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University of Toronto, Faculty of Arts and Science
Final Examinations, Wed, April 11, 2018, 14:00–17:00,
EM 119

APM346–Partial Differential Equations,
Section L0201

Instructor: Prof. Victor Ivrii

Duration — 3 hours

All 7 problems are independent. Total marks for this paper: 105.

The paper constitutes 40% of the final mark.

A list of useful formulas is attached in the last two-sided page.

No other aids allowed.

Last name
First name
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Term mark	
Exam Mark	
Final Mark	

#	points	Mark
1	[15]	
2	[15]	
3	[15]	
4	[15]	
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6	[15]	
7	[15]	
Total	[105]	

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Problem 1 (15 pts). Solve by the method of characteristics the BVP for a wave equation

$$u_{tt} - 16u_{xx} = 0, \quad 0 < x < \infty, \ t > 0 \quad (1.1)$$

$$u(x, 0) = f(x), \quad (1.2)$$

$$u_t(x, 0) = g(x), \quad (1.3)$$

$$(u_x - u)(0, t) = h(t) \quad (1.4)$$

with $f(x) = 4e^{-2x}$, $g(x) = 16e^{-2x}$ and $h(t) = e^{-8t}$. You need to find a continuous solution.

Problem 2 (15 pts). Solve IVP for the heat equation

$$2u_t - u_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \quad (2.1)$$

$$u|_{x=0} = 0, \quad (2.2)$$

$$u|_{t=0} = f(x) \quad (2.3)$$

with $f(x) = e^{-x}$.

Solution should be expressed through $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z^2} dz$.

Solution. We can consider Cauchy problem with odd function $f(x) = \sigma(x)e^{-|x|}$, $\sigma(x) = \pm 1$ for $x \gtrless 0$. Thus

$$\begin{aligned} u(x, t) = & \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-y} e^{-\frac{1}{2t}(y-x)^2} dy - \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^0 e^y e^{-\frac{1}{2t}(y-x)^2} dy = \\ & \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-x+\frac{t}{2}} e^{-\frac{1}{2t}(y+t-x)^2} dy - \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^0 e^{x+\frac{t}{2}} e^{-\frac{1}{2t}(y-t-x)^2} dy = \end{aligned}$$

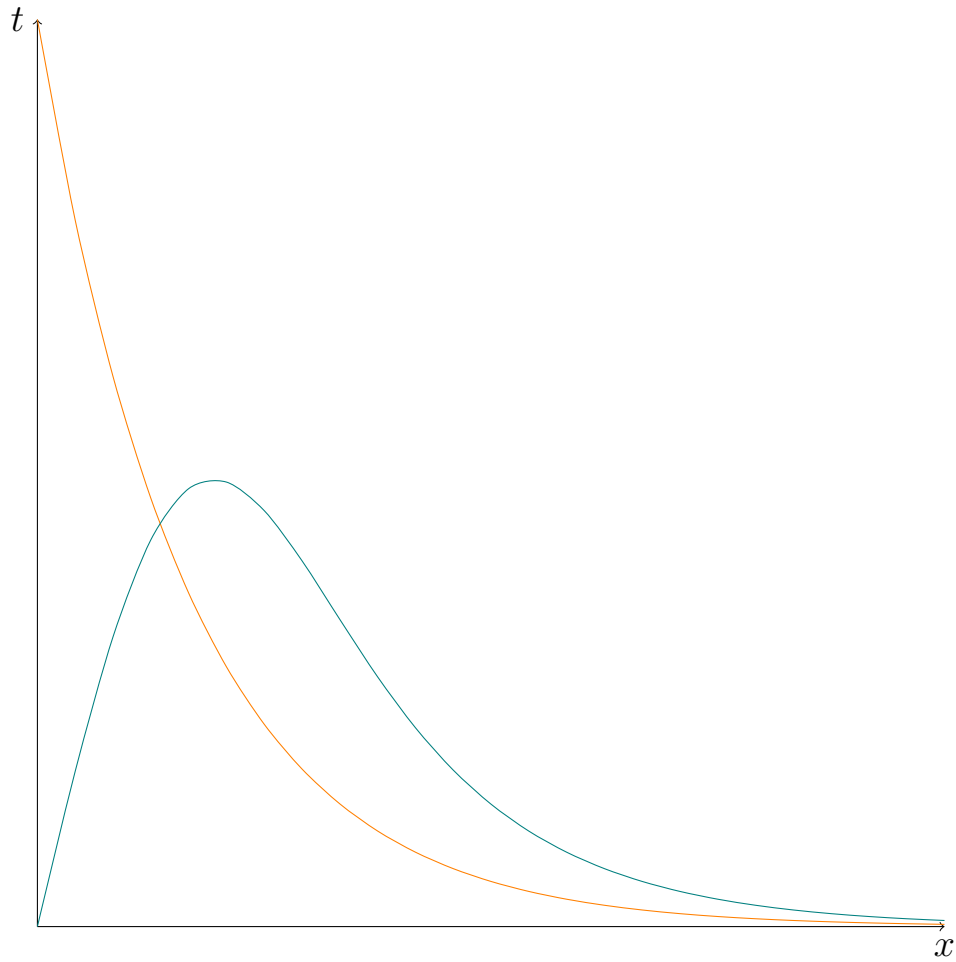
and setting $z := \pm(y-x \pm t)/\sqrt{2t}$ in the first and second integrals respectively

$$\frac{1}{\sqrt{\pi}} e^{-x+\frac{t}{2}} \int_{\frac{t-x}{\sqrt{2t}}}^\infty e^{-z^2} dz - \frac{1}{\sqrt{\pi}} e^{x+\frac{t}{2}} \int_{\frac{t+x}{\sqrt{2t}}}^\infty e^{-z^2} dz,$$

and finally

$$u(x, t) = \frac{1}{2} \left(1 - \operatorname{erf}\left(\frac{t-x}{\sqrt{2t}}\right) \right) e^{-x+\frac{t}{2}} - \frac{1}{2} \left(1 - \operatorname{erf}\left(\frac{t+x}{\sqrt{2t}}\right) \right) e^{x+\frac{t}{2}}.$$

□



Continued

Problem 3 (15 pts). Solve by the method of separation of variables

$$u_{tt} - u_{xx} + 4u = 0, \quad 0 < x < \pi, \quad t > 0, \quad (3.1)$$

$$u(0, t) = u(\pi, t) = 0, \quad (3.2)$$

$$u(x, 0) = f(x), \quad (3.3)$$

$$u_t(x, 0) = g(x) \quad (3.4)$$

with $f(x) = 0$ and $g(x) = x^2 - \pi x$. Write the answer in terms of Fourier series.

Solution. Separating variables $u(x, t) = X(x)T(t)$ we get

$$X'' + \lambda X = 0, \quad (3.5)$$

$$X(0) = X(\pi) = 0, \quad (3.6)$$

$$T'' + (\lambda + 4)T = 0. \quad (3.7)$$

Problem (3.5)–(3.6) has solution

$$\lambda_n = n^2, \quad X_n = \sin(nx), \quad n = 1, 2, \dots \quad (3.8)$$

and therefore

$$T_n = A_n \cos((n^2 + 4)^{1/2}t) + B_n((n^2 + 4)^{1/2}t), \quad (3.9)$$

and

$$u = \sum_{n=1}^{\infty} \left[A_n \cos((n^2 + 4)^{1/2}t) + B_n((n^2 + 4)^{1/2}t) \right] \sin(nx). \quad (3.10)$$

Plugging to (3.3)–(3.4) we get

$$\begin{aligned} \sum_{n=1}^{\infty} A_n \sin(nx) &= 0, \\ \sum_{n=1}^{\infty} (n^2 + 4)^{1/2} B_n \sin(nx) &= x^2 - \pi x. \end{aligned}$$

and therefore $A_n = 0$,

$$\begin{aligned}
(n^2 + 4)^{1/2} B_n &= \frac{2}{\pi} \int_0^\pi (x^2 - \pi x) \sin(nx) dx = \\
&= -\frac{2}{\pi n} \int_0^\pi (x^2 - \pi x) d \cos(nx) = \frac{2}{\pi n} \int_0^\pi (2x - \pi) \cos(nx) dx = \\
&= \frac{2}{\pi n^2} \int_0^\pi (2x - \pi) d \sin(nx) = -\frac{4}{\pi n^2} \int_0^\pi \sin(nx) dx = \\
&= \frac{4}{\pi n^3} \cos(nx) \Big|_{x=0}^{x=\pi} = \begin{cases} 0 & n = 2m, \\ -\frac{8}{\pi(2m+1)^3} & n = 2m+1. \end{cases}
\end{aligned}$$

Then

$$u(x, t) = - \sum_{m=0}^{\infty} \frac{8}{(2m+1)^3 \pi \sqrt{(2m+1)^2 + 4}} \sin((2m+1)x) \sin(\sqrt{(2m+1)^2 + 4}t).$$

□

Problem 4 (15 pts). Consider the Laplace equation in the sector

$$u_{xx} + u_{yy} = 0 \quad \text{in } \frac{1}{4} \leq x^2 + y^2 < 4, y > 0, \quad (4.1)$$

with the boundary conditions

$$u = 1 \quad \text{for } x^2 + y^2 = 4, \quad (4.2)$$

$$u = -1 \quad \text{for } x^2 + y^2 = \frac{1}{4}, \quad (4.3)$$

$$u = 0 \quad \text{for } y = 0, \quad (4.4)$$

where θ is a polar angle.

(a) Look for solutions u in the form of $u(r, \theta) = R(r)P(\theta)$ (in polar coordinates) and derive a set of ordinary differential equations for R and P . Write the correct boundary conditions for P .

(b) Solve the eigenvalue problem for P and find all eigenvalues.

(c) Solve the differential equation for R .

(d) Find the solution u of (4.1)–(4.3).

Solution. In polar coordinates $\{y = 0\}$ is $\theta = 0, \pi$.

Separating variables we get

$$\frac{r^2 R'' + r R'}{R} + \frac{P''}{P} = 0 \implies P'' + \lambda P = 0, \quad (4.5)$$

$$P(0) = P(\pi) = 0, \quad (4.6)$$

$$r^2 R'' + r R' + \lambda R = 0. \quad (4.7)$$

This problem has solutions $\lambda_n = n^2$, $X_n = \sin(n\theta)$, $n = 1, 2, \dots$

Then $r^2 R'' + r R' + n^2 R = 0 \implies R_n = A_n r^n + B_n r^{-n}$. Then

$$u = \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \sin(n\theta)$$

and using (4.2), (4.3)

$$u|_{r=2} = \sum_{n=1}^{\infty} (2^n A_n + 2^{-n} B_n) \sin(n\theta) = 1,$$

$$u|_{r=\frac{1}{2}} = \sum_{n=1}^{\infty} (2^{-n} A_n + 2^n B_n) \sin(n\theta) = -1$$

which implies

$$\sum_{n=1}^{\infty} (2^n + 2^{-n})(A_n + B_n) \sin(n\theta) = 0 \implies (A_n + B_n) = 0$$

$$\sum_{n=1}^{\infty} (2^n - 2^{-n})(A_n - B_n) \sin(n\theta) = 2 \implies (2^n - 2^{-n})(A_n - B_n) =$$

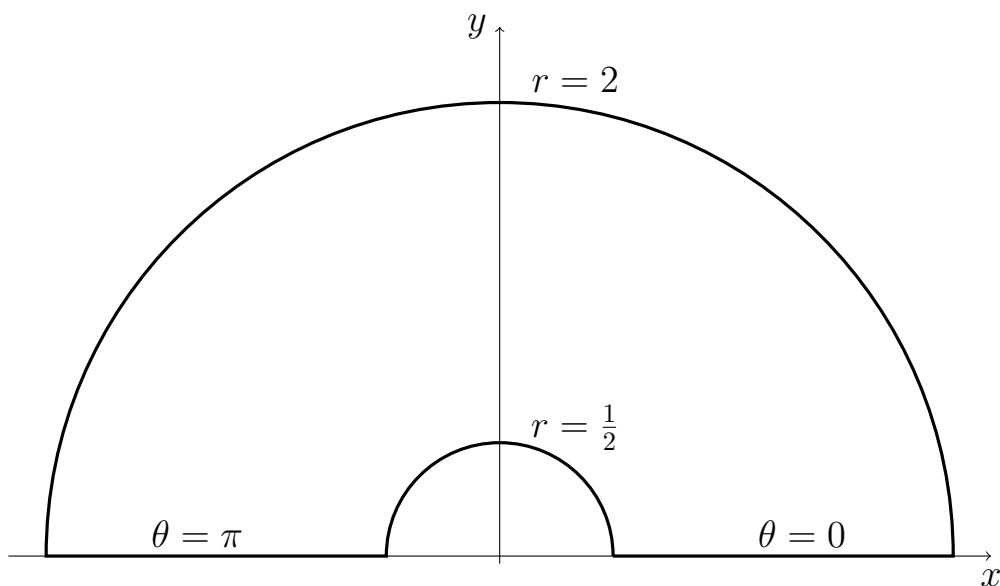
$$\frac{4}{\pi} \int_0^{\pi} \sin(n\theta) d\theta = \begin{cases} \frac{8}{\pi(2m+1)} & n = 2m+1, \\ 0 & n = 2m \end{cases}$$

which implies $A_{2m} = B_{2m} = 0$ and

$$A_{2m+1} = -B_{2m+1} = \frac{4}{\pi(2m+1)(2^{2m+1} - 2^{-2m-1})}$$

Finally

$$u = \sum_{m=0}^{\infty} \frac{4}{\pi(2m+1)(2^{2m+1} - 2^{-2m-1})} (r^{2m+1} - r^{-2m-1}) \sin((2m+1)\theta).$$



□

Problem 5 (15 pts). Consider Laplace equation in the half-strip

$$u_{xx} + u_{yy} = 0 \quad y > 0, \quad 0 < x < \frac{\pi}{2} \quad (5.1)$$

with the boundary conditions

$$u_x(0, y) = u_x\left(\frac{\pi}{2}, y\right) = 0, \quad (5.2)$$

$$(u_y - u)(x, 0) = g(x) \quad (5.3)$$

with $g(x) = 1$ and condition $\max |u| < \infty$.

- (a) Write the associated eigenvalue problem.
- (b) Find all eigenvalues and corresponding eigenfunctions.
- (c) Write the solution in the form of a series expansion.

Solution. Separating variables $u = X(x)Y(y)$ we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \implies X'' + \lambda X = 0, \quad (5.4)$$

$$Y'' - \lambda Y = 0, \quad (5.5)$$

$$X(0) = X\left(\frac{\pi}{2}\right) = 0. \quad (5.6)$$

Solving (5.4), (5.6) we have

$$\lambda_n = 4n^2, \quad X_n = \sin(2nx) \quad n = 1, 2, 3, \dots,$$

and then solving (5.5) we get

$$Y_n A_n e^{-2ny} + B_n e^{2ny} \quad n = 1, 2, 3, \dots,$$

and the last term as growing for $y > 0$ we need to drop:

$$Y_n = A_n e^{-2ny} \quad n = 1, 2, 3, \dots$$

Then

$$u(x, y) = \sum_{n=1}^{\infty} A_n e^{-2ny} \sin(2nx). \quad (5.7)$$

Plugging to (5.3) we get

$$\sum_{n=1}^{\infty} A_n (-1 - 2n) \sin(2nx) = 1$$

and

$$A_n = -\frac{4}{(2n+1)\pi} \int_0^{\frac{\pi}{2}} \sin(2nx) dx = \frac{2}{n(2n+1)\pi} \cos(2nx) \Big|_{x=0}^{x=\frac{\pi}{2}} =$$

$$\frac{2}{n(2n+1)\pi} (\cos(n\pi) - 1) = \begin{cases} 0 & n = 2m, \\ -\frac{4}{(2m+1)(4m+3)\pi} & n = 2m+1 \quad m = 0, 1, \dots \end{cases}$$

Then

$$u(x, y) = -\sum_{n=1}^{\infty} \frac{4}{(2m+1)(4m+3)\pi} e^{-(4m+2)y} \cos((4m+2)x). \quad (5.8)$$

□

Problem 6 (15 pts). Solve as $t > 0$

$$u_{tt} - \Delta u = 0, \quad (6.1)$$

with initial conditions

$$u(x, y, z, 0) = 0, \quad u_t(x, y, z, 0) = \begin{cases} r^{-1} \sin(r) & r := \sqrt{x^2 + y^2 + z^2} < \pi, \\ 0 & r \geq \pi, \end{cases} \quad (6.2)$$

and solve by a separation of variables.

HINT. Use spherical coordinates, observe that solution must be spherically symmetric: $u = u(r, t)$ (explain why).

Also, use equality

$$ru_{rr} + 2u_r = (ru)_{rr}. \quad (6.3)$$

Solution. Solution is spherically symmetric because the problem is. Then

$$u_{tt} - \left(u_{rr} + \frac{2}{r}u_r\right) = 0 \quad r > 0, t > 0. \quad (6.4)$$

Multiplying by r and using (6.3) we arrive to the first equation below:

$$v_{tt} - v_{rr} = 0 \quad r > 0, \quad (6.5)$$

$$v(0, t) = 0, \quad (6.6)$$

$$v(r, 0) = 0, \quad v_t(r, 0) = h(r) = \begin{cases} \sin(r) & 0 < r < \pi, \\ 0 & r \geq \pi, \end{cases} \quad (6.7)$$

Continuing $h(r)$ as an odd function $\tilde{h}(r) = \begin{cases} \sin(r) & -\pi < r < \pi, \\ 0 & |r| \geq \pi, \end{cases}$ and solving Cauchy problem we get

$$v(r, t) = \frac{1}{2} \int_{r-t}^{r+t} \tilde{h}(z) dz$$

which is 0 for $r > |t| + \pi$ and for $0 < r < |t| - \pi$. Otherwise it is

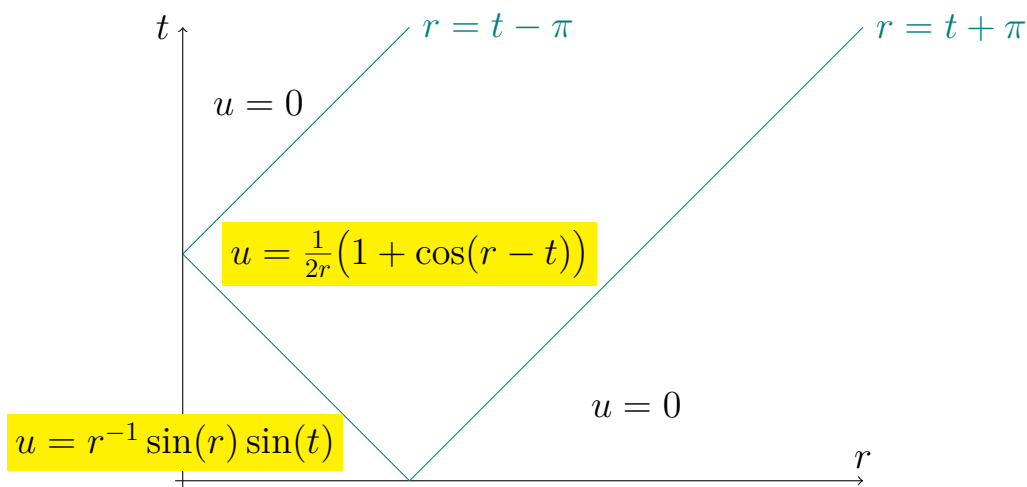
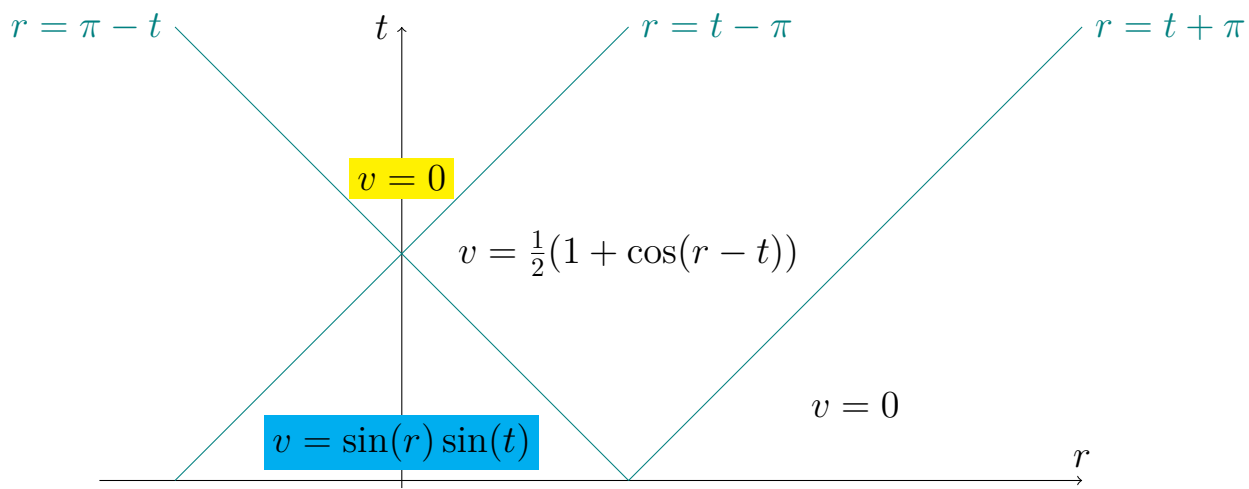
$$v(r, t) = \frac{1}{2} \int_{z_-}^{z_+} \sin(z) dz = \frac{1}{2} (\cos(z_-) - \cos(z_+)) = \sin\left(\frac{1}{2}(z_+ + z_-)\right) \sin\left(\frac{1}{2}(z_+ - z_-)\right);$$

$z_- = \max(r - t, -\pi)$, $z_+ = \min(r + t, \pi)$. Then, for $r > 0$, $t > 0$, we have

$$v(r, t) = \begin{cases} 0 & r > t + \pi, \\ \frac{1}{2}(1 - \cos(r - t)) & t - \pi < r < t + \pi, r + t > \pi, \\ \sin(r) \sin(t) & r + t < \pi, \\ 0 & 0 < r < t - \pi. \end{cases}$$

and, finally,

$$u(r, t) = r^{-1}v(r, t) = \begin{cases} 0 & r > t + \pi, \\ \frac{1}{2}r^{-1}(1 + \cos(r - t)) & t - \pi < r < t + \pi, r + t > \pi, \\ r^{-1} \sin(r) \sin(t) & r + t < \pi, \\ 0 & 0 < r < t - \pi. \end{cases}$$



□

Problem 7 (15 pts). Solve using (partial) Fourier transform with respect to y

$$\Delta u := u_{xx} + u_{yy} = 0, \quad x > 0, \quad (7.1)$$

$$u_x|_{x=0} = h(y), \quad (7.2)$$

$$u \rightarrow 0 \quad \text{as } x \rightarrow +\infty \quad (7.3)$$

with $h(y) = \frac{4y}{(y^2+1)^2}$.

HINT. Fourier transform of $g(y) = \frac{2}{y^2+1}$ is $\hat{g} = e^{-|\eta|}$ and $h(y) = -g'(y)$.

Solution. Making Fourier transform we get

$$\hat{u}_{xx} - \eta^2 \hat{u} = 0, \quad x > 0, \quad (7.4)$$

$$\hat{u}|_{x=0} = \hat{h}(\eta), \quad (7.5)$$

$$\hat{u} \rightarrow 0 \quad \text{as } x \rightarrow +\infty \quad (7.6)$$

and solving (7.4) we see that $\hat{u} = A(\eta)e^{-|\eta|x} + B(\eta)e^{|\eta|x}$; (7.6) implies that $B(\eta) = 0$ and (7.5) implies then $A(\eta) = -|\eta|^{-1}\hat{h}(\eta)$.

Due to hint and properties of Fourier transform $\hat{h}(\eta) = -i\eta\hat{g}(\eta)$.

Then $\hat{u}(x, \eta) = i\sigma(\eta)e^{-|\eta|(1+x)}$ with $\sigma(\eta) = \pm 1$ for $\eta \gtrless 0$, and

$$\begin{aligned} u(x, y) &= \int_{-\infty}^{\infty} \hat{u}(x, \eta) e^{i\eta y} d\eta = i \int_{-\infty}^{\infty} \sigma(\eta) e^{-|\eta|(1+x+i\eta y)} d\eta \\ &= -i \int_{-\infty}^0 e^{\eta(1+x+yi)} d\eta + i \int_0^{\infty} e^{-\eta(1+x-yi)} d\eta \\ &= -\frac{i}{1+x+yi} + \frac{i}{1+x-yi} = -\frac{2y}{(1+x)^2 + y^2}. \end{aligned}$$

□

Appendix: Some useful formulas.**Not exam problems.****You may detach this page**

1. The 2D Laplacian in polar coordinates and 3D Laplacian in spherical coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2},$$

$$\Delta f = \frac{\partial^2 f}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial f}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{\rho^2} \cot(\phi) \frac{\partial f}{\partial \phi} + \frac{1}{\rho^2 \sin^2(\phi)} \frac{\partial^2 f}{\partial \theta^2}.$$

2. The n -dimensional Stokes theorem

$$\int_D \frac{\partial f}{\partial x_i} dx = \int_{\partial D} f \nu_i d\sigma,$$

where ν (with components ν_i) is the unit normal vector pointing outside.

3. The complex Fourier series of a periodic function $f(x)$ of period $2l$, defined on the interval $(-l, l)$ is

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{\pi i n x / l}$$

with the coefficients c_n given by the formula

$$c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\pi i n x / l} dx.$$

4. The Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx.$$

The inverse Fourier transform is

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) dk.$$

Here some of its properties:

Continued

- (a) if $g(x) = f(ax)$, then $\hat{g}(k) = \frac{1}{|a|} \hat{f}(\frac{k}{a})$;
- (b) $\widehat{f'}(k) = ik\hat{f}(k)$;
- (c) if $g(x) = xf(x)$ then $\hat{g}(k) = -i\hat{f'}(k)$;
- (d) if $g(x) = f(x - a)$, then $\hat{g}(k) = e^{-iak} \hat{f}(k)$;
- (e) if $g(x) = f(x)e^{ixb}$, then $\hat{g}(k) = \hat{f}(k - b)$;
- (f) if $h = f * g$, then $\hat{h}(k) = 2\pi \hat{f}(k)\hat{g}(k)$;
- (g) if $f(x) = e^{-x^2/2}$, then $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2/2}$.