pg. 57 2. Let x_A = number of model A machines bought.
x_B = number of model B machines bought.
These can fold $30x_A + 50x_B$ boxes per minute, require $x_A + 2x_B$ attendants, and cost \$$15000x_A$ + \$$20000x_B$.

An appropriate model is:

Minimize $Z = 3x_A + 4x_B$ subject to

the constraints
$$3x_A + 5x_B \geq 32$$
$$x_A + 2x_B \leq 12$$
$$x_A \geq 0,\ x_B \geq 0;\ x_A, x_B \text{ integral.}$$

Note: This is an integer programming problem. Also, the first constraint is equivalent to $30x_A + 50x_B \geq 320$ and similarly, the minimum cost = \$5000 · the minimum value of Z.

Canonical form:
Maximize $z = -3x_A - 4x_B$ subject to

the constraints
$$3x_A + 5x_B - s_1 = 32$$
$$x_A + 2x_B + s_2 = 12$$
← integers
$$x_A \geq 0,\ x_B \geq 0,\ s_1 \geq 0,\ s_2 \geq 0;$$
$$x_A, x_B, s_1, s_2 \text{ integral.}$$
(s_1, s_2 are slack variables; integral because x_A, x_B are.)

Standard form:
Maximize $z = -3x_A - 4x_B$ subject to

the constraints
$$-3x_A - 5x_B \leq -32$$
$$x_A + 2x_B \leq 12$$
$$x_A \geq 0,\ x_B \geq 0;\ x_A, x_B \text{ integral.}$$

pg. 57 4. Let x_1 = number of acres of corn, x_2 = number of acres of soybeans, x_3 = number of acres of oats.

Two appropriate models are:

Maximize $z = 40x_1 + 30x_2 + 20x_3$

subject to the constraints

$$x_1 + x_2 + x_3 \leq 12$$
$$6x_1 + 6x_2 + 2x_3 \leq 48$$
$$36x_1 + 24x_2 + 18x_3 \leq 360$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

and

Maximize $z = 4x_1 + 3x_2 + 2x_3$

subject to the constraints

$$x_1 + x_2 + x_3 \leq 12$$
$$3x_1 + 3x_2 + x_3 \leq 24$$
$$6x_1 + 4x_2 + 3x_3 \leq 60$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

Both are already in standard form. The second model has the equivalent canonical form:

Maximize $z = 4x_1 + 3x_2 + 2x_3$ subject to

the constraints

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 12 \\ 3x_1 + 3x_2 + x_3 + x_5 &= 24 \\ 6x_1 + 4x_2 + 3x_3 + x_6 &= 60 \end{aligned}$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0.$$

(x_4, x_5, x_6 are slack variables.)

Pg. 58 6. Let C denote the number of barrels (per week, say) of cement, currently being produced; C is <u>not</u> a decision variable in the models to follow. Let x_1 = the fraction of current production to be processed using four-field precipitators.

x_2 = the fraction of current production to be processed using five-field precipitators.

Pg. 58 6. (cont'd)

Thus, $x_1 C$ = number of barrels per week to be processed using four-field precipitators.

$x_2 C$ = number of barrels per week to be processed using five-field precipitators.

and $x_1 + x_2 \leq 1$ will be a constraint in the standard-form model to follow.

Currently, the plant emits $2C$ pounds of dust per week. Given values of the decision variables, x_1 and x_2, the four-field precipitators will reduce emissions by $1.5 x_1 C$ pounds per week and the five-field type will reduce emissions by $1.8 x_2 C$ pounds per week. The EPA requirement is that $1.5 x_1 C + 1.8 x_2 C \geq .84 \cdot 2C$. (That is, total emission reduction must be $\geq 84\%$ of current emissions.)

Also, given x_1 and x_2, the weekly cost of operating the precipitators is $.14 x_1 C + .18 x_2 C$ (\$ per week).

To summarize, an appropriate model is:

Minimize $z = .14 x_1 + .18 x_2$ subject to

the constraints
$$x_1 + x_2 \leq 1$$
$$1.5 x_1 + 1.8 x_2 \geq 1.68$$
$$x_1 \geq 0, \ x_2 \geq 0.$$

(To find minimal weekly cost (in \$) multiply the optimal value of z by C.)

Pg. 58 6. (concluded)

In canonical form: Maximize $z = -.14x_1 - .18x_2$

subject to the constraints
$$x_1 + x_2 + x_3 = 1$$
$$1.5x_1 + 1.8x_2 - x_4 = 1.68$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0.$$

(x_3 and x_4 are slack variables. x_3 represents the fraction of current production which is not processed by either precipitator. $\frac{x_4}{2}$ represents the percentage of emission reduction by which the plant exceeds EPA requirements.)

In standard form: Maximize $z = -.14x_1 - .18x_2$

subject to the constraints
$$x_1 + x_2 \leq 1$$
$$-1.5x_1 - 1.8x_2 \leq -1.68$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Pg. 58 8. Let x_1 = amount invested in utilities stock
x_2 = amount invested in electronics stock
x_3 = amount invested in bonds

The limitation on investment in stocks may be expressed as $x_1 + x_2 \leq \frac{1}{2}(x_1 + x_2 + x_3)$ so an appropriate model is:

Maximize $z = .09x_1 + .04x_2 + .05x_3$ s.t.
$$x_1 + x_2 + x_3 \leq 200000$$
$$\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 \leq 0$$
$$x_1 \leq 40000$$
$$x_3 \geq 70000$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

(assuming intervals between dividends and bond coupons are all equal)

Pg. 58 8. (cont'd) Canonical form:

Maximize $Z = .09x_1 + .04x_2 + .05x_3$ subject to the constraints

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 \quad &= 200000 \\ \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 \quad + x_5 &= 0 \\ x_1 \quad + x_6 &= 40000 \\ x_3 \quad - x_7 &= 70000 \end{aligned}$$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, x_7 \geq 0.$

Standard form: Maximize $Z = .09x_1 + .04x_2 + .05x_3$

subject to the constraints

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 200000 \\ \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &\leq 0 \\ x_1 &\leq 40000 \\ -x_3 &\leq -70000 \end{aligned}$$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

Pg. 59 10. For $i = 1, 2$ and $j = 1, 2, 3, 4$ let x_{ij} = the amount of component j used to make mixture i according to the table:

mixtures ↓ / components →	alkylate	cat. cracked	st. run	isopentane
high octane	x_{11}	x_{12}	x_{13}	x_{14}
low octane	x_{21}	x_{22}	x_{23}	x_{24}

The octane constraints for each mixture may be expressed

$$100 = \frac{108x_{11} + 94x_{12} + 87x_{13} + 108x_{14}}{x_{11} + x_{12} + x_{13} + x_{14}}$$ for the high octane mixture

$$90 = \frac{98x_{21} + 87x_{22} + 80x_{23} + 100x_{24}}{x_{21} + x_{22} + x_{23} + x_{24}}$$ for the low octane mixture

Pg. 59 10. (cont'd) These two constraints are satisfied provided the x_{ij} satisfy the linear equations

$$8x_{11} - 6x_{12} - 13x_{13} + 8x_{14} = 0 \quad \text{and}$$
$$8x_{21} - 3x_{22} - 10x_{23} + 10x_{24} = 0$$

The two vapor pressure constraints may be expressed

$$7 = \frac{5x_{i1} + 6.5x_{i2} + 4x_{i3} + 18x_{i4}}{x_{i1} + x_{i2} + x_{i3} + x_{i4}} \qquad (i = 1, 2)$$

which similarly are equivalent to a pair of linear equations.

Profit = revenue − cost (we assume the table, page 59, gives <u>unit</u> revenues and costs)

$$= 6.50(x_{11} + x_{12} + x_{13} + x_{14}) + 7.50(x_{21} + x_{22} + x_{23} + x_{24})$$
$$- 7.20(x_{11} + x_{21}) - 4.35(x_{12} + x_{22}) - 3.80(x_{13} + x_{23}) - 4.30(x_{14} + x_{24})$$

An appropriate model is then:

Maximize $z = -.70x_{11} + 2.15x_{12} + 2.70x_{13} + 2.20x_{14} + .30x_{21} + 3.15x_{22} + 3.70x_{23} + 3.20x_{24}$ subject to

the constraints

$$\begin{array}{llllllllll}
8x_{11} & -6x_{12} & -13x_{13} & +8x_{14} & & & & & = 0 \\
2x_{11} & +.5x_{12} & +3x_{13} & -11x_{14} & & & & & = 0 \\
x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & = 1300 \\
 & & & & 8x_{21} & -3x_{22} & -10x_{23} & +10x_{24} & = 0 \\
 & & & & 2x_{21} & +.5x_{22} & +3x_{23} & -11x_{24} & = 0 \\
 & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & = 800 \\
x_{11} & & & & +x_{21} & & & & \le 700 \\
 & x_{12} & & & & +x_{22} & & & \le 600 \\
 & & x_{13} & & & & +x_{23} & & \le 900 \\
 & & & x_{14} & & & & +x_{24} & \le 500
\end{array}$$

"demands must be met exactly" →

supply limitations →

$$x_{ij} \ge 0; \quad i = 1, 2, \quad j = 1, 2, 3, 4.$$

Pg. 59 10. (cont'd)

In view of the fact that demands are to be met exactly, revenue will equal $6.50 \times 1300 + 7.50 \times 800$ for any feasible solution and a second appropriate model could have the same constraints but different objective function: Maximize

$$z = -7.20x_{11} - 4.35x_{12} - 3.80x_{13} - 4.30x_{14} - 7.20x_{21} - 4.35x_{22} - 3.80x_{23} - 4.30x_{24}.$$

To write an equivalent problem in canonical form, use either objective function and replace the $\leq$ constraints with the system

$$\begin{aligned}
x_{11} \qquad\qquad\qquad + x_{21} \qquad\qquad\qquad + y_1 &= 700\\
x_{12} \qquad\qquad\qquad + x_{22} \qquad\qquad + y_2 &= 600\\
x_{13} \qquad\qquad\qquad + x_{23} \qquad + y_3 &= 900\\
x_{14} \qquad\qquad\qquad + x_{24} + y_4 &= 500\\
y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 &\geq 0.
\end{aligned}$$

To write an equivalent problem in standard form, use either objective function and replace the first six constraints on page 6 with

$$\begin{aligned}
8x_{11} - 6x_{12} - 13x_{13} + 8x_{14} &\leq 0\\
-8x_{11} + 6x_{12} + 13x_{13} - 8x_{14} &\leq 0\\
2x_{11} + .5x_{12} + 3x_{13} - 11x_{14} &\leq 0\\
-2x_{11} - .5x_{12} - 3x_{13} + 11x_{14} &\leq 0\\
x_{11} + x_{12} + x_{13} + x_{14} &\leq 1300\\
-x_{11} - x_{12} - x_{13} - x_{14} &\leq -1300\\
8x_{21} - 3x_{22} - 10x_{23} + 10x_{24} &\leq 0\\
-8x_{21} + 3x_{22} + 10x_{23} - 10x_{24} &\leq 0\\
2x_{21} + .5x_{22} + 3x_{23} - 11x_{24} &\leq 0\\
-2x_{21} - .5x_{22} - 3x_{23} + 11x_{24} &\leq 0\\
x_{21} + x_{22} + x_{23} + x_{24} &\leq 800\\
-x_{21} - x_{22} - x_{23} - x_{24} &\leq -800.
\end{aligned}$$