Challenge Set 3
Solution Sketches

Page 247

1.b.

Formal products are of the form $x^{e_1} x^{e_2} x^{e_3}$, where $0 \le e_i \le 4$. We are interested in those of the form $x^{e_1 + e_2 + e_3}$ where $e_1 + e_2 + e_3 = 4$. There are 15 formal products in all. The ordered 3-tuples (e_1, e_2, e_3) which satisfy this are: $(4, 0\ 0), (0,4,0), (0,0,4), (2,2,0), (2,0,2),\ (0,2,2), (2,1,1), (1,2,1), (1,1,2), (3, (3,0,1), (0,3,1), (0,1,3), (1,0,3),$ and $(1,3,0)$.

d. In the expansion of $(1 + x + x^2 + \Lambda +)^3$, we have product of the form $x^{e_1} x^{e_2} x^{e_3}$ where $e_i \ge 0$, for $i = 1,2,3$. We need not consider any $e_i > 4$. If so, then $e_1 + e_2 + e_3 > 4$. We only need to consider the e_i's such that $e_1 + e_2 + e_3 = 4$ where $0 \le e_i \le 4$, $i = 1,2,3$. There are 15 formal product in all, and they are the same as those in part b.

2.

a. $e_1 + e_2 + e_3 + e_4 + e_5 = r$, $0 \le e_i \le 4$. Generating function required is $g(x) = (1 + x + x^2 + x^3 + x^4)^5$.

b. $e_1 + e_2 + e_3 = r$, $0 < e_i < 4$. Generating function required is $g(x) = (x + x^2 + x^3)^3$.

c. $e_1 + e_2 + e_3 + e_4 = r$, $2 \le e_i \le 8$. Generating function required is
$(x^2 + x^4 + x^6 + x^8)(x^3 + x^5 + x^7)(x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^2$.

d. $e_1 + e_2 + e_3 + e_4 = r$, $0 \le e_i$. Generating function required is $(1 + x + x^2 + \Lambda)^4$.

e. $e_1 + e_2 + e_3 + e_4 = r$, $e_i > 0$, e_2, e_4 odd, and $e_4 \le 3$. Generating function required is
$(x + x^2 + x^3 + \Lambda)(x + x^3 + x^5 + \Lambda)^2 (x + x^3)$.

3.

a. This is the same as the number of integer solutions to $e_1 + e_2 + e_3 = r$, $0 \leq e_1 \leq 3$, $0 \leq e_2 \leq 4$, and $0 \leq e_3 \leq 4$. To generate all formal products of the form $x^{e_1} x^{e_2} x^{e_3}$ with the given constraints, we need the generating function, $g(x) = (1 + x + x^2 + x^3)(1 + x + x^2 + x^3 + x^4)^2$.

b. This is the same as the number of integer solutions to $e_1 + e_2 + e_3 = r$, where $1 \leq e_1 \leq 5$, $1 \leq e_2 \leq 3$, $1 \leq e_3 \leq 8$. To generate all formal products of the form $x^{e_1} x^{e_2} x^{e_3}$ with the given constraints, we need the generating function $g(x) = (x + x^2 + x^3 + x^4 + x^5)(x + x^2 + x^3)(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8)$.

c. This is the same as the number of integer solutions to $e_1 + e_2 + e_3 + e_4 = r$, $0 \leq e_i$, $i = 1,2,3,4$. To generate all formal products of the form $x^{e_1} x^{e_2} x^{e_3}$, with the given constraints, we need the generating function $g(x) = (1 + x + x^2 + \Lambda)^4$.

d. This is the same as the number of integer solutions to $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 = r$, $0 \leq e_1 = $ od $0 \leq e_2 = $ odd, $0 \leq e_i$, $i = 3,4,5,6,7$. To generate all formal products of the form $x^{e_1} x^{e_2} x^{e_3}$ with the given constraints, we need the generating function $(x + x^3 + x^5 + \Lambda)^2 (1 + x + x^2 + \Lambda)^5$.

6. This is the same as the number of integer solutions to $e_1 + e_2 + e_3 + e_4 + e_5 = r$, where $e_i \geq 0$. To generate formal products of the form $x^{e_1} x^{e_2} x^{e_3} x^{e_4} x^{e_5}$ with the given constraints, we need the generating function, $g(x) = (1 + x + x^2 + \Lambda)^5$.

8.a. This is like the number of ways to distribute 27 identical balls into four boxes which at most 27 ball in any one box. The balls represent votes, and since all votes for one candidate are identical, this mode will work. We could also look at this as the number of integer solutions to $e_1 + e_2 + e_3 + e_4 = r$ where $0 \leq e_i$, $i = 1,2,3,4$. The generating function required is $g(x) = (1 + x + x^2 + \Lambda)^4$ and we require the coefficient of x^{27}.

b. Since each candidate votes for him / herself, it follows that the boxes in our model have at least one ball in them. Considering our number of integer solutions model, we have $e_1 + e_2 + e_3 + e_4 = r$, where $e_i \geq 1$, $i = 1,2,3,4$. This time, our generating function is $g(x) = (x + x^2 + x^3 + \Lambda)^4$ and we still want the coefficient of x^{27}

c. If no candidate can receive a majority of the vote (i.e. 14 votes), then no box can have more than 13 balls in it. Our number of integer solutions model will now be, $e_1 + e_2 + e_3 + e_4 = r$, $0 \leq e_i \leq 13$, $i = 1,2,3,4$. The generating function will be $g(x) = (1 + x + x^2 + \Lambda + x^{13})^4$, and we still want the coefficient of x^{27}.

9. You can think of this as the number of ways of distributing r - identical objects into n - distinct boxes (where each element of the n - set represents a box) such that any box can have as many objects as it likes, or as the number of integer solutions to $e_1 + e_2 + \Lambda + e_n = r$, $0 \le e_i$, $i = 1,2,...,n$.
The required generating function is $g(x) = (1 + x + x^2 + \Lambda \ )^n$.
Aside
1: Let n = 3 and consider the 3 - set S = {1,2,3}. Any r - combination with repetition will be of the form, 1,1,...,1 , 2,2,...,2 , 3,3,...,3, where we have e_1 1's, e_2 2's and e_3 3's. (Since this is a combination, order doesn't matter). Suppose you have three distinct boxes. If you treat the 1's, 2's and 3's as identical objects, then the number of ways you can put these "identical" objects into three distinct boxes equals the number of r - combinations with repetition.

2: If repetition was not allowed, (i.e. each element of the n - set can be used at most once) then this is the same as the number of integer solutions to $e_1 + e_2 + \Lambda + e_n = r$, $0 \le e_i \le 1, i = 1,2,...,n$
and the required generating function is $g(x) = (1 + x)^n$.

14. a. This is like finding the number of integer solutions to $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = r$, where e_1, e_2, e_3 are even, and e_4, e_5, e_6 are odd. The generating function for this is
$g(x) = (x + x^3 + x^5)^3 (x^2 + x^4 + x^6)^3$.

   b. This is like finding the number of integer solutions to $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = r$, where $e_i \ne i$.
The generating function for this is
$$\sum_{i=1}^{6} \left[(x + x^2 + x^3 + x^4 + x^5 + x^6) - x^i \right] =$$
$(x^2 + x^3 + x^4 + x^5 + x^6)(x + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^5 + x^6)$
$(x + x^2 + x^3 + x^4 + x^6)(x + x^2 + x^3 + x^4 + x^5)$.

15. To generate all formal products of the form $x^{e_1} x^{e_2} x^{e_3} x^{e_4}$ with the given constraints, we need the generating function $(x^{-3} + x^{-2} + x^{-1} + x^0 + x^1 + x^2 + x^3)^4$.

16. This is like the number of ways to distribute r identical balls into 6 boxes with at most 9 balls in each box. For example, 3 balls in box 1, 2 balls in box 2 and no balls in the rest of the boxes represents the number 320000. Any number between 0 and 999999 can be represented using balls in boxes. We can look at this as the number of integer solutions to $e_1+e_2+e_3+e_4+e_5+e_6 = r$ with $0 \le e_i \le 9$. The generating function for this is
$g(x)=(1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9)^6$.

17. This is like the number of integer solutions to $e_1+e_2+e_3+e_4+e_5+e_6+e_7+e_8 = r$ where $e_i=5k$, $k=0,1,2,\ldots$. The generating function for this is $g(x) = (1+x^5+x^{10}+\ldots)^8$.

19. To model this problem, select r-integers such that no two are consecutive, i.e. $n_1,n_2,\ldots, n_r$. The n-r non-selected integers can be distributed into r+1 "boxes", before, between and after the chosen integers. Once they are put in the box, they can ordered from least to greatest. The first and last box can have none or some numbers in them. The boxes in the middle must have at least one number in them (if not then at least one of the two numbers n_1, $n_2,\ldots n_r$ must be consecutive). The generating function required is $(1+x+x^2+\ldots)(x+x^2+x^3+\ldots)^4(1+x+x^2+\ldots)$. For the case where n=20 we are interested in the coefficient of x^{15} {r=5, and we are distributing 15 numbers into 6 boxes}. For general n, we are interested in the coefficient of x^{n-5}.

Aside:
To show you how this works, let us consider the case where n=5 and r=2. There are six ways to select 2 nonconsecutive from the integers {1,2,3,4,5}, i.e. {1,3}, {1,4}, {1,5},{2,4},{2,5},and {3,5}. As we discussed above, this process is equivalent to the number of ways of putting 3 balls into 3 boxes with the middle box having at least one ball in it. There are six ways to do this, they are: (0,3,0), (0,2,1),(0,1,2), (1,1,1),(1,2,0) and (2,1,0) where the coordinates represent the number of balls in box 1, box2 and box3 respectively.

24.
a. This is the same as the number of integer solutions to $e_1+e_2+\ldots+e_{50} = r$, where $0 \le e_i \le 1$. The generating function required is $(1+x)^{50}$.
b. Similiar to part a except that this time, $0 \le e_i \le 3$. The generating function is $(1+x+x^2+x^3)^{50}$.

25. This is the same as the number of integer solutions to $e_1+e_2+e_3+e_4+e_5=r$ (for the chocolate bars) where $e_i \ge 0$ and $f_1+f_2+f_3+f_4+f_5 = s$ (for the lollipops) where $0 \le f_j \le 3$. To generate the formal products of the form

$x^{e_i f_j}$ where $e_i \ge 0$, $0 \le f_j \le 3$ we need the two variable generating function
$g(x,y) = (1+x+x^2+\ldots)(1+y+y^2+y^3)^5$.

Page 256

1. Find the coefficient of x^8 in the expansion of $(1 + x + x^2 + ...)^n$.

$1 + x + x^2 + \Lambda = \dfrac{1}{1-x}$, so $(1 + x + x^2 + \Lambda)^n = \left(\dfrac{1}{1-x}\right)^n = \dfrac{1}{(1-x)^n}$. The coefficient

of x^8 is $\dbinom{8 + n - 1}{8} = \dbinom{7 + n}{8}$.

2. $(x^5 + x^6 + x^7 + ...)^8 = \left(x^5(1 + x + x^2 + ...)\right)^8 = x^{40}(1 + x + x^2 + \Lambda)^8 = x^{40} \dfrac{1}{(1-x)^8} =$

$x^{40}\left(1 + \dbinom{1 + 8 - 1}{1}x + \Lambda \dbinom{r + 8 - 1}{r}x^r + ...\right)$. We need the coefficient of x^{r-40} in the expansion

of $\dfrac{1}{(1-x)^8}$. The coefficient is $\dbinom{r - 40 + 8 - 1}{r - 40}$.

3. $\left(1 + x^2 + x^4\right)(1 + x)^m$. From the binomial theorem,

$(1 + x)^m = \left(1 + \dbinom{m}{1}x + \dbinom{m}{2}x^2 + \Lambda + \dbinom{m}{m}x^m\right)$. Let $f(x) = 1 + x^2 + x^4$ and $g(x) = (1 + x)^m$.

The coefficient of x^9 in $f(x)g(x) = 1\dbinom{m}{9} + 1\dbinom{m}{7} + 1\dbinom{m}{5}$.

5. $\left(x^2 + x^3 + x^4 + x^5 + x^6 + x^7\right)^4 = \left(x^2(1 + x + x^2 + x^3 + x^4 + x^5)\right)^4 =$

$x^8(1 + x + x^2 + x^3 + x^4 + x^5)^4 = x^8 \dfrac{(1 - x^6)^4}{(1-x)^4} = x^8(1 - x^6)^4 \dfrac{1}{(1-x)^4} =$

$x^8(1 - 4x^6 + 6x^{12} - 4x^{18} + x^{24})\left(1 + \dbinom{1 + 4 - 1}{1}x + \Lambda + \dbinom{r + 4 - 1}{r}x^r + \Lambda \right) =$

$(x^8 - 4x^{14} + 6x^{20} - 4x^{26} + x^{32})\left(1 + \dbinom{1 + 4 - 1}{1}x + \Lambda + \dbinom{r + 4 - 1}{r}x^r + \Lambda \right).$

The coefficient of x^{18}

$= 1\dbinom{10 + 4 - 1}{10} - 4\dbinom{4 + 4 - 1}{4} = \dbinom{13}{10} - 4\dbinom{7}{4}.$

7

b. $\dfrac{x^2 - 3x}{(1-x)^4} = (x^2 - 3x)\dfrac{1}{(1-x)^4} = (x^2 - 3x)\left(1 + \dbinom{1+4-1}{1}x + \Lambda + \dbinom{r+4-1}{r}x^r + \Lambda\right)$

The coefficient of x^{12} is $\dbinom{10+4-1}{10} - 3\dbinom{11+4-1}{11} = \dbinom{13}{10} - 3\dbinom{14}{11}$.

c. $(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

$\dfrac{1}{(1-x)^5} = 1 + \dbinom{1+5-1}{1}x + \dbinom{2+5-1}{2}x^2 + \Lambda + \dbinom{r+5-1}{r}x^r + \Lambda$

$\dfrac{(1+x)^5}{(1-x)^5} = (1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)\left(1 + \dbinom{1+5-1}{1}x + \Lambda + \dbinom{r+5-1}{r}x^r + \Lambda\right)$.

The coefficient of x^{12} is,

$1\dbinom{12+5-1}{12} + 5\dbinom{11+5-1}{11} + 10\dbinom{10+5-1}{10} + 10\dbinom{9+5-1}{9} + 5\dbinom{8+5-1}{8} + 1\dbinom{7+5-1}{7}$.

9. $\left(x^2 + x^3 + x^4 + x^5\right)^5 = \left(x^2(1 + x + x^2 + x^3)\right)^5 = x^{10}(1 + x + x^2 + x^3)^5$. The coefficient

of x^9 is 0 because the smallest power of x would be 10.

10 For $(1 + x^3 + x^6 + x^9 + \Lambda)^7$, let $u = x^3$. Now $(1 + u + u^2 + \Lambda)^7 = \dfrac{1}{(1-u)^7} = $

$\left(1 + \dbinom{1+7-1}{1}u + \dbinom{2+7-1}{2}u^2 + \Lambda + \dbinom{r+7-1}{r}u^r + \Lambda\right) = 1 + \dbinom{1+7-1}{1}x^3 + \Lambda + \dbinom{r+7-1}{r}x^{3r} + \Lambda$

coefficient of x^{18} is when r = 6, i.e. $\dbinom{6+7-1}{6} = \dbinom{12}{6}$.

11.

b. $\dfrac{1}{1+x} = (1+x)^{-1} = \dfrac{1}{1+(-x)} = 1 + (-x) + (-x)^2 + \Lambda + (-x)^r + \Lambda$. Coefficient of x^{12} is

$(-1)^{12} = 1$.

d. $(1-4x)^{-5} = \dfrac{1}{(1-4x)^5} = 1 + \dbinom{1+5-1}{1}4x + \dbinom{2+5-1}{2}(4x)^2 + \Lambda + \dbinom{r+5-1}{r}(4x)^r + \Lambda$ =

$1 + \dbinom{1+5-1}{1}4x + \dbinom{2+5-1}{1}16x^2 + \Lambda + \dbinom{r+5-1}{r}4^r x^r + \Lambda$. The coefficient of x^{12} is

$\dbinom{12+5-1}{12}4^{12}$.

13

a. The generating function which models this problem is

$g(x) = (x^2 + x^3 + x^4 + \Lambda + x^{10} + \Lambda)^3 = \left(x^2(1 + x + x^2 + \Lambda)\right)^3 = x^6(1 + x + x^2 + \Lambda)^3 = \dfrac{x^6}{(1-x)^3}$.

We want the coefficient of x^{10} .

$\dfrac{x^6}{(1-x)^3} = x^6\,\dfrac{1}{(1-x)^3} = x^6\left(1 + \dbinom{1+3-1}{1}x + \Lambda + \dbinom{r+3-1}{r}x^r + \Lambda\right)$ so the coefficient of $x^{10} =$

$\dbinom{4+3-1}{4} = \dbinom{6}{4}$.

b. The generating function which models this problem is

$g(x) = (1 + x + x^2)(1 + x + x^2 + \Lambda)^2 = (1 + x + x^2)\left(\dfrac{1}{1-x}\right)^2 = (1 + x + x^2)\dfrac{1}{(1-x)^2} =$

$(1 + x + x^2)\left(1 + \dbinom{1+2-1}{1}x + \Lambda + \dbinom{r+2-1}{r}x^r + \Lambda\right)$. We want the coefficient of x^{10} .

The coefficient of $x^{10} = \dbinom{10+2-1}{10} + \dbinom{9+2-1}{9} + \dbinom{8+2-1}{8} = \dbinom{11}{10} + \dbinom{10}{9} + \dbinom{9}{8}$.

c. The generating function which models this problem is

$$g(x) = (1 + x + x^2 + \Lambda\;)^2(1 + x^2 + x^4 + \Lambda\;) = (1 + x^2 + x^4 + \Lambda\;)\frac{1}{(1-x)^2} =$$

$$(1 + x^2 + x^4 + \Lambda\;)\left(1 + \binom{1+2-1}{1}x + \Lambda + \binom{r+2-1}{r}x^r + \Lambda\;\right). \text{ We want the coefficient of } x^{10}. \text{ This}$$

coefficient is $\binom{10+2-1}{10} + \binom{8+2-1}{8} + \binom{6+2-1}{6} + \binom{4+2-1}{4} + \binom{2+2-1}{2} + 1 \quad =$

$$\binom{11}{10} + \binom{9}{8} + \binom{7}{6} + \binom{5}{4} + \binom{3}{2} + 1.$$

18. This is like the number of integer solutions to

$e_1 + e_2 + e_3 + \Lambda + e_{10} = r$ where $1 \le e_i \le 6$ for $i = 1, 2, \ldots, 10$. The generating function

required is $g(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^{10}$. We want the coefficient of x^{25}.

$$g(x) = (x(1 + x + x^2 + x^3 + x^4 + x^5))^{10} = x^{10}\left(\frac{1-x^6}{1-x}\right)^{10} = x^{10}(1-x^6)^{10}\frac{1}{(1-x)^{10}} =$$

$$\sum_{j=0}^{10}\binom{10}{j}(-1)^j x^{6j+10}\left(1 + \binom{1+10-1}{1}x + \Lambda + \binom{r+10-1}{r}x^r + \Lambda\;\right). \text{ The coefficient of } x^{25} =$$

$$\binom{10}{0}(-1)^0\binom{15+10-1}{15} + \binom{10}{1}(-1)^1\binom{9+10-1}{9} + \binom{10}{2}(-1)^2\binom{3+10-1}{3} = \binom{24}{15} - 10\binom{18}{9} + 45\binom{12}{3}.$$

21. The generating function which models this problem is,

$$g(x) = (x^2 + x^3 + x^4)(x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + \Lambda + x^9).$$

Once you give two books of type I to each teacher, there are two books of type I left.
So each teacher can only end up with at most 4 books of type I. Once you give two
books of type II to each teacher, there are three books of type II left. So each teacher
can only end up with at most five books of type II. Once you give two books of type III
to each teacher, there are seven books of type III left. So each teacher can only end up
with at most nine books of type III. We want the coefficient of x^{12}.

$$g(x) = x^6\left(\frac{1-x^3}{1-x}\right)\left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^8}{1-x}\right) = x^6(1-x^3)(1-x^4)(1-x^8)\frac{1}{(1-x)^3} =$$

$$x^6(1-x^{15}+x^{12}+x^{11}-x^8+x^7-x^4-x^3)\left(1+\binom{1+3-1}{1}x+\Lambda+\binom{r+3-1}{r}x^r+\Lambda\right).$$ The

coefficient of x^{12} is $\binom{6+3-1}{6}-\binom{3+3-1}{3}-\binom{2+3-1}{3}=\binom{8}{6}-\binom{5}{3}-\binom{4}{2}.$

24. Lay out the 8T's, and put boxes before, after, and in between the T's. There
are 9 boxes in all. We distribute the H's into these boxes. Since we don't want a run
of seven or more heads, each box can have no more than 6 H's. The generating
function for this is $g(x)=(1+x+x^2+x^3+x^4+x^5+x^6)^9$. We want the coefficient of

x^{17}. $g(x)=\left(\dfrac{1-x^7}{1-x}\right)^9=\dfrac{(1-x^7)^9}{(1-x)^9}=\sum_{i=0}^{9}\binom{9}{i}(-1)^i x^{7i}\left(1+\binom{1+9-1}{1}x+\Lambda+\binom{r+9-1}{r}x^r+\Lambda\right).$

The coefficient of x^{17} is $\binom{17+9-1}{17}-\binom{10+9-1}{10}\binom{9}{1}+\binom{14+9-1}{14}\binom{9}{2}=$

$\binom{25}{17}-\binom{9}{1}\binom{18}{10}+\binom{9}{2}\binom{22}{14}$. The number of ways to flip a coin 25 times with 8T's and 17H's

is $\dfrac{25!}{17!\,8!}=\binom{25}{8}$. Therefore, the probability is $\dfrac{\binom{25}{17}-\binom{9}{1}\binom{18}{10}+\binom{9}{2}\binom{22}{14}}{\binom{25}{8}}$

26. If a die is rolled 2 times, then the generating function which represents all the possible outcomes is $g_2(x) = (x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6)$.

First roll   Second roll
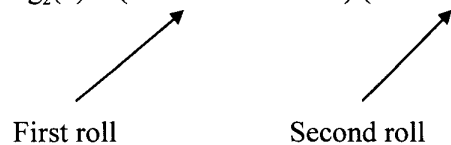
We can generalize this to n rolls with the generating function $g_n(x)=(x+x^2+x^3+x^4+x^5+x^6)^n$. It is not hard to see that $g_0(x)=1$ and $g_1(x)= x+x^2+x^3+x^4+x^5+x^6$. Since our die can be rolled any number of times, we have to add up all these generating functions, i.e.
$1+ (x+x^2+x^3+x^4+x^5+x^6)+ (x+x^2+x^3+x^4+x^5+x^6)^2+\ldots+(x+x^2+x^3+x^4+x^5+x^6)^n+\ldots$ .If we let $u=(x+x^2+x^3+x^4+x^5+x^6)$, we have $g(x)= 1+u+u^2+\ldots = 1/(1-u) = 1/(1-x-x^2-x^3-x^4-x^5-x^6)= (1-x-x^2-x^3-x^4-x^5-x^6)^{-1}$.

19.
The key here is that any combination of 15 boxes will yield 300 chocolates. If we look at this way, then this is equivalent to the number of integer solutions to

$e_1 + e_2 + \Lambda + e_7 = r$ where $1 \le e_i \le 5$. The generating function which models this is

$g(x) = (x + x^2 + \Lambda + x^5)^7$, we want the coefficient of x^{15}.

$$g(x) = x^7(1 + x + x^2 + x^3 + x^4)^7 = x^7 \frac{(1-x^5)^7}{(1-x)^7} = x(1-x^5)^7 \frac{1}{(1-x)^7} =$$

$x^7(1 - 7x^5 + 21x^{10} - 35x^{15} + 35x^{20} - 21x^{25} + 7x^{30} - x^{35})\sum_{j=0}^{\infty}\binom{j+7-1}{j}x^j$. The coefficient of

x^{15} is $\binom{8+7-1}{8} - 7\binom{3+7-1}{3}$.

23.

Collect \$24 from 4 children and 6 adults. Each child gives at least one dollar and atmost four dollars. Therefore we need $(x + x^2 + x^3 + x^4)^4$ as part of our generating function. Each adult gives at least one dollar and atmost seven dollars. Therefore we need $(x + x^2 +\Lambda +x^7)^6$ as part of our generating function. The generating function which models this is

$g(x) = (x + x^2 + x^3 + x^4)^4 (x + x^2 +\Lambda +x^7)^6 = x^4(1 + x + x^2 + x^3)^4 x^6(1 + x + x^2 +\Lambda +x^6)^6 =$

$x^{10} \dfrac{(1-x^4)^4}{(1-x)^4} \dfrac{(1-x^7)^6}{(1-x)^6} = x^{10}(1-x^4)^4(1-x^7)^6 \dfrac{1}{(1-x)^{10}}$. We need the coefficient of x^{14} in

$(1-x^4)^4(1-x^7)^6 \displaystyle\sum_{j=0}^{\infty} \binom{j+10-1}{j} x^j$.

$(1-x^4)^4(1-x^7)^6 \displaystyle\sum_{j=0}^{\infty} \binom{j+10-1}{j} x^j =$

$(1-4x^4 + 6x^8 - 4x^{12} + x^{16})(1-6x^7 +15x^{14} - 20x^{21} +15x^{28} - 6x^{35} + x^{42})\displaystyle\sum_{j=0}^{\infty} \binom{j+10-1}{j} x^j$.

The coefficient of $x^{14} =$

$\binom{14+10-1}{14} - 6\binom{7+10-1}{7} + \binom{6}{2} - 4\binom{10+10-1}{10} + 24\binom{3+10-1}{3} + 6\binom{6+10-1}{6} - 4\binom{2+10-1}{2}$.

27.

$\binom{m+n}{r}$ is the coefficient of x^r in the expansion of $(1+x)^{m+n}$. Suppose $m \le n$.

$(1+x)^{m+n} = (1+x)^m (1+x)^n = \displaystyle\sum_{i=0}^{m} \binom{m}{i} x^i \sum_{j=0}^{n} \binom{n}{j} x^j$. The coefficient of x^r is,

$\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} +\Lambda + \binom{m}{k}\binom{n}{r-k} +\Lambda + \binom{m}{m}\binom{n}{r-m}$. Hence we conclude that

$\binom{m+n}{r} = \binom{n}{r} + \binom{m}{1}\binom{n}{r-1} +\Lambda + \binom{m}{k}\binom{n}{r-k} +\Lambda + \binom{m}{m}\binom{n}{r-m}$.

28.

$$\frac{(1-x^2)^n}{(1-x)^n} = (1+x)^n.$$

LHS $= (1-x^2)^n \dfrac{1}{(1-x)^n} = \displaystyle\sum_{i=0}^{n}(-1)^i\binom{n}{i}x^{2i}\sum_{j=0}^{\infty}\binom{j+n-1}{j}x^j$. The general term of this

expansion is $(-1)^i\binom{n}{i}\binom{j+n-1}{j}x^{j+2i}$. For the coefficient of x^m, we need i and j such that

$m = j + 2i$. For m even, the coefficient of x^m is the sum over all pairs of (i,j) such that

$i + j = m$, i.e. $\displaystyle\sum_{k=0}^{\frac{m}{2}}(-1)^k\binom{n}{k}\binom{n+m-2i-1}{n-1}$. The coefficient of x^m in the RHS $= \binom{n}{m}$.

$\binom{n+m-2i-1}{m-2i} = \binom{n+m-2i-1}{n+m-2i-1-m+2i} = \binom{n+m-2i-1}{n-1}$. Therefore we can conclude

$$\sum_{k=0}^{\frac{m}{2}}(-1)^k\binom{n}{k}\binom{n+m-2i-1}{n-1} = \binom{n}{m}.$$

29.
a.

Evaluate $\displaystyle\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r+k}$ where r is fixed. If we write $\binom{m}{k} = \binom{m}{m-k}$, by using the symmetry

property of binomial coefficients, then $\displaystyle\sum_{k=0}^{m}\binom{m}{k}\binom{n}{r+k} = \sum_{k=0}^{m}\binom{n}{r+k}\binom{m}{m-k}$. Combinatorially,

this is $\binom{m+n}{r+m}$, because if we want to select $r + m$ objects out of $n + m$ objects, we can split the

$n + m$ object into two sets of m and n objects. We can select $r + k$ objects out of the set with
n objects and $m - k$ objects out of the set with m objects . Summing up all the possible cases
gives our desired result.

c. $\displaystyle\sum_{k=0}^{n}2^k\binom{n}{k} = $ the binomial expansion of $(1+2)^n = 3^n$.

33. $g(x) = a_0 + a_1 x + a_2 x^2 + ... + a_r x^r + ... + a_n x^n$.

We need the following result about derivatives, $\dfrac{d^n}{dx^n}(x^k) = \begin{cases} k! & \text{if } n = k \\ 0 & \text{if } n > k \\ (k-1)(k-2)...(k-n+1)x^{k-n} & \text{if } n < k \end{cases}$

Hence $\dfrac{d^r}{dx^r}[g(x)] = a_r r! + a_{r+1} \dfrac{d^r}{dx^r} x^{r+1} + \Lambda$. The terms with powers larger than r

will have positive powers of x left. Hence

$\dfrac{d^r}{dx^r}[g(x)]$ at $x = 0$, $= a_r r! + 0 + 0 + \Lambda$. Hence $\dfrac{1}{r!}\dfrac{d^r}{dx^r}[g(x)] = \dfrac{a_r r!}{r!} = a_r$.

39.

How can we get a sum of r and count the number of ways to get it?

Let i be the out put of the first die where i= 1,2,3,4,5,6. Once we know this, the number

of ways to get a sum of r is the number of integer solutions to $e_1 + e_2 + ... + e_i = r$ where

$1 \le e_1 \le 6$, $1 \le e_2 \le 6$,..., $1 \le e_i \le 6$. The generating function for this is $(x + x^2 + x^3 + x^4 + x^5 + x^6)^i$.

Since i can go from 1 to 6, the generating function we want is the sum of the generating

functions, $(x + x^2 + x^3 + x^4 + x^5 + x^6) + (x + x^2 + x^3 + x^4 + x^5 + x^6)^2 + ... +$

$(x + x^2 + x^3 + x^4 + x^5 + x^6)^6 = f(f(x))$ where $f(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)$.

40.

$P_x(t) = \displaystyle\sum_{i=0}^{\infty} a_i t^i$ where a_i = probability that it takes i minutes to save a customer.

$P_y(t) = \displaystyle\sum_{j=0}^{\infty} b_j t^j$ where b_j = probability that j people line up to be served each minute.

$P_z(t) = \displaystyle\sum_{k=0}^{\infty} c_k t^k$ where c_k = probability that k people line up to be served.

$c_k = a_i b_j$ where ij = k.

$b_i b_j = P(Y = i \text{ and } Y = j) = 0$ unless i = j. $b_i^2 = b_i$ because $P(Y = i \text{ and } Y = i) = P(y = i)$.

A discrete random variable can't have values i and j at the same time.

$$\left(b_0 + b_1t + b_2t^2 + \Lambda\right)^2 = b_1{}^2t^2 + b_2{}^2t^4 + \Lambda = b_1t^2 + b_2t^4 + \Lambda$$

$$\left(b_0 + b_1t + b_2t^2 + \Lambda\right)^3 = b_1{}^3t^3 + b_2{}^3t^6 + \Lambda = b_1t^3 + b_2t^6 + \Lambda$$

So $P_z(t) = \sum_{k=0}^{\infty} c_k t^k = \sum_{k=0}^{\infty}\sum_{\substack{i,j \\ ij=k}} a_i b_j t^k = \sum_{i=0}^{\infty} a_i(b_0 + b_1t + b_2t^2 + \Lambda)^i = P_x(P_y(t)).$

$a_1(b_1t + b_2t^2 + \Lambda) + a_2(b_1t^2 + b_2t^4 + \Lambda) + a_3(b_1t^3 + b_2t^6 + \Lambda) + \Lambda \ a_i(b_1t^i + b_2t^{2i} + b_3t^{3i} + \ldots + \Lambda$