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Subsets of a Set [n]

1) How many k-subsets of [n] are there?

$k \geq 0$, integer

$n = 4$	12	13	14
$k = 2$	34	23	24

let x be the # of k-subsets
Each such subset can be arranged in k! ways.
Thus, $x \cdot k!$ counts the number of ordered k-subsets of [n], which is just $n^{\underline{k}}$

$$\therefore \; x \cdot k! = n^{\underline{k}}$$

$$\Rightarrow x = \frac{n^{\underline{k}}}{k!} \equiv \binom{n}{k}$$

What is $\binom{n}{0}$? $\binom{0}{0}$? $\binom{3}{4}$?

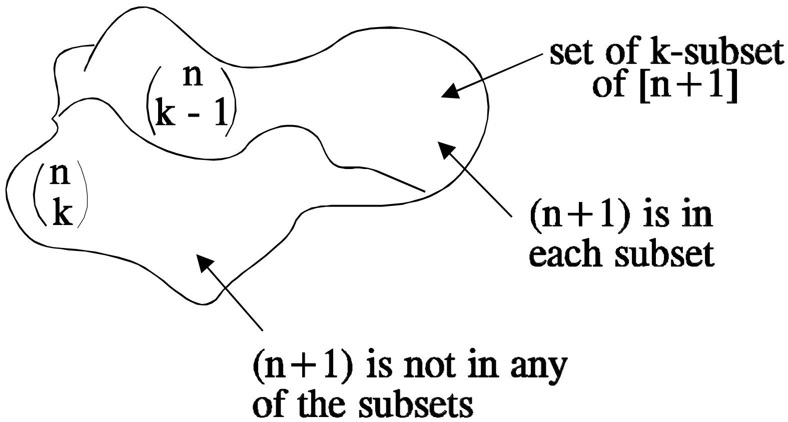
Notice: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

(This is called the triangle formula for binomial coefficients.)
Fix your eye on the element. (n + 1):
(n + 1) is in or out of any subset

$\binom{n}{k}$ counts all k-subsets where (n + 1) is <u>OUT</u> (because these are just k-subsets of [n]).

$\binom{n}{k-1}$ counts all k-subsets where (n + 1) is in.

By the SUM rule, this counts all k-subsets of [n + 1].



"Algebraic" Proof of the above identity:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$=$$

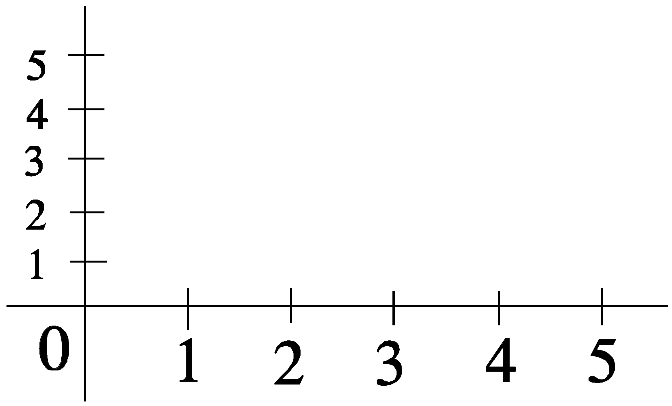
$$= \frac{n!}{(k-1)!(n-k)!} \left[\frac{n+1}{k(n-k+1)}\right]$$

$$= \frac{(n+1)!}{k!(n-k+1)!} \equiv \binom{n+1}{k}$$

Note: $\binom{n}{k} = \binom{n}{n-k}$

Each choice of a k-subset leaves behind an (n - k) subset.

Graphs of Binomial Coefficients

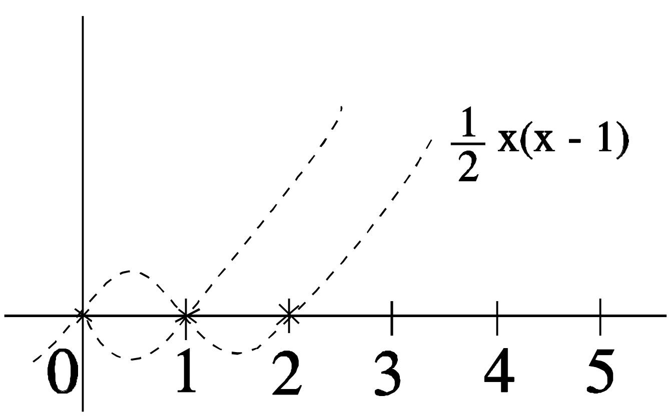


$f_2(n) = \binom{n}{2}$

$f_3(n) = \binom{n}{3}$

$g_2(k) = \binom{2}{k}$

$g_3(k) = \binom{3}{k}$

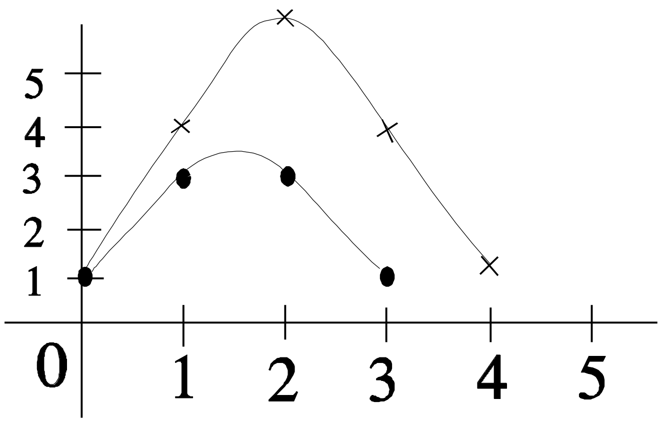


$f_2(n) = \binom{n}{2} = \frac{n(n-1)}{2}$

$f_3(n) = \binom{n}{3} = \frac{1}{6}\, n(n-1)(n-2)$

$g_3(r) = \binom{3}{r}$

$g_6(r) = \binom{6}{r}$



Unimodal:up/down
Single or Double maximum

Array of Binomial Coefficients

$\binom{n}{0} = 1 \qquad n \geq 0$

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

$56 \cdot 36 \cdot 210 = 28 \cdot 120 \cdot 126 = 42360$ (Hexagon Property)

$35 \cdot 6 \cdot 10 = 20 \cdot 21 \cdot 5$

(i) $\sum_{j=1}^{n} j = \binom{n+1}{2}$

(1,2) (1,3) (1,4), ...(1, n + 1) n
(2,3) (2,4) .... (2, n + 1) n - 1
(3,4) .... (3, n + 1) n - 2

(ii) $\sum_{0 \leq i \leq n} \binom{i}{k} = \binom{n+1}{k+1}$

Note $\sum_{0 \leq i \leq n} \binom{i}{k} \equiv \sum_{i=k}^{n} \binom{i}{k}$

$$= \binom{k}{k} + \binom{k+1}{k} + \_ + \binom{k+(n-k)}{k}$$

"Combinatorial argument" goes like this:
RHS counts all (k + 1) - subsets of [n + 1].

Suppose (n + 1) is <u>in</u> such a subset. Remaining elements chosen in $\binom{n}{k}$ ways.

Suppose (n + 1) <u>not in</u>; now suppose n <u>is</u> <u>in</u>. Remaining elements chosen in $\binom{n-1}{k}$ ways.

And so on. Use SUM RULE since these are "or" possibilities. This counts all ways to get (k + 1) - subset, and is just LHS.

<u>Exercise</u>: Prove using triangle formula for binomial coefficients.

(iii) $\binom{n}{k} = \binom{n}{k}\binom{n-1}{k-1}$, $k \_ 0$

This is called the absorption identity

A more general identify: $k\binom{n}{k} = n\binom{n-1}{k-1}$

(Holds for k = 0)

Exercise: Show that $(n-k)\binom{n}{k} = n\binom{n-1}{k}$

(Hint: multiply both sides by (n - k), simplify right hand side)

(iv) $$\sum_{k \le n}\binom{m+k}{k} = \binom{n+m+1}{n}$$

$$= (-1)^k \frac{n(n+1)\ \_\ (n+k-1)}{k!}$$

$$= (-1)^k \binom{n+k-1}{k} \equiv (-1)^k \frac{n^{\overline{k}}}{k!}$$

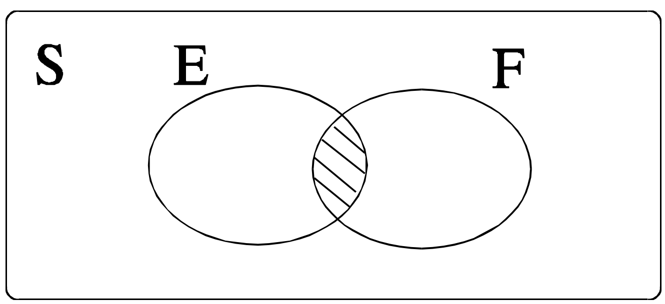
(vi) Eq is missing, should be sum (0 to n) of k to the m lower equals (n+1) to the (m+1) lower all divided by (m+1)

$$\left(\text{Looks a little like } \int x^m \, dx = \frac{x^{m+1}}{m+1}\right)$$

Probabilistic Notions

Sample space: set of possible outcomes
Event - subset of the set of outcomes (subset of sample space)

Prob (Event) = “Size of Event”/”Size of Sample Space”



DISCRETE CASE
"Size of Sample Space" = total no. of possible outcomes
"Size of Event" = outcomes corresponding to event.
Example: Toss a fair coin 5 times. What is prob. of precisely 2 heads?
Solution: Sample space is all 5-sequences of H, T. Those with exactly 2 H, 3 T constitute the event we seek.

Sample space has $2^5 = 32$ 5-sequences.

Event has $\binom{5}{2}$ 5-sequences (just choose the 2 places for the H).

Prob = $\frac{10}{32} = \frac{5}{16}$

<u>NOTE</u>: All 5-sequences are equi-probable.

Example: Choose 2 numbers from {0,1,2, … , 9}
(repetition allowed). Find prob that sum = 10.
Solution: There $10 \times 10 = 100$ 2-tuples. Of these, precisely 9 have the required property

{(1,9), (2,8), … , (5,5),(6,4), … , (9,1)} so 9/100.

Prob that E does not occur = 1 - P(E) = $P(E^c)$

NOTE: $S = E \cup E^c$, $E \cap E^c = \emptyset$.
$1 = P(S) = P(E) + P(E^c)$.
Prob that E <u>or</u> F occurs is $P(E \cup F)$
Prob that E <u>and</u> F occurs is $P(E \cap F)$
If $E \cap F = \emptyset$, $P(E \cup F) = P(E) + P(F)$
In general, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

<u>Distribution and Occupancy Problems (“Balls in Boxes”)</u>

General idea is to count the number of ways to place r balls into n boxes.

The catch is that the balls and boxes may be <u>distinct</u> (distinguishable) or <u>nondistinct</u> (you can’t tell them part). Further, in each case, there are 3 possible restrictions on the number of balls in each box:
(i) as many balls as you like (including none)
(ii) no more than 1 ball in each box
(iii) no box can be empty (that is, at least 1 ball in each box)

a) # of ways to place r distinct balls
in n distinct boxes:

(i) as many balls as you like in each
(ii) no more than 1 ball in each
(iii) no box can be empty
(≡ at least 1 ball in each)

(i) $\underbrace{n \times n \times n \ldots \times n}_{r \text{ factors}} = n^r$

(ii) n (n - 1) … (n - r + 1) =
(iii) We'll do this later!

b) # of ways to place r nondistinct ball in n distinct boxes:
(i) Since the balls are nondistinct, while the boxes are distinct, all that matters is the number of balls in each distinct box.

Suppose the distinct boxes are numbered 1,2, … , n. Associate with each distribution of the r balls in the n boxes an r-tuple of the numbers of the boxes in which each ball is placed. For example, if r = 4 and n = 3, and 2 balls are in box 3, and 1 ball in each of boxes 1 and 2, then thr 4-tuple would be 1,2,3,3.

Thus, our problem is equivalent to counting the number of r-tuples which can be made from [n], where we allow the same element of [n] to occur as many times as we like (that is, we allow repetition of elements) and where the order of the elements of the r-tuple doesn't matter.

Here is the key idea. Since order doesn't matter, let's arrange the elements of the r-tuple in ascending order. Let these elements be

$a_1 \le a_2 \le \ldots \le a_r$ (r tuple on [n], repetition allowed)
$\leftrightarrow a_1 < a_2 + 1 < a_3 + 2 < \ldots a_r + (r - 1)$
(r tuple on [n + r - 1], no repetition
↑
# of choices of latter is $\binom{n + r - 1}{r} = \frac{n^{\bar{r}}}{r!}$

Since this correspondence between increasing r-tuples and strictly increasing r-tuples is 1:1, this solves the original problem.

Example: Choose a dozen bagels of different types: onion, garlic, regular. How many ways?

$$n = 3, \quad r = 12 \quad \binom{3 + 12 - 1}{12} = \binom{14}{12} = 91$$

(ii) $\binom{n}{r}$

(iii) Put 1 in each box. Then distribute (r - n) balls left. Since the balls are nondistinct, use the formula from part (i).

$$\binom{n + (r - n) - 1}{r - n} = \binom{r - 1}{r - n}$$