GENERATING FUNCTIONS

Solve an infinite number of related problems in one swoop.
*Code the problems, manipulate the code, then decode the answer!
Really an algebraic concept but can be extended to analytic basis for interesting results.

(i) Ordinary Generating Functions

$\{a_0, a_1, \ldots, a_k, \ldots\}$ sequence where the kth term is the solution of some problem, for every k.

Create the **object** ("formal power series")

$\sum_{k=0}^{\infty} a_k \ x^k$ where x^k is like a place-holder for a_k. This looks like an analytic power series but it's NOT (not yet, anyway).

Rules of Operation: just do what comes naturally.

$$\sum a_k \ x^k \pm \sum b_k \ x^k = \sum (a_k \pm b_k) \ x^k$$

$$\left(\sum a_k \ x^k\right) \left(\sum b_k \ x^k\right) = \sum c_k \ x^k$$

where $c_k = \sum_{j=0}^{k} a_j \ b_{k-j}$

Examples

(I) $a_k = 1 \,, 0 \le k \le n$
$a_k = 0 \quad k > n$

$$\sum_{k=0}^{\infty} a_k \ x^k = 1 + x + \_ + x^n = \frac{1 - x^{n+1}}{1 - x}$$

why is last equality true? Because ....

$$(1 + x + \ldots + x^n)(1 - x) = 1 + x + \ldots + x^n$$
$$= \frac{- x - \_ - x^n - x^{n+1}}{1 - x^{n+1}}$$

(ii) $a_k = 1 \quad \forall k.$

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$$

$$\left((1 - x)\sum_{k=0}^{\infty} x^k = \sum_{k\geq 0} x^k \sum_{k\geq 0} x^{k+1} = \sum_{k\geq 0} x^k - \sum_{k\geq 1} x^k = 1 \right)$$

NOTE: You don't need anything about convergence! At the same time, you shouldn't think of "x" as a variable into which you substitute values (not yet, anyway) but soon it will be OK).

(iii) If we have the o.g.f., we can find the sequence: e.g. Suppose the o.g.f. is $(1 + x)^n$ then the sequence is found as follows:

$$(1 + x)^n = \sum_{k\geq 0} \binom{n}{k} x^k$$

Thus, $a_k = \binom{n}{k}$ (note that $a_k = 0$ for $k > n$).

Exponential Generating Function

$$\{a_0, a_1, \dots , a_k, \dots \} \leftarrow \sum_{k\geq 0} \frac{a_k}{k!} x^k.$$

$$\left(\sum_k \frac{a_k}{k!} x^k\right)\left(\sum_k \frac{b_k}{k!} x^k\right) = \sum_k \frac{c_k}{k!} x^k$$

, where

$$c_k = \sum_{j=0}^{k} \binom{j}{k} a_j \, b'_{k-j} \qquad\qquad c'_k = \sum_{j=0}^{k} a'_j \, b'_{k-j}$$

$$\frac{c_k}{k!} = \sum_{j=0}^{k} \frac{a_j}{j!} \frac{b_k}{(k\bullet j)!}$$

Examples

(i) $a_k = 1,\ 0 \leq k \leq n$; $a_k = 0$ for $k > n$

$$\sum_{k=0}^{n} \frac{x^k}{k!}$$

(ii) $a_k = 1 \qquad \forall k$

$$\sum_{k\geq 0} \frac{x^k}{k!} = e^x$$

(iii) $a_k = n^{\underline{k}}$ ($\equiv$ number of k-perms of an n-set)

(iv) If we have the e.g.f. sin x then

$$\sin x = \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

so $a_{2k+1} = (-1)^k \quad k \geq 0$
$a_{2k} = 0 \qquad k \geq 0$

<u>Some Generating Function Manipulations</u>

Suppose $A(x) = \sum_k a_k x^k$, $B(x) = \sum_k x^k$

- Then $A(x) B)(x) = \sum_k d_k x^k$ where

$$c_k = \sum_{j=0}^{k} a_j \bullet 1 = \sum_{j=0}^{k} a_j$$

- Similarly, $A^2(x) = \sum_k d_k x^k$ where

$$d_k = \sum_{j=0}^{k} a_j a_{k-j}$$

e.g. $a_k = \binom{n}{k}$ then $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

- Also, $\frac{1}{x}[A(x) - a_0] = \sum_{k\geq 0} a_{k+1} x^k$, which is

the o.g.f. for the sequence $\{a_1, a_2, \ldots, a_k, \ldots\}$ which is the original sequence shifted one place to the left (and the first term dropped off).

By contrast, $xA(x) = \sum_{k\geq 0} a_k x^{k+1}$ which is

$0 + \sum_{k\geq 0} a_{k-1} x^k$ or the o.g.f. for the sequence

$\{0, a_0, a_1, \ldots, a_k, \ldots\}$

$\downarrow$

$(k + 1)^{th}$ place.

$$\frac{d}{dx} \sum_{k\geq 0} a_k x^k \equiv \sum_{k\geq 1} ka_k x^{k-1}$$

which is the o.g.f. for $\{a_1, 2a_2, 3a_3, \ldots, ka_k, \ldots$

All of those ideas carry over to e.g.f. in an analogous way.

Applications to Counting Problems

Here the coefficients of the powers of x can provide the answers we seek, e.g.

1. Ordinary Generating Functions

How can we enumerate all possible selections from 3 distinct objects a,b,c?

$1 + ax \equiv$ a is chosen (ax) or not (1)
$1 + bx \equiv$ b is chosen (bx) or not (1)
$1 + cx \equiv$ c is chosen (cx) or not (1)

By product rule,

$(1 + ax)(1 + bx)(1 + cx) \equiv$ all possible selections

$1 + (a + b + c)x + (ab + bc + ac)x^2 + (abc)x^3$

$\downarrow$ $\downarrow$ $\downarrow$

one object, either a,b,c 2 objects 3 objects

Suppose we only wish the number of selections of 1,2,3 of the objects. Then we can count this number by weighting each selection with weight 1. This is equivalent to setting

$a = 1 = b = c$.

$$1 + 3x + 3x^2 + x^3 = (1 + x)^3$$

How many ways to select r balls from 2 green, 3 gold, 4 blue, 8 red?

Here the order of the selection of the balls doesn't matter. You can select 0, 1, 2 or 3 green balls, so $1 + x + x^2 + x^3$ enumerates these choices. But this is the same for the gold balls. For the blue it is $1 + x + x^2 + x^3 + x^4$, for the red
$1 + x + \ldots + x^8$. Hence solution is <u>coefficient of x^r in</u>

$$(1 + x + x^2 + x^3)^2 (1 + x + x^2 + x^3 + x^4) (1 + x + \ldots + x^8)$$

In general we have:

Suppose we have p types of objects, with n_i indistinguishable objects of type i, $i = 1, 2, \ldots, p$. The number of ways to pick k objects if we can pick any number of objects of each type is given by the coefficient of x^k in

$$(1+x+\_+x^{n_1})(1+x+\_+x^{n_2})\_\,(1+x+\_+x^{n_p})$$

Suppose the number of each type of object is "infinite" (think of this as solutions with repetition allowed). Then the above formula becomes

$$(1 + x + \ldots + x^k + \ldots)^p = \left(\frac{1}{1-x}\right)^p = (1-x)^{-p}$$

The coefficient of x^k in this is $\binom{-p}{k}(-1)^k$ or $\binom{p+k-1}{k}$

(non-distinct balls, distinct boxes)

<u>Exercise:</u> Find the number of ways to distribute r identical balls into 5 distinct boxes with an even number of balls, not exceeding 10, in each of the first two boxes, and between 3 and 5 balls in the other 3 boxes.

Let the number in box i be e_i. Then $\sum_{i=1}^{5} e_i = r$
$0 \le e_1, e_2, \le 10$, e_1, e_2 even
$3 \le e_3, e_4, e_5 \le 5$
The generating function for the solution is
$(1 + x^2 + x^4 + x^6 + x^8 + x^{10})^2 (x^3 + x^4 + x^5)^3$
and the solution is the coefficient of x^r in the above generating functions

<u>Exercise</u>: Find number of ways to distribute 25 identical balls into 7 distinct boxes if first box has up to 10 balls, other boxes any number.

$$[x^{25}]\, G(x) = [x^{25}]\, (1 + x + \ldots x^{10})(1 + x + x^2 + \ldots)^6$$

$= [x^{25}]\left(\frac{1-x^{11}}{1-x}\right)\left(\frac{1}{1-x}\right)^{6}$

$= [x^{25}]\ (1 - x^{11})\ (1 - x)^{-7}$

$= [x^{25}]\ (1 - x^{11})\ \sum_{r}\binom{7+r-1}{r}x^{r}$

$= [x^{25}]$

$= \binom{7+25-1}{25} - \binom{7+14-1}{14}$

$= \binom{31}{25} - \binom{20}{14}$

N.B. Could have argued this directly, viz., fill the boxes

without constraint in $\binom{7+25-1}{25}$ ways, how many of these

have at least 11 balls in box 1 is $\binom{7+14-1}{14}$, now subtract. So g.f. not always required.

Evaluate: $\sum_{k\geq 0} k\binom{n}{k}^{2}$

Recall that (Absorption) $k\binom{n}{k} = n\binom{n-1}{k-1}$

$\therefore \sum_{k\geq 0} k\binom{n}{k}^{2} = \sum_{k\geq 0} n\binom{n-1}{k-1}\binom{n}{k}$

$$= \quad n \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{n}{k+1}$$

$$= \quad n \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{n}{n-1-k}$$

The sum is the coefficient of x^{n-1} in the product

$(1 + x)^{n-1} (1 + x)^{n} = (1 + x)^{2n-1}$, which is just $\binom{2n-1}{n-1}$. Thus

$$\sum_{k\geq 0} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$$

Find the number of integer solutions to

$$x_1 + x_2 + \ldots + x_n = k$$

with $0 \leq a_i \leq x_i \leq b_i$

Generating function for variable x_i is

$A_i (z) = A_i(z) = z^{a_i} + z^{a_{i+1}} \_ + z^{b_i}$

The composite generating function is

$A(z) = A_1(z)\, A_2(z) \ldots A_n(z)$

The solution is the coefficient of z^k in A(z), $[z^k]$ A(z)

<u>Note</u>: Technically we can extend the above to allow <u>negative</u> values of the x_i. Thus, to find the number of integer solutions of

$$x_1 + x_2 + \ldots + x_{10} = n\ , \ -2 \leq x_i \leq 2\ ,$$

the 'ogf' is $(z^{-2} + z^{-1} + 1 + z + z^2)^{10}$, and the solution is

$[z^n](z^{-2} + z^{-1} + 1 + z + z^{2)10}$. Verify by hand that this works for n = 4.

'Making Change'

How many ways to make change for a buck using nickels, dimes, and quarters?

ogf nickels $= A_1(z) = 1 + z^5 + z^{10} + z^{15} + \ldots$
ogf dimes $= A_2(z) = 1 + z^{10} + z^{20} + z^{30} + \ldots$
ogf quarters $= A_3(z) = 1 + z^{25} + z^{50} + z^{75} + \ldots$

Required ogf = $A(z) = A_1(z)\, A_2(z)\, A_3(z)$

Solution is $[z^{100}]$ A(z)

Here, to determine coefficient, must multiply out (truncate each series at z^{100}).

Exercise: Suppose we also used pennies. How would the solution then relate to the ogf found above.

Solution: It would be the sum of all the coefficients in $A(z)$ above for terms with degree ≤ 100. Prove this.

Partitions of an Integer

* partition of n is unordered collection of positive integers that sum to n. For convenience, often put “parts” in decreasing order. (“summands”)
Eg. Partitions of 4 : 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1
$p(n)$ = number of partitions of n; $p(4) = 5$

As distribution problem, this is “indistinguishable balls into indistinguishable into indistinguishable boxes”.
Notice how this differs from integer solutions to an equation (where boxes (variables) distinguishable).
To partition a number, we have to know how many of the summands are 1, 2, 3, … . The ogf for the 1's is

$$1 + z + z^2 + z^3 + \ldots = \frac{1}{1 - z}$$

for the 2's is $1 + z^2 + z^4 + \ldots \quad = \frac{1}{1 - z^2}$

and so on. Thus

$$p(z) = \left(\frac{1}{1-z}\right)\left(\frac{1}{1-z^2}\right)\left(\frac{1}{1-z^3}\right)\ldots$$

(infinite product)

Exercise: Find the number of partitions of n into summands ≤ 6.
Solution:

Exercise: Find the ogf for a_n, the number of ways to partition n into distinct summands.

Solution: There can be at most 1 of any type of summand, so ogf is

$$A(z) = \underset{\substack{\uparrow \\ \text{0 or 1} \\ \text{one}}}{(1 + z)} \quad x \quad \underset{\substack{\uparrow \\ \text{0 or 1} \\ \text{two}}}{(1 + z^2)} \quad x \quad \underset{\substack{\uparrow \\ \text{0 or 1} \\ \text{three}}}{(1 + z^3)}x\ldots$$

Notice:

$$A(z) = \infty\ \Pi\ \frac{1-z^{2r}}{1-z^r} = \infty\ \Pi \left(\frac{1}{1-z^{2r-1}}\right)$$

r = 1 r = 1

which is the ogf for partitions with only odd summands.

2. Exponential Generating Function

Suppose we have p types of objects with n_i indistinguishable objects of type i, i = 1, 2, … , p. The number of perms of length k with up to n_i objects of type i is the coefficient of

$\frac{x^k}{k!}$ in the egf

$$\left(1 + x + \frac{x^2}{2!} + \_ + \frac{x^{n_1}}{n_1}\right)\left(1 + x + \frac{x^2}{2!} + \_ + x\mathrm{SUP}\,\frac{n_2}{n_2!}\right)\_$$

$$\ldots\left(1 + x + \frac{x^2}{2!} + \_ + \frac{x^{n_p}}{n_p!}\right)$$

Notice that if there is an unlimited number of objects of each type, the egf is $(e^x)^p$ and the number of perms of length k with an arbitary number of objects of type i is the

coefficient of $\frac{x^k}{k!}$ in e^{px} , i.e.

(1st place can be filled in p ways, same for the second, third, etc!!)

To 'prove' this, note that the coefficient of $\frac{x^k}{k!}$ in the product

is $\sum \frac{k!}{k_1!\,k_2!\_\,k_p!}$ where the sum is over all possible

$k_1, k_2, \ldots, k_p \ni k_1 + k_2, + \ldots + k_p = k, k_i \geq 0$. This is because the way to find a term in the product with x^k is to have a product

$\frac{x^{k_1}}{k_1!} \bullet \frac{x^{k_2}}{k_2!} - \frac{x^{k_p}}{k_p!}$ where $k_1 + \ldots + k_p = k$. Now mult. top and bottom by k! and sum over all possible such terms.

Exercise: Find the number of arrangements of r items selected from n distinct items, no repetition allowed.

EGF for item i is $1 + x$, $i = 1, 2 \ldots, n$
EGF for all is $(1 + x)^n$,

Exercise: Find the number of ways to place 25 people into 3 different rooms with at least one person in each room.

Solution: For the first room, there is only 1 way to place any number of persons in that room. Hence the e.g.f. is

$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \_ \;=\; e^x - 1$. Thus, since the 3 rooms are different, the e.g.f. for the 3 rooms is

$(e^x - 1)^3 = e^{3x} - 3e^{2x} + 3e^x - 1.$

Thus, $\left[\frac{x^{25}}{25!}\right](e^x - 1)^3 \;=\; \left[\frac{x^{25}}{25!}\right]\sum_r (3^r - 3\bullet 2^r + 3)\frac{x^r}{r}$

$$= 3^{25} - 3\cdot 2^{25} + 3$$

In the terms of the earlier result, p = 3. What we count is the number of perms of length 25 on {1,2,3} in which all of 1,2,3 appear (no room empty). The reason these are <u>perms</u> (and not merely selections as for o.g.f.) is that the people are distinct - to each assignment of rooms to people we correspond an arrangement of 1's, 2's, 3's.

Find the number of strings of length n that can be constructed using {a,b,c,d,e} if:

a) b occurs an odd number of times

b) both a and b occur an odd number of times.

a) Order is important, so use e.g.f.

egf for b: $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \_ \;=\; \frac{e^x - e^{-x}}{2}$

egf for all others $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \_ \;=\; e^x$

$\therefore$ egf = $(e^x)^4\left(\frac{e^x - e^{-x}}{2}\right) \;=\; \frac{1}{2}(e^{5x} - e^{3x})$

$[x^n]$ is $\frac{1}{2}(5^n - 3^n)$.

b) By similar reasoning, solution is

$$[x^n] \text{ is} \left[\frac{1}{2}(e^x - e^{-x})\right]^2 e^{3x} \quad = [x^{n]}$$

=

$$= \qquad \frac{1}{4}(5^n - 2 \bullet 3^n + 1)$$

<u>Distinguishable Balls, Distinguishable Boxes Revisited: Stirling No. of 2nd Kind</u>

Number of ways to put n distinct balls into k non-distinct boxes ≡ S(n,k) (no box empty)
If the boxes are <u>distinct</u> there are k!S(n,k) distributions.

Suppose ball i goes into box C(i). This gives a sequence C(1), C(2), … , C(n) using the numbers 1, 2, … , k with each number used at least once. The e.g.f. for each number is

$$\left(x + \frac{x^2}{2!} + \_+ \frac{x^n}{n!} + \_\right) = e^x - 1$$

so for all k the e.g.f. is $(e^x - 1)^k$. The coefficient of $\frac{x^n}{n!}$ in

$(e^x - 1)^k$ is precisely k! S(n,k); we can compute it as follows:

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^i e^{(k-i)x} = (e^x - 1)^k = H(x)$$

.

Substitute (k - i)x for x in the usual series for $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$ to get

$$H(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \sum_{n\geq 0} \frac{1}{n!} (k-i)^n \text{ xSUPn}$$

$$= \sum_{n\geq 0} \frac{x^n}{n!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

$$\therefore\ k!\,S(n,k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

$$S(n,k) = \frac{1}{\ } \sum_{i=0}^{k} (-1\ ^{i} \binom{k}{\ } (k-i\ ^{n}$$

<u>Exercise</u> Find the number of r-digit quatenary requences (digits 0,1,2,3) with an even number of zeros. Off number of 1's.

$\uparrow$ Egf for 0's $\qquad\qquad$ $\uparrow$ egf for 1's

=

=

$$= \frac{1}{4} \bullet 4^r = 4^{r-1} \text{ if } r > 0$$

Simple form $\Rightarrow$ combinatorial argument exists. Can you find one?