Existence and Uniqueness Theorem

Elements of the Real Analysis

Definition 1. Let

(i) $C^0([a,b]) := \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$ be a space of continuous functions on $[a,b]$ with a norm $\|f\| \equiv \max_{[a,b]} |f(x)|$ and a *distance* $\text{dist}(f,g) \equiv \|f-g\| = \max_{[a,b]} |f(x)-g(x)|$, $f, g \in C^0([a,b])$;

(ii) $C^1([a,b]) := \{f : [a,b] \to \mathbb{R} \mid df/dx \text{ exists and it is continuous}\}$ be a space of continuously differentiable functions on $[a,b]$;

(iii) A *Cauchy sequence* in a metric space (i.e. a set with a distance satisfying triangle inequality and such that $\text{dist}(f,g) = \text{dist}(g,f)$ and $\text{dist}(f,g) = 0 \iff f = g$) is a sequence $\{f_n\}_{n\geq 1}$ such that

$$\lim_{n,m\to\infty} \text{dist}(f_n, g_m) = 0.$$

($C^0([a,b])$ is an example of a metric space);

(iv) A *complete* metric space is a metric space such that for every Cauchy sequence $\{f_n\}_{n\geq 1}$, there exists a point $f := \lim_{n\to\infty} f_n$, in that space such that

$$\lim_{n\to\infty} \text{dist}(f_n, g) = 0.$$

Cauchy theorem from 1st year Calculus says that the real numbers form a complete metric space.

Theorem 2. *$C^0([a,b])$ is complete (with respect to $\text{dist}(f,g)$).*

Proof. Assume $\{g_n\}_{n\geq 1}$ is a Cauchy sequence in $C^0([a,b])$. This implies that $\{g_n(x)\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, for any $x \in [a,b]$. By Cauchy's theorem, this last sequence converges to a number, which we denote with $g(x)$. We obtain in this way a function g defined on $[a,b]$. Moreover,

$$\lim_{n\to\infty} \text{dist}(g_n; g) = 0$$

since for any $\epsilon > 0$, there is N such that if $m, n \geq N$,

$$|g_n(x) - g_m(x)| < \epsilon/2 \text{ for all } x \in [a,b].$$

Taking the limit for $m \to \infty$ one obtains

$$|g_n(x) - g(x)| \le \epsilon/2 < \epsilon \text{ for all } x \in [a,b].$$

Finally, g is continuous: Given $\epsilon > 0$, take N for which

$$|g_n(x) - g(x)| < \epsilon/3, \qquad n \ge N, \text{ for all } x \in [a,b].$$

Select $n \ge N$ as above. Since $g_n(x)$ is continuous, one has that for any $x, y \in [a,b], 0 < |x-y| < \delta$,

$$|g_n(x) - g_n(y)| < \epsilon/3$$

for an appropriate $\delta = \delta(\epsilon/3) > 0$. It follows immediately that

$$|g(x) - g(y)| < \epsilon$$

for $0 < |x-y| < \delta, x, y \in [a,b]$, and hence g is continuous. □

Lemma 3. *Let $f(x,y)$ be a function with $\frac{\partial f}{\partial y}$ continuous. Put*

$$\nabla f(x, y_1, y_2) := \frac{f(x,y_1) - f(x,y_2)}{y_1 - y_2} = \int_0^1 \frac{\partial f}{\partial y}(x, sy_1 + (1-s)y_2) ds.$$

(It follows from $h(y_1) - h(y_2) = \int_{y_1}^{y_2} h'(t)dt$ by a change of variable $t = sy_1 + (1-s)y_2$.)

Denote

$$B = \max_{|x|,|y| \le b} \left| \frac{\partial f}{\partial y}(x,y) \right|.$$

Then

$$\nabla f(x, y_1, y_2) \le B \text{ for all } |x|, |y_1|, |y_2| \le b.$$

Proof. Show this yourself! It is easy! □

Existence and Uniqueness theorem

Theorem 4. *Let $f(x,y)$ be continuos and $\frac{\partial f}{\partial y}$ exist and be bounded in the "box" $|x - \bar{x}| \le b, |y - \bar{y}| \le b$. Then Cauchy problem*

$$y' = f(x,y), \tag{1}$$

$$y(\bar{x}) = \bar{y} \tag{2}$$

has a unique solution $y = y(x)$ on interval $(\bar{x} - a', \bar{x} + a')$ with sufficiently small $a' > 0$.

Proof. Denote

$$A = \max_{|x-\bar{x}|,|y-\bar{y}|\le b} |f(x,y)| \qquad B = \max_{|x-\bar{x}|,|y-\bar{y}|\le b} \left|\frac{\partial f}{\partial y}(x,y)\right|$$

and let us redefine $a = a' = \min\{b/A, 1/2B\}$ (so that $a \cdot A \le b$ and $a \cdot B \le 1/2$).

(i) First of all we claim that (1)–(2) is equivalent to a single integral equation

$$y(x) = I(y)(x) := \bar{y} + \int_{\bar{x}}^{x} f(s, y(s))ds. \tag{3}$$

Really, if y satisfies (1)–(2) then integrating (1) from $\bar{x}$ to x we arrive to $y(x) - y(\bar{x}) = I(y)(x)$ and using (2) we arrive to (3). Conversely if y satisfies (3) then $y \in C^1(\bar{x} - a, \bar{x} + a)$ (because $I(y)$ is a continuously differentiable) and differentiating (3) we arrive to (1); plugging $x = \bar{x}$ into (3) we arrive to (1).

(ii) Note that for any $y, z \in C^0([\bar{x} - a, \bar{x} + a])$ such that $|y(x) - \bar{y}| \le b$, $|z(x) - \bar{y}| \le b$ we have $\operatorname{dist}(y, z) \le 2b$.

(iii) $I(y)$ defined above is a *contraction*, that is

$$\operatorname{dist}(I(y), I(z)) \le q \operatorname{dist}(y, z) \tag{4}$$

for some $q < 1$. In fact, due to the lemma 3:

$$|I(y)(x) - I(z)(x)| = |\int_{\bar{x}}^{x} \nabla f(s, y(s), z(s))(y(s) - z(s))ds| \le aB \cdot \operatorname{dist}(y, z)$$

and, since $aB \le 1/2$, we can take $q = 1/2$.

Remark 1. It follows from (iii) that $I(g_i) = g_i$ for $i = 1, 2$ implies $g_1 = g_2$. Show this yourself. This proves uniqueness.

(iv) Any sequence composed of $y_0 \in C^0([\bar{x} - a, \bar{x} + a])$ with $\|y(x) - \bar{y} \le b$ (for instance $y_0 \equiv 0$), $y_n := I(y_{n-1}), n \ge 1$, is a Cauchy sequence: indeed, because $q = 1/2$,

$$\lim_{n\to\infty} q^n = 0$$

Take $n(\epsilon)$ such that $q^n < \epsilon/2b$ for all $n \ge n(\epsilon)$. Let $m \ge n \ge n(\epsilon)$. Then

$$\operatorname{dist}(g_m, g_n) = \|I^n(y_{m-n} - y_0)\| \le q^n \|y_{m-n} - y_0\| \le q^n 2b < (\epsilon/2b) \cdot 2b = \epsilon.$$

(v) By making use of theorem 2, there exists $y \in C^0([-a,a])$ such that $\lim_{n\to\infty} \operatorname{dist}(y_n, y) = 0$, and hence $|y(x) - \bar{y}| \le b$ for $|x - x_0| \le a$.

Since

$$\begin{aligned}\operatorname{dist}(y, I(y)) &\le \operatorname{dist}(y, y_n) + \operatorname{dist}(y_n, I(y)) \\ &= \operatorname{dist}(y, y_n) + \operatorname{dist}(I(y_{n-1}), I(y)) \le \operatorname{dist}(y, y_n) + \frac{1}{2}\operatorname{dist}(y_{n-1}, y)\end{aligned}$$

and

$$\lim_{n\to\infty} \operatorname{dist}(y, y_n) = 0 = \lim_{n\to\infty} \operatorname{dist}(y_{n-1}, y),$$

it follows $\operatorname{dist}(y, I(y)) = 0$, i.e. $y = I(y)$.

□

Existence theorem

One can prove

Theorem 5. *Let $f(x,y)$ be continuos in the "box" $|x - \bar{x}| \le b, |y - \bar{y}| \le b$. Then Cauchy problem (1)–(2) has a solution $y = y(x)$ on interval $(\bar{x} - a', \bar{x} + a')$ with sufficiently small $a' > 0$.*

Sketch of the proof. Consider Euler approximations with the step h:

$$y_{h,n+1} = y_{h,n} + f(x_n, y_{h,n})h, \qquad x_n = \bar{x} + nh, \qquad y_{h,0} = \bar{y} \tag{5}$$

and on "step" intervals (x_n, x_{n+1}) we apply a linear approximation $y_h(x) = y_{h,n} + f(x_n, y_{h,n})(x - x_n)$. Here we take $n = 0, 1, 2, \ldots$ but we can go also in the opposite direction (replacing h by $-h$). So, we get a piecewise linear function $y_h(x)$.

One can prove that

(a) Functions $y_h(x)$ are defined on interval $[\bar{x} - a, \bar{x} + a]$ with a redefined as $a' = \min(a, b/A)$ (see proof of theorem 4) and are uniformly bounded there: $|y_h(x) - \bar{y}| \le b$;

(b) Functions $y_h(x)$ are uniformly continuous i.e. for each $\epsilon > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta \implies |y_h(x) - y_h(x')| < \epsilon$; indeed, $\delta = \epsilon/A$ works. "Uniformly" here and above means that bound and δ do not depend on h;

(c) $|y_h(x) - I(y_h)(x)| \le \varepsilon_h$ for all $x \in [\bar{x} - a, \bar{x} + a]$ with $\varepsilon_h \to 0$ as $h \to 0$.

Let us take $h_m = 2^{-m} \to +0$ as $m \to \infty$.

Now we apply Arzelá–Ascoli theorem from Real Analysis:

Theorem 6. *From sequence of functions $y_{h_m}(x)$ satisfying (a)–(b) one can select a subsequence $y_{h_{m_k}}(x)$ converging in $C([\bar{x}-a, \bar{x}+a])$: $y_{h_{m_k}}(x) \to y(x)$. Since step $h_{m_k} \to 0$ the limit is by no means piecewise linear!*

Then obviously $I(y_{h_{m_k}}) \to I(y)$. Further, (c) implies that $y = I(y)$ and therefore y satisfies (3) and thus it satisfies (1)–(2). □

Remark 7. (i) In contrast to theorem 4 theorem 6 does not imply uniqueness of solution; indeed, example $y' = y^{\frac{1}{3}}$ analyzed in the lectures shows the lack of uniqueness;

(ii) Both theorems 4 and 6 are based on fixed point equation $y = I(y)$ but existence of the fixed point y is due to different ideas: in theorem 4 it exists and is unique because map $y \to I(y)$ is contractive; in theorem 6 it exists (but is not necessarily unique) because map $y \to I(y)$ is compact (we did not define this notion).