1. Consider the l.p.p. $Ax = b$ with $x \geq 0$ where $A = \begin{pmatrix} 10 & 12 & 5 & 7 \\ 4 & 5 & 2 & 3 \\ \cancel{8} & \cancel{10} & \cancel{4} & \cancel{6} \end{pmatrix}$, $b' = (b_1, b_2, \cancel{2b_2})$, and the cost vector is $c' = (3, 1, 1, 4)$. Suppose the basic variables correspond to the columns A_3, A_4.

(a) (0 points) Hint: A basis matrix B must be square! Write the matrix B that you will use here.

$\boxed{B = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}}$ $B^{-1} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$

(b) (3 points) Compute the two basic direction d vector(s).

$d_B^1 = -B^{-1}A_1 = -\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 10 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$d_B^2 = -B^{-1}A_2 = -\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 12 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

$d^1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}$ $d^2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}$

(c) (1 point) Write the two $\boldsymbol{d_B}$ vectors, one for each of the two $\boldsymbol{d}$ vector(s).

$d_B^1 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$ $d_B^2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

(d) (3 points) Compute the reduced cost $\overline{c}_k$ for all $1 \leq k \leq 4$ (including the basic variables).

$\bar{c}_1 = c_1 - c_B B^{-1} A_1 = 3 - (1\ 4)\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 10 \\ 4 \end{pmatrix} = 3 + (1\ 4)\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \boxed{1 = \bar{c}_1}$

$\bar{c}_2 = c_2 - c_B B^{-1} A_2 = 1 - (1\ 4)\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 12 \\ 5 \end{pmatrix} = 1 + (1\ 4)\begin{pmatrix} -1 \\ -1 \end{pmatrix} = 1 - 1 - 4 = \boxed{-4 = \bar{c}_2}$

$\bar{c}_3 = c_3 - c_B B^{-1} A_3 = 1 - (1\ 4)\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 1 - (1\ 4)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{0 = \bar{c}_3}$

$\bar{c}_4 = c_4 - c_B B^{-1} A_4 = 4 - (1\ 4)\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 4 - (1\ 4)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boxed{0 = \bar{c}_4}$

(e) (1 point) For which j's is the j-th basic direction vector $\boldsymbol{d}$ a direction of cost decrease?

$j = 2$

2. Consider the standard l.p.p. $Ax = b$ with $x \geq 0$ where

- A is a 4×6 matrix with full row rank.
- The basic variables correspond to the last four columns A_3, A_4, A_5, A_6.
- $x' = (0, 0, 8, 7, 4, 3)$ is the current basic feasible solution.
- The $j = 2$ basic direction vector d is a feasible direction of cost decrease with $d' = (r, s, -10, -7, -5, 3)$

(a) (1 point) What must r and s be equal to?

$r = 0, \qquad s = 1$

(b) (2 points) Compute θ^*.

$$\theta^* = \min_{\substack{i \in basis \\ d_i < 0}} \left\{ \frac{-x_{B(i)}}{d_{B(i)}} \right\} = \min\left\{ \frac{-8}{-10}, \frac{-7}{-7}, \frac{-4}{-5} \right\}$$

$$= \min\left\{ \frac{4}{5}, 1, \frac{4}{5} \right\}$$

$$= \frac{4}{5}$$

(c) (2 points) Find the set of minimizing indices l for θ^*.

$l^* \in \{1, 3\}$

(d) (1 point) State whether the new basic feasible y is degenerate or not and why.

The new basic feasible y <u>is degenerate</u> because x_3 and x_5 will become null and only one of them will leave the basis, thus <u>one basic variable will be zero</u>.

(e) (2 points) Solve for y.

$$y = x' + \theta^* d'$$

$$= \begin{pmatrix} 0 \\ 0 \\ 8 \\ 7 \\ 4 \\ 3 \end{pmatrix} + \frac{4}{5} \begin{pmatrix} 0 \\ 1 \\ -10 \\ -7 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4/5 \\ 0 \\ 7/5 \\ 0 \\ 27/5 \end{pmatrix}$$

Notice:

$y_{B(1)} = y_3 = 0$

$y_{B(3)} = y_5 = 0$

$y_j = \theta^*, \quad j = 2.$

3. Consider the standard l.p.p. $Ax = b$ with $x' = (x_1, x_2, x_3, x_4) \geq 0'$ where $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \boxed{0} & 0 & \frac{1}{3} \end{pmatrix} \Leftarrow l$.

Suppose that:

- the only minimizing index for θ^* is $l = 3$
- the $u = -d_B$ vector corresponding to the entering variable $j = 2$ is $u' = (a, b, c)$ with unknown numbers a, b, c.

(a) (1 point) Which variable x_i exits the basis? $x_{B(3)} = x_4$

(b) (1 point) Find the B^{-1} matrix. (Hint: find B first.)

x_2 entering $\Rightarrow$ non basic so $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$.

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(c) (3 points) Write down the simplex sequence of elementary row operations (in terms of the unknown numbers a, b, c) that will update the B^{-1} matrix to become the $\overline{B}^{-1}$ matrix. (Please use the notation R_1, R_2, R_3 to denote the rows of the B^{-1} matrix)

$\bar{l}_1 = l_1 - \frac{a}{c} l_3$ { Multiply the third row by $-\frac{a}{c}$ and add it to the first row.

$\bar{l}_2 = l_2 - \frac{b}{c} l_3$ { Multiply the third row by $-\frac{b}{c}$ and add it to the second row.

$\bar{l}_3 = \frac{l_3}{c}$ { Divide the third row by c.

(d) (3 points) Compute the $\overline{B}^{-1}$ matrix in terms of the unknown numbers a, b, c.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \underset{l_1 - \frac{a}{c} l_3}{\sim} \begin{pmatrix} 1 & 0 & -\frac{3a}{c} \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \underset{l_2 - \frac{b}{c} l_3}{\sim} \begin{pmatrix} 1 & 0 & -\frac{3a}{c} \\ 0 & 2 & -\frac{3b}{c} \\ 0 & 0 & 3 \end{pmatrix}$$

$$\underset{\frac{l_3}{c}}{\sim} \begin{pmatrix} 1 & 0 & -\frac{3a}{c} \\ 0 & 2 & -\frac{3b}{c} \\ 0 & 0 & \frac{3}{c} \end{pmatrix} = \overline{B}^{-1}$$