

⇐ Enter the 1st letter of your last name. MAKE IT LARGE !

Last name .....
First name .....
ID .....

#	points	Mark
<b>1</b>	[4]	
<b>2</b>	[4]	
<b>3</b>	[4]	
<b>4</b>	[4]	
<b>5</b>	[4]	
<b>Total</b>	[20]	

**Constitutes 20% of the Final Mark**

**Only solutions will be evaluated (not just the answers)!**

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**Problem 1** (4pt). Solve by Fourier method

$$u_{tt} - u_{xx} = 0 \quad 0 < x < \pi, \tag{1.1}$$

$$u|_{x=0} = 0, \quad (u_x + \alpha u)|_{x=\pi} = 0 \tag{1.2}$$

$$u|_{t=0} = \sin(x), \quad u_t|_{t=0} = 0 \tag{1.3}$$

with  $\alpha \in \mathbb{R}$ .

HINT: We know that  $\lambda_n$  are real but since we do not know the sign of  $\alpha$  we do not know if it all  $\lambda_n \geq 0$ ; so you must consider the case of some of  $\lambda_n < 0$ .

NOTE: Only find equations for eigenvalues.

*Solution.* Separation of variables leads to

$$X'' + \lambda X = 0, \tag{1.4}$$

$$X(0) = 0, \quad (X' + \alpha X)(\pi) = 0 \tag{1.5}$$

$$T'' + \lambda T = 0. \tag{1.6}$$

(a) Consider  $\lambda = k^2 > 0$ . Then  $X = A \cos(kx) + B \sin(kx)$  and plugging to boundary conditions we get  $A = 0$ ,  $B(k \cos(k\pi) + \alpha \sin(k\pi)) = 0$ , and to have a nontrivial solution we need  $k \cos(k\pi) + \alpha \sin(k\pi) = 0$  which for  $\alpha \neq 0$  is equivalent to

$$k = -\alpha \tan(k\pi). \tag{1.7}$$

Solving graphically

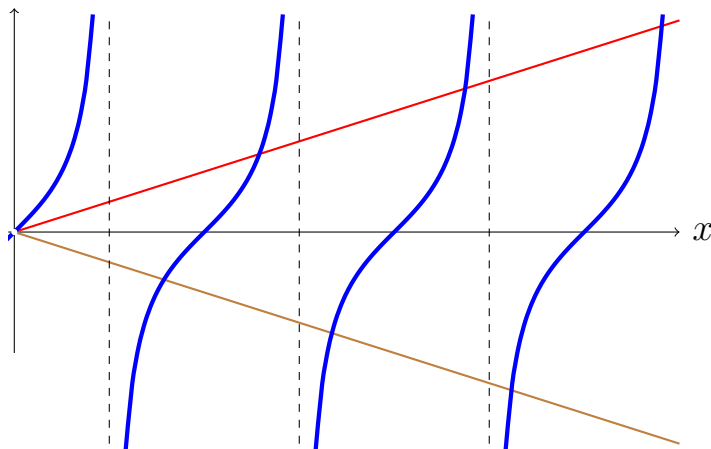


Figure 1: Brown line for  $\alpha > 0$ , red line for  $\alpha < 0$ .

(b) If  $\lambda = 0$ , then  $X = A + Bx$ ; plugging into boundary conditions we get  $A = 0$ ,  $B(1 + \alpha\pi) = 0$  and we have nontrivial solutions only for  $\alpha = -1/\pi$ .

(c) Let  $\lambda = -k^2 < 0$ . Then  $X = A \cosh(kx) + B \sinh(kx)$ ; plugging into boundary conditions we get  $A = 0$ ,  $B(k \cosh(k\pi) + \alpha \sinh(k\pi)) = 0$ , and we have nontrivial solutions only for  $k \cosh(k\pi) + \alpha \sinh(k\pi) = 0$ , which is equivalent to

$$k = -\alpha \tanh(k\pi). \tag{1.8}$$

It has only one solution and only for  $\alpha < -1/\pi$ .

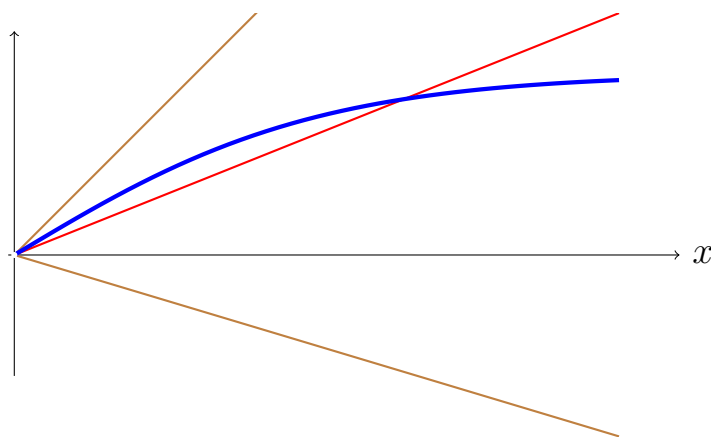


Figure 2: Brown line for  $\alpha > -1/\pi$ , red line for  $\alpha < -1/\pi$ .

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**Problem 2** (4pt). Solve

$$u_{xx} + u_{yy} = 0 \quad -\infty < x < \infty, \quad 0 < y < \infty, \quad (2.1)$$

$$(u_y + \alpha u)|_{y=0} = g(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| > 1 \end{cases} \quad (2.2)$$

$$\max |u| < \infty. \quad (2.3)$$

HINT: Use partial Fourier transform with respect to  $x$ . Write solution as a Fourier integral without calculating it.

Find restriction to  $\alpha$ , so that there will be no singularities.

*Solution.* After partial Fourier transform

$$-k^2 \hat{u} + \hat{u}_{yy} = 0 \quad 0 < y < \infty, \quad (2.4)$$

$$(\hat{u}_y + \alpha \hat{u})|_{y=0} = \hat{g}(k). \quad (2.5)$$

One can calculate easily  $\hat{g}(k) = \frac{\sin(k)}{\pi k}$ .

Solving (2.4) we get  $\hat{u} = A(k)e^{-|k|y} + B(k)e^{|k|y}$  and  $B(k) = 0$  due to (2.3) and plugging to boundary condition we get  $A(k)(-|k| + \alpha) = \frac{\sin(k)}{\pi k} \implies A(k) = \frac{\sin(k)}{\pi k(\alpha - |k|)}$ ,

$$\hat{u} = \frac{\sin(k)}{\pi k(\alpha - |k|)} e^{-|k|y}. \quad (2.6)$$

and

$$u(x, y) = \int_{-\infty}^{\infty} \frac{\sin(k)}{\pi k(\alpha - |k|)} e^{-|k|y + ikx} dk. \quad (2.7)$$

There will be singularity for  $|k| = \alpha$  which is excluded by  $\alpha < 0$ . □

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**Problem 3** (4pt). Using Fourier method find eigenvalues and eigenfunctions of Laplacian in the rectangle  $\{0 < x < a, 0 < y < b\}$  with the boundary conditions:

$$u_{xx} + u_{yy} = -\lambda u \quad 0 < x < a, 0 < y < b, \tag{3.1}$$

$$u_x|_{x=0} = u_x|_{x=a} = u|_{y=0} = u|_{y=b} = 0. \tag{3.2}$$

*Solution.* Separating variables  $u = X(x)Y(y)$  we arrive to

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0 \tag{3.3}$$

Then

$$X'' + \mu X = 0, \tag{3.4}$$

$$X'(0) = X'(a) = 0 \tag{3.5}$$

and

$$Y'' + \nu Y = 0, \tag{3.6}$$

$$Y(0) = Y(b) = 0 \tag{3.7}$$

and

$$\lambda = \mu + \nu. \tag{3.8}$$

Next

$$\mu_m = \frac{\pi^2 m^2}{a^2}, \quad X_m = \cos\left(\frac{\pi m x}{a}\right), \quad m = 0, 1, 2, \dots, \tag{3.9}$$

$$\nu_n = \frac{\pi^2 n^2}{b^2}, \quad Y_n = \sin\left(\frac{\pi n y}{b}\right), \quad n = 1, 2, \dots, \tag{3.10}$$

and finally

$$\lambda_{mn} = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \quad u_{mn} = \cos\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n y}{b}\right), \tag{3.11}$$

with  $m = 0, 1, 2, \dots$ , and  $n = 1, 2, \dots$  □

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**Problem 4** (4pt). Consider Laplace equation in the disc with a cut

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad r < 9, 0 < \theta < 2\pi \quad (4.1)$$

with the Dirichlet boundary conditions as  $\theta = 0$  and  $\theta = 2\pi$

$$u|_{\theta=0} = u|_{\theta=2\pi} = 0 \quad (4.2)$$

and the Neumann boundary condition as  $r = 9$

$$u_r|_{r=9} = 1. \quad (4.3)$$

Using separation of variables find solution as a series.

*Solution.* Separating variables  $u(r, \theta) = R(r)\Theta(\theta)$  we get

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''}{\Theta} = 0$$

and therefore both terms are constant:

$$\Theta'' + \lambda\Theta = 0, \quad (4.4)$$

$$\Theta(0) = \Theta(2\pi) = 0 \quad (4.5)$$

and therefore  $\lambda_n = \frac{n^2}{4}$ ,  $\Theta_n = \sin(\frac{n\theta}{2})$ ,  $n = 1, 2, \dots$ . Then

$$r^2R'' + rR' - \frac{n^2}{4}R = 0 \quad (4.6)$$

and  $R = Ar^{n/2} + Br^{-n/2}$  where we drop the last term as it is singular at  $r = 0$ . So  $u_n = A_n r^{n/2} \sin(\frac{n\theta}{2})$  and

$$u = \sum_{n=1}^{\infty} A_n r^{n/2} \sin(\frac{n\theta}{2}). \quad (4.7)$$

Plugging into (4.3) we get

$$\sum_{n=1}^{\infty} A_n \frac{n}{2} 3^{n-2} \sin(\frac{n\theta}{2}) = 1; \quad (4.8)$$

then

$$A_n = \frac{2}{n} 3^{2-n} \times \frac{1}{\pi} \int_0^{2\pi} \sin\left(\frac{n\theta}{2}\right) d\theta = \begin{cases} 0 & n = 2m, \\ \frac{8 \cdot 3^{1-2m}}{(2m+1)^2\pi} & n = 2m+1 \end{cases} \quad (4.9)$$

and

$$u = \sum_{m=0}^{\infty} \frac{8 \cdot 3^{1-2m}}{(2m+1)^2\pi} r^{(2m+1)/2} \sin\left(\frac{(2m+1)\theta}{2}\right). \quad (4.10)$$

□

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**Problem 5** (4pt). Find Fourier transforms of the function

$$f(x) = \cos^2(x)e^{-|x|} \tag{5.1}$$

and write this function as a Fourier integral.

*Solution.* The simplest:

$$\begin{aligned} \hat{f}(k) &= \frac{1}{\pi} \operatorname{Re} \left( \int_0^\infty \cos^2(x) e^{-|x|-ikx} dx \right) = \\ &= \frac{1}{2\pi} \operatorname{Re} \left( \int_0^\infty (1 + \cos(2x)) e^{-x-ikx} dx \right) = \\ &= \frac{1}{4\pi} \operatorname{Re} \left( \int_0^\infty (2 + e^{2ix} + e^{-2ix}) e^{-x-ikx} dx \right) = \\ &= \frac{1}{4\pi} \operatorname{Re} \left( \int_{-\infty}^0 (2e^{-x(1+ik)} + e^{-x(1+ki-2i)} + e^{-x(1+ki+2i)}) dx \right) = \\ &= \frac{1}{4\pi} \operatorname{Re} \left( \frac{2}{1+ik} + \frac{1}{1+ki-2i} + \frac{1}{1+ki+2i} \right) = \\ &= \frac{1}{2\pi} \left( \frac{2}{1+k^2} + \frac{1}{1+(k-2)^2} + \frac{1}{1+(k+2)^2} \right). \end{aligned}$$

□



