The number of ways of picking two pairs of shoes =  $n(B) = C(10,2)$ . Using these results, it follows that
 
$$P(\text{selecting at least one pair of shoes}) = \frac{C(10,1) \times (C(18,2) - 9)}{C(20,4)} + \frac{C(10,2)}{C(20,4)}.$$

Another way to approach this problem is to first calculate the probability that no pair of shoes is selected and then subtract this from one. This will yield the probability that at least one pair of shoes is selected. The number of ways to select no pair of shoes is  $C(10,4) \times C(2,1) \times C(2,1) \times C(2,1) \times C(2,1)$ .  $C(10,4)$  counts the number of ways to select four shoes out of ten with order being irrelevant. Once the pairs have been selected, then out of each pair we can select one shoe, and this

can be done in  $(C(2,1))^4$  ways. The probability of selecting at least one pair of shoes is  $1 - \frac{C(10,4) \times 2^4}{C(20,4)}$ .

4. Fix the six individuals in some order and suppose we have two Monday, Wednesday and Friday labels. We will place one label on each person. The end result will be two accidents with Monday labels, two accidents with Wednesday and two accidents with Friday labels. This is like having six distinct items, two each of three kinds, and we want all the possible arrangements. The number of such arrangements is  $\frac{6!}{2!2!2!}$ .

You can think of this as the number of ways to arrange the letters MM WW FF (i.e. permutations with repetition). Using the permutation with repetitions formula, the number of arrangements is  $\frac{6!}{2!2!2!}$ . For part b, choose the three days in  $C(7,3)$  ways, then multiply the solution obtained in part a by the number. The final answer is  $C(7,3) \times \frac{6!}{2!2!2!}$ .

5. If we were arranging the people on a bench then there would be  $10!$  ways to do this. When they are sitting on a circular table, it is possible to rotate the table in 10 ways without changing the permutation. (I.e. you still have the same two people to your right and left.).  $10!$  counts 10 times the number of distinct arrangements around a circular table. Thus, the solution is  $\frac{10!}{10}$ .

6. page 221, 9a. Prove by a combinatorial argument that  $C(2n,2) = 2 \times C(n,2) + n^2$ . The LHS counts the number of unordered pairs of numbers from the set  $\{1, 2, \dots, 2n\}$ . This can be counted as follows. Split the set into two sets of  $n$  numbers. Choose a pair of numbers from either the first set of  $n$  numbers or the second set of  $n$  numbers. By the "or" rule this can be done in  $C(n,2) + C(n,2)$  ways. The only other way to choose a pair of numbers is to choose one number from each of the  $n$  number sets. This can be done in  $C(n,1) \times C(n,1)$  ways. We add these two cases together and we get  $2 \times C(n,2) + n^2$ .

page 221, 14c. This is the binomial expansion of  $(1+x)^n$  where  $x=2$ . Hence

$$\sum_{k=0}^n 2^k C(n,k) = 3^n.$$

page 221, 14f.

$$\begin{aligned} \frac{1}{k+1} C(n,k) &= \frac{1}{k+1} \frac{n!}{(n-k)! k!} = \frac{n!}{(n-k)!(k+1)!} = \frac{1}{n+1} \frac{(n+1)!}{(n-k)!(k+1)!} = \\ &= \frac{1}{n+1} C(n+1, k+1). \text{ Hence } \sum_{k=0}^n \frac{1}{k+1} C(n,k) = \sum_{k=0}^n \frac{1}{n+1} C(n+1, k+1) = \\ &= \frac{1}{n+1} (C(n+1,1) + C(n+1,2) + \dots + C(n+1, n+1)) = \\ &= \frac{1}{n+1} (C(n+1,0) + C(n+1,1) + C(n+1,2) + \dots + C(n+1, n+1) - C(n+1,0)) = \\ &= \frac{1}{n+1} ((1+1)^{n+1} - 1) = \frac{1}{n+1} (2^{n+1} - 1). \end{aligned}$$

page 221, 15.

From the triangle identity of the binomial coefficients,

$$C(k+1, r+1) - C(k, r+1) = C(k, r).$$

$$\begin{aligned} \sum_{k=m}^n C(k, r) &= \sum_{k=m}^n [C(k+1, r+1) - C(k, r+1)] = C(m+1, r+1) - C(m, r+1) + C(m+2, r+1) - \\ &= C(m+1, r+1) + \dots + C(n, r+1) - C(n-1, r+1) + C(n+1, r+1) - C(n, r+1). \\ &= C(n+1, r+1) - C(m, r+1) \text{ (telescoping series.)} \end{aligned}$$

7. page 201, 3.

3a. There are three choices for each digit, using the multiplication principle there are  $3^6$  such numbers. For 3b. the number of distinct

arrangements of the digits 3 3 5 5 7 7 is  $\frac{6!}{2!2!2!}$ . Each distinct

arrangement gives us a different six digit number. Thus the fraction of

such numbers is  $\frac{6!}{3^6 \cdot 2!2!2!}$ .

page 201, 5.

This is the number of ways to select ten identical objects (coins) with repetition from four types. Apply the formula given in theorem 2, page 197 of Tucker, and our answer is  $C(10+4-1, 10) = C(13, 10)$ .

page 201, 13.

There are two types of sequences we can get here:

type I. An eight-digit sequence with six different digits with one digit appearing three times (i.e. 2 1 4 5 6 7 5 5).

type II. An eight-digit sequence with six different digits with two distinct digits repeated (i.e. 2 1 4 5 6 7 2 7).

The number of sequences of type I is  $C(10,6) \times C(6,1) \times \frac{8!}{3!}$ .  $C(10,6)$  counts the number of ways of selecting six digits out of 10 with order being irrelevant.  $\frac{8!}{3!}$  counts the number of eight-digit sequences with one digit appearing three times and  $C(6,1)$  counts the number of ways of selecting the digit which repeats.

The number of sequences of type II is  $C(10,6) \times C(6,2) \times \frac{8!}{2!2!}$ . This time,  $\frac{8!}{2!2!}$  counts the number of eight-digit sequences with two distinct digits repeated and  $C(6,2)$  counts the number of ways of selecting the digits that repeat. Using the “or” principle, we add these two cases together and get the answer  $C(10,6) \times (C(6,1) \frac{8!}{3!} + C(6,2) \frac{8!}{2!2!})$ .

page 201,15.

We must classify the possible types of five letter “words”.

type I. Four letters of one kind and another different letter.

type II. Three letters of one kind and two of another.

type III. Three letters of one kind and two other letters each of a different kind.

type IV. Two letters of one kind, two letters of another kind and one other letter.

type V. Two letters of one kind, and three different letters.

Mississippi has one M, four I's , four S's and two P's.

For type I, there are  $C(2,1) \times C(3,1) \times \frac{5!}{4!}$  such “words”.

For type II, there are  $C(2,1) \times C(2,1) \times \frac{5!}{3!2!}$  such “words”.

For type III, there are  $C(2,1) \times C(3,2) \times \frac{5!}{3!}$  such “words”.

For type IV, there are  $C(3,2) \times C(2,1) \times \frac{5!}{2!2!}$  such “words”.

For type V, there are  $C(3,1) \times \frac{5!}{2!}$  such “words”. Using the “or” principle, our answer will be the sum of all these cases.

8. page 189, 12

A standard 52 card deck has 13 types of cards, each type comes in four suits.

- a. The number of five-card poker hands with four aces, is  $C(4,4) \times C(48,1)$ .  $C(4,4)$  counts the number of ways to select four aces out of the four aces available. After the four aces are selected there are  $C(48,1)$  ways to select a non-ace.
- b. The number of five-card poker hands with four of a kind is  $C(13,1) \times C(4,4) \times C(48,1)$ . From thirteen types of cards, pick one of them, i.e.  $C(13,1)$  ways, next pick four of the four cards of available of that type, i.e.  $C(4,4)$  ways, and with the forty-eight cards left pick one them, i.e.  $C(48,1)$  ways.
- c. The number of five-card poker hands with exactly two different pairs is  $C(13,2) \times C(4,2) \times C(4,2) \times C(44,1)$ . From thirteen types of cards, pick two of them, i.e.  $C(13,2)$  ways, next pick two of the four cards available of the first type, do the same with the second type, i.e.  $C(4,2) \times C(4,2)$  ways to do both, and with the forty-four cards remaining select one more ,i.e.  $C(44,1)$  ways.

- d. The number of five-card poker hands that are a full house is  $C(13,1) \times C(4,3) \times C(12,1) \times C(4,2)$ . From thirteen types of cards, pick one of them which will form the three of a kind, i.e.  $C(13,1)$  ways, next select three cards from the four available of this type. There are now twelve types of cards left, pick one of them to form the two of a kind, i.e.  $C(12,1)$  ways, next select two of the four cards available of this type.
- e. There are nine possible ways to get consecutive values. They are 2 3 4 5 6, 3 4 5 6 7, 4 5 6 7 8, 5 6 7 8 9, 6 7 8 9 10, 7 8 9 10 J, 8 9 10 J Q, 9 10 J Q K, 10 J Q K A. Each sequence can be obtained in  $C(4,1) \times C(4,1) \times C(4,1) \times C(4,1) \times C(4,1)$  ways, because each card comes in four suits. Using the “or” principle, we sum all of these up, and get  $9 \times 4^5$  as our answer. Note that this answer includes straight flushes.
- f. The number of five-card poker hands that have no pairs, is  $C(13,5) \times C(4,1) \times C(4,1) \times C(4,1) \times C(4,1) \times C(4,1)$  ways. From the thirteen types of cards available, choose five of them, i.e.  $C(13,5)$  ways. From each of the types you selected, choose one card from each.

page 189, 22.

Call the teams, team A and team B. Since there are five men and three women, the only way to have at least one woman on each team is for one team to have two women and the other team to have one woman. The number of ways to do this is  $C(2,1) \times C(3,2) \times C(5,2) \times C(3,3) \times C(1,1)$ . Pick the team which will have the two women on it, i.e.  $C(2,1)$  ways, next select two of the three women which will be on this team, i.e.  $C(3,2)$  ways, from the five men available choose two of them, i.e.  $C(5,2)$  ways, and from the remaining four people, create the next team( only one way to do this ,  $C(3,3) \times C(1,1)$ ). If the teams don't matter, then the answer will be the one we obtained divided by two.

Another way to approach this problem is to count the number of ways to create two teams of four such that one teams doesn't have a woman and subtract this from the total number of ways to create two teams of four from five men and three women without restriction. If a team doesn't

have a woman, then it must consist of four men. There are  $C(2,1) \times C(5,4) \times C(4,4)$ . Pick the team which will have four men on it, i.e.  $C(2,1)$  ways, next select four out of the five men available to be on this team, and from the four remaining people, create the next team. The number of ways to create two teams of four without restriction from eight people is  $C(8,4) \times C(4,4)$ . Pick four people out of eight and create the first team and with the four remaining, create the other team. Our answer will be  $C(8,4) \times C(4,4) - C(2,1) \times C(5,4) \times C(4,4)$ . If the teams don't matter, then divide this answer by two.

page 189, 33.

The word **INSTRUCTOR** has the vowels, **I, U, O** and the consonants **N,S,T,T,R,R,C**.

a. Put the vowels in a box, and call it the new vowel **V**. We are now counting the number of arrangements of the letters **V,N,S,T,T,R,R,C**.

The number of such arrangements is  $\frac{8!}{2!2!} \times 3!$ .  $\frac{8!}{2!2!}$  counts the number of arrangements of eight letters with a couple of letters repeated.  $3!$  counts the number of arrangements of the vowels in **V**. The fraction of

all such arrangements is  $\frac{\frac{8!}{2!2!}}{\frac{10!}{2!2!}}$ .

b. This can be split up into two cases. Case one has been done above, and what remains is to find the number of arrangements in which *exactly* two vowels are together. Out of the three vowels available choose two of them. This can be done in  $C(3,2)$  ways. Put those two vowels in a box, and call it the new vowel **V**. Call the remaining vowel **V<sub>1</sub>**. We are now trying to find the arrangements of the letters,

**V, V<sub>1</sub>, N, S, T, T, R, R, C**. The number of these arrangements is

$\frac{9!}{2!2!} \times 2! = \frac{9!}{2!} \cdot \frac{2!}{2!2!}$  counts the number of ways to arrange nine letters

with two distinct letters repeating.  $2!$  counts the number of arrangements of the new vowel **V**. We must take away the number of arrangements with all the vowels together since we are only considering the case with two vowels together. The number of arrangements with the three vowels together can occur in four ways with **VV<sub>1</sub>V** and **V<sub>1</sub>VV**. Remember that the vowels in **V** can be arranged two ways. To give an

example, suppose  $V = I U$  and  $V_1 = O$ . The four ways this can occur are as follows:  $I U O$ ;  $U I O$ ;  $O I U$ ;  $O U I$ . The number of arrangements with all three vowels together and our two specified vowels together is  $\frac{8!}{2!2!} \times 4 = 8!$ . The number of arrangements with exactly two vowels together is  $C(3,2) \times [\frac{9!}{2!} - 8!]$ . The number of arrangements with at least two vowels together is this answer plus the one we obtained in part a. Divide this by  $\frac{10!}{2!2!}$  to get the fraction of such arrangements.

### **Problems from page 176**

6. There are  $n$  ways to pick the man. Once the man is picked there are  $n-1$  women who are not his wife. Pick one of them in  $n-1$  ways. By the fundamental principle of counting, there are  $n(n-1)$  ways to pick a man and woman who are not husband and wife.

7.

- a. There are 20 books in total so there are 20 ways in which one book could be selected.
- b.  $8 \times 7 \times 5$ . From the eight English books pick one of them, from the seven French books pick one, and from the five German books pick one of them.
- c. There are 6 cases to consider here.
  - i. Two French and one English.
  - ii. Two French and one German.
  - iii. Two German and one English.
  - iv. Two German and one French.
  - v. Two English and one German.
  - vi. Two English and one French.
- i. Select two French books in  $C(7,2)$  ways and one English book in 8 ways. Once they are selected, there are  $3!$  ways to arrange them.  $C(7,2) \times 8 \times 3! = 1008$ .

- ii. Select two French books in  $C(7,2)$  ways and one German book in 5 ways. Once they are selected, there are  $3!$  ways to arrange them.  
 $C(7,2) \times 5 \times 3! = 630$ .
- iii. Select two German books in  $C(5,2)$  ways and one English book in 8 ways. Once they are selected, there are  $3!$  ways to arrange them.  
 $C(5,2) \times 8 \times 3! = 480$ .
- iv. Select two German books in  $C(5,2)$  ways and one French book in 7 ways. Once they are selected, there are  $3!$  ways to arrange them.  
 $C(5,2) \times 7 \times 3! = 420$ .
- v. Select two English books in  $C(8,2)$  ways and one German book in 5 ways. Once they are selected, there are  $3!$  ways to arrange them.  
 $C(8,2) \times 5 \times 3! = 840$ .
- vi. Select two English books in  $C(8,2)$  ways and one French book in 7 ways. Once they are selected, there are  $3!$  ways to arrange them.  
 $C(8,2) \times 7 \times 3! = 1176$ .

Adding these all up, we get 4554.

10. 3 A's and 5B's. A collection of A's and B's can only be distinguished by the number of A's and B's in the collection. Any collection can have 0,1,2, or 3 A's, or 0,1,2,3,4, or 5 B's. There are  $4 \times 6 - 1 = 23$  non empty subcollections of A's and B's.

13.

- a. The first digit can't be zero, so we only have 9 choices for this digit. We have 10 choices for each of the other digits. Therefore, there are  $9 \times 10^4$  five digit numbers.
- b. A number is even if and only if the last digit is even. There are 5 choices for the last digit, and 9 choices for the first digit. After taking care of these restrictions, the other three digits can be selected in 10 ways. There are  $9 \times 5 \times 10^3 = 45000$  even five digit numbers.
- c. The number of such five digit numbers depends upon the position of the 3. If 3 is the first digit, then each of the other four digits can be selected in 9 ways ( we can't pick another 3). From this case we have  $9^4$  such five digit numbers. If the 3 is in any other position, we have 8 choices for the first digit, and 9 choices for the other digits. From these four cases we get  $4 \times 8 \times 9^3$  such five digit numbers. Adding up these cases we have  $9^4 + 4 \times 8 \times 9^3$  five digit numbers with exactly one three.

20. There are 21 numbers between 20 and 40 inclusive. 11 of them are even and 10 of them are odd. If the sum of two numbers is even, then the numbers must be both even or both odd. Case 1: If they are both even, then there are  $11 \times 10$  ways to choose the numbers. Case 2: If they are both odd then there are  $10 \times 9$  ways to choose the numbers. Adding these together, we have  $110 + 90 = 200$  ways in which two numbers selected from the numbers 20 to 40 can add to an even number. There are  $21 \times 21$  ways in which the numbers could be selected, so the probability of this happening is  $200 / 21^2$ .
30. A binary sequence is a sequence composed of just two digits, 0 and 1. If there are no pairs of consecutive digits which are the same, then there are only two possible sequences of this type, ie 01010101.... and 10101010.... .

**Problems from page 208.**

1.
  - a. Model: selection with repetition. Distribute 40 identical balls into 4 distinct boxes. This can be done in  $C(40+4-1, 40) = C(43, 40)$  ways.
  - b. Only one way. Just give each child 10 beans. There is only such way because the beans are indistinguishable.
  - c. Give one bean to each child. There is only one way to do this since the beans are indistinguishable. We are free to distribute the remaining 36 beans any way we wish, i.e. in  $C(36+4-1, 36) = C(39, 36)$  ways.
  
5. Give two donuts of each type to each principal. There is only one way to do this. Now we are free to distribute the remaining 10 chocolate, 4 cinnamon and 6 powdered sugar donuts without restriction, i.e. in  $C(10+4-1, 10) \times C(4+4-1, 4) \times C(6+4-1, 4)$  ways.
  
6. We have to distribute 24 different objects into three different boxes with twice as many objects in one box as in the other two combined. This means that one box has to have 16 objects and the other two must have a total of 8. There are 3 ways to choose the box which will have 16 objects in it,  $C(24, 16)$  ways to choose the 16 objects to go in it, and  $2^8$  ways to distribute the remaining 8 balls into the other two boxes. Our answer is  $3 \times C(24, 16) \times 2^8$ .

7. Take any arrangement of the letters V, S, T, N, and G. We can put a position in front, in between the letters and one at the end for a total of 6 distinct positions. To ensure that there are no consecutive I's, we choose 3 of these positions in  $C(6,3)$  ways and distribute the I's into those positions. Since the I's are identical, there is only one way to arrange them once they are in position. There are  $5!$  ways to arrange the letters V,S,T,N,G, so there are a total of  $C(6,3) \times 5!$  arrangements of the letters in the word VISITING with no pair of consecutive I's.
8. Take any arrangement of the 21 consonants. We can put a position in front, in between the consonants, and one at the end for a total of 22 positions. To ensure there are no pairs of vowels appearing consecutively we choose 5 positions out of the 22 in  $C(22,5)$  ways and place the vowels in these positions. Once they are in position they can be arranged in  $5!$  ways. The consonants can be arranged in  $21!$  ways. Our answer is  $C(22,5) \times 5! \times 21!$ .
9. Lay out the 14 H's. There is only one way we can do this. Put a position in front, in between and at the end of the H's for a total of 15 positions. To ensure that there are no consecutive tails, we choose 6 positions out of the 15 in  $C(15,6)$  ways and put a T in each position. There is only one way to arrange the T's once they are in position. Our answer is  $C(15,6)$ .
- 10.
- Model: selection with repetition. The number of solutions to  $x_1+x_2+x_3+x_4+x_5 = 28$  equals the number of ways to distribute 28 identical balls into five boxes without restriction. This can be done in  $C(28+5-1,28)$  ways.
  - $x_i > 0$ . Each box has at least one ball in it. Distribute one ball into each box. This can only be done in one way. The remaining 23 balls can be distributed without restriction in  $C(23+5-1,23)$  ways.
  - $x_i \geq i$ . Put one ball in box 1, two balls in box 2, three balls in box 3, four balls in box 4, and five balls in box 5. This can only be done in one way. The remaining 13 balls can be distributed without restriction in  $C(13+5-1, 13)$ ways.

11. Let  $x_i = u_i - 5$ .  $u_i = x_i + 5 \geq -5 + 5 = 0$ .  $u_1 + u_2 + u_3 = 15$ . This is equivalent to the number of ways to distribute 15 identical balls into 3 distinct boxes with no restrictions. This can be done in  $C(15+3-1, 15)$  ways.

12. For positive integer solutions,  $x_i > 0$  or  $x_i \geq 1$ . We want the number of integer solutions to  $x_1 + x_2 + x_3 + x_4 \leq 99$  where  $x_i \geq 1$  for  $i = 1, 2, 3, 4$ . The number of solutions to  $x_1 + x_2 + x_3 + x_4 = k$  with  $x_i \geq 1$  is  $C(k-4+4-1, k)$ . Summing up all the cases where  $k=1, 2, \dots, 99$  we get

$$\sum_{k=1}^{99} C(k-1, k).$$

15. We have three different teddy bears and nine identical lollipops.

- There are  $4^3$  ways to distribute the teddy bears, and  $C(9+4-1, 9)$  ways of distributing the lollipops. There is a total of  $4^3 \times C(9+4-1, 9)$  ways to distributing these without restriction.
- Each child can have at most one teddy bear. There only three teddy bears, so one child must go without. The teddy bears can be distributed to three children in  $3!$  ways. Distribute the lollipops without restriction in  $C(9+4-1, 9)$  ways. There are 4 ways to choose the child who will not receive a teddy bear. Putting these together we have  $4 \times 3! \times C(9+4-1, 9)$ .
- Distribute the teddies in  $4^3$  ways. We can ensure that each child gets 3 goodies by filling out each child with the lollipops.

16. The letters of EFFLORESCENCE, can be written as E E E E F F C C L O R S N. Put the C's together in a box and count them like one letter. Do the same with the F's. Consider any arrangement of the letters L O R S N CC and FF. We can put a position in front, in between and at the end of the arrangement for a total of 8 positions. To ensure that there are no consecutive E's, we choose 4 of these positions in  $C(8, 4)$  ways and put the E's in them. The letters L O R S N CC and FF can be arranged in  $7!$  ways, so the number of arrangements with no pair of consecutive E's and with consecutive C's and F's is  $C(8, 4) \times 7!$ . The total number of arrangements of the letters of EFFLORESCENCE is

$\frac{13!}{4!2!2!}$ , so the fraction of such arrangements is  $\frac{C(8,4) \times 7!}{13!}$ .

17. The letters of MISSISSIPPI can be written as M I I I I S S S S P P.

Consider any arrangement of the letters M I I I I P and P. We can put a position in front, in between and at the end of the arrangement for a total of 8 positions. We can ensure no consecutive S's by choosing 4 positions out of the 8 in  $C(8,4)$  ways. The letters M I I I I P and P can be arranged in  $\frac{7!}{4!2!}$  ways. The total number of arrangements with no consecutive S's in  $C(8,4) \times \frac{7!}{4!2!}$ .

18. We break this down into cases based on the number of objects in box 4. Box 4 can have  $3k$  balls in it, where  $k = 0, 1, 2, 3, 4, 5$ . If box 4 has  $3k$  balls in it, then we distribute the other  $15-3k$  balls into the first three boxes without restriction in  $C(15-3k+3-1, 15-3k)$  ways. Summing up all the cases we get that the number of such distributions is

$$\sum_{k=0}^5 C(15-3k+3-1, 15-3k).$$

20. We are distributing 8 balls into six different boxes.

a. Balls are identical. Choose  $k$  balls, where  $k=0, 1, 2, 3, 4$ . There is only one way to do this since the balls are identical. There are  $C(k+2-1, k)$  ways to distribute the balls into the first two boxes. The remaining  $8-k$  balls can be distributed in  $C(8-k+4-1, 8-k)$  ways. Summing up over all the cases, we get

$$\sum_{k=0}^4 [C(k+2-1, k)C(8-k+4-1, 8-k)] \text{ as our answer.}$$

b. Balls are distinct. Choose  $k$  balls where  $k = 0, 1, 2, 3, 4$ . There are  $C(8, k)$  ways to do this. There are  $2^k$  ways to distribute these balls into the first two boxes. The remaining  $8-k$  balls can be distributed in  $4^{8-k}$  ways. Putting these together we have  $C(8, k) \times 4^{8-k} \times 2^k$  as our answer.

32. This is equivalent to the number of ways there are to distribute 20 identical balls into 6 distinct boxes such that the first three boxes have a total of 7 balls in them. Choose 7 balls out of the 20 available. There is only one way to do this since the balls are identical. There are  $C(7+3-1,7)$  ways to distribute them into the first three boxes. The remaining 13 balls can be distributed in  $C(13+3-1,13)$  ways.
- 35.
- This is equivalent to the number of unordered collections of 4 red balls, 5 blue balls and 7 black balls. There are  $5 \times 6 \times 8$  such collections with the empty collection interpreted as all the balls going into one box. There is one to one correspondence between these splittings and the unordered collections of the balls.
  - Take away the two cases above where one box gets all the balls and we have  $5 \times 6 \times 8 - 2$  as our answer.
52. Choose the 11 players who are to have this distribution of cards in  $C(13,11)$  ways. Choose 11 hearts in  $C(13,11)$  ways, 11 spades in  $C(13,11)$  ways, 11 clubs in  $C(13,11)$  ways, and 11 diamonds in  $C(13,11)$  ways. The cards can be ordered in  $11!$  once they are chosen. The remaining two players have to be dealt cards. There are 2 hearts, 2 clubs, 2 diamonds, and 2 spades left. We don't want either of them to have one card of each type, otherwise we have more than 11 players with this distribution of cards. There are  $C(8,4)$  ways to deal four cards to these two players and there are  $C(2,1)C(2,1)C(2,1)C(2,1)$  ways in which they could be dealt a hand with one card of each type. So there are  $C(8,4) - 2^4$  ways to deal these two players a four card hand such that the card hand doesn't contain one card from each suit. Putting this all together we have  $[C(13,11)]^5 \times (11!)^4 \times [C(8,4) - 2^4]$ .