Please note: handouts are not only obligatory but allow you to solves problems faster. See problem 3 in particular.

1 [6 points]. Find integrating factor and solve

$$x\,dx + y(1+x^2+y^2)\,dy = 0.$$

SOLUTION. As $M = x$, $N = y(1+x^2+y^2)$, calculating $M_y - N_x = -2xy$ we see that it is not 0 but $(M_y - N_x)/N = -2y$ does not depend on x and therefore we are looking for an integrating factor $\mu = \mu(y)$ satisfying

$$(\log \mu)' = 2y \implies \mu = e^{y^2}$$

and therefore multiplying by e^{y^2} we get

$$xe^{y^2}\,dx + ye^{y^2}(1+x^2+y^2)\,dy = 0 \implies \Psi_x = xe^{y^2} \implies$$
$$\Psi = \frac{1}{2}x^2e^{y^2} + \chi(y) \implies \Psi_y = x^2ye^{y^2} + \chi'(y) = e^{y^2}(1+x^2+y^2)y \implies$$
$$\chi'(y) = y(y^2+1)e^{y^2} \implies \chi(y) = \frac{1}{2}y^2e^{y^2} \implies \Psi = \frac{1}{2}(x^2+y^2)e^{y^2}$$

ANSWER: $(x^2+y^2)e^{y^2} = C$.

2a [2 points]. Consider equation

$$(\cos(t) + t\sin(t))y'' - t\cos(t)y' + y\cos(t) = 0.$$

Find wronskian $W = W[y_1, y_2](t)$ of two solutions such that $W(0) = 1$.

2b [2 points]. Check that one of the solutions is $y_1(t) = t$. Find another solution y_2 such that $W[y_1, y_2](\pi/2) = \pi/2$ and $y_2(\pi/2) = 0$.

SOLUTION. (a) Consider the equation for W:

$$W'/W = \frac{t\cos(t)}{\cos(t) + t\sin(t)} \implies \ln W = \int \frac{t\cos(t)}{\cos(t) + t\sin(t)}\,dt =$$
$$\ln(\cos(t) + t\sin(t)) + \ln C \implies W = C(\cos(t) + t\sin(t))$$

To evaluate the above integral use the substitution $u = \cos(t) + t\sin(t)$. Then $W(0) = 1 \implies C = 1 \implies W = (\cos(t) + t\sin(t))$.

(b) Note that $\frac{\pi}{2} = \cos(\frac{\pi}{2}) + \frac{\pi}{2}\sin(\frac{\pi}{2}) = W[y_1, y_2](\frac{\pi}{2})$
Plugging $y_1 = t$ into the Wronskian, and using the integrating factor $\mu = \frac{1}{t}$, we get

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t & y_2 \\ 1 & y_2' \end{vmatrix} = ty_2' - y_2 = \big(\cos(t) + t\sin(t)\big) \implies$$

$$(y_2t^{-1})' = (\cos(t) + t\sin(t))t^{-2} \implies y_2t^{-1} = \int \big(t^{-2}\cos(t) + t^{-1}\sin(t)\big)\, dt =$$
$$- t^{-1}\cos(t) + C_1 \implies y_2 = -\cos(t) + C_1 t.$$

$y_2(\pi/2) = 0 \implies 0 = 0 + \frac{\pi}{2}C_1 \implies C_1 = 0.$
ANSWER: $y_2 = -\cos(t)$.

3 [6 points]. Find the general solution for equation

$$y'' + 4y' + 5y = te^{-2t} + e^{-2t}\cos(t).$$

SOLUTION. The characteristic equation is

$$r^2 + 4r + 5 = 0 \implies (r+2)^2 = -1 \implies r_{1,2} = -2 \pm i.$$

General solution to homogeneous equation is $y = e^{-2t}\big(C_1\cos(t) + C_2\sin(t)\big)$.
"SHORT" SOLUTION USING HANDOUT #8:

$$L(y) = y'' + 4y' + 5y; \qquad g(t) = te^{-2t} + e^{-2t}\cos(t).$$

STEP 1: $Q(r) = r^2 + 4r + 5$, $Q'(r) = 2r + 4$; $Q(-2) = 1 \implies m = 1$, $Q'(-2) = 0$,
$Q(-2+i) = 4 - 4i - 1 - 8 + 4i + 5 = 0$, $Q'(-2+i) = -4 + 2i + 4 = 2i \implies m = 1$

STEP 2: $L(te^{-2t}) = te^{-2t} + 0$
$L(te^{(-2+i)t}) = 2ie^{(-2+i)t} = 2e^{-2t}(-\sin(t) + i\cos(t))$

STEP 3: $y_p(t) = te^{-2t} + \frac{t}{2}e^{-2t}\sin(t)$.

ANSWER: $y(t) = e^{-2t}\big(C_1\cos(t) + C_2\sin(t)\big) + te^{-2t} + \frac{t}{2}e^{-2t}\sin(t)$

"LONG" SOLUTION USING UNDETERMINED COEFFICIENTS:
As $f_1 = te^{-2t}$ we look for a solution $\bar{y}_1 = (at+b)e^{-2t}$ and find $\bar{y}_1 = te^{-2t}$. Indeed,

$$\begin{aligned} &\bar{y}_1' = (a - 2b - 2at)e^{-2t} \quad\text{and}\quad \bar{y}_1'' = (-4a + 4b + 4at)e^{-2t} \\ \Longrightarrow\ & te^{-2t} = \bar{y}_1'' + 4\bar{y}_1' + 5\bar{y} = e^{-2t}(at+b) \\ \Longrightarrow\ & a = 1 \text{ and } b = 0 \\ \Longrightarrow\ & \bar{y}_1 = te^{-2t} \end{aligned}$$

As $f_2 = e^{-2t}\cos(t)$, which appears in the solution to the homogeneous equation, we look for a solution $\bar{y}_2 = te^{-2t}(c\cos(t) + d\sin(t))$ and find $\bar{y}_2 = \frac{t}{2}e^{-2t}\sin(t)$. Indeed,

$$\begin{aligned} \bar{y}_2' &= e^{-2t}\big((c + dt - 2ct)\cos(t) + (d - ct - 2dt)\sin(t)\big), \\ \bar{y}_2'' &= e^{-2t}\big((2d - 4c - 4dt + 3ct)\cos(t) + (-2c - 4d + 4ct + 3dt)\sin(t)\big). \end{aligned}$$

Thus,

$$\begin{aligned} & e^{-2t}\cos(t) = \bar{y}_2'' + 4\bar{y}_2' + 5\bar{y}_2 = e^{-2t}\big(2d\cos(t) - 2c\sin(t)\big) \\ \Longrightarrow\ & c = 0 \quad\text{and}\quad d = \frac{1}{2} \\ \Longrightarrow\ & \bar{y}_2 = \frac{t}{2}e^{-2t}\sin(t) \end{aligned}$$

ANSWER: $y = e^{-2t}\big(C_1\cos(t) + C_2\sin(t)\big) + te^{-2t} + \frac{t}{2}e^{-2t}\sin(t)$.

4 [4 points]. Find solution of

$$y^{(4)} + 8y'' + 16y = 0$$

satisfying initial conditions

$$y(0) = 1,\ y'(0) = y''(0) = y'''(0) = 0.$$

SOLUTION. $r^4 + 8r^2 + 16 = (r^2+4)^2 \implies r_1 = r_2 = 2i, r_3 = r_4 = -2i \implies$

$$y(t) = C_1\cos(2t) + C_2 t\cos(2t) + C_3\sin(2t) + C_4 t\sin(2t).$$

Then

$$\begin{aligned} y'(t) &= (-2 + C_4 - 2tC_2)\sin(2t) + (2C_3 + C_2 + 2tC_4)\cos(2t), \\ y''(t) &= (-4 + 4C_4 - 4tC_2)\cos(2t) + (-4C_2 - 4C_3 - 4tC_4)\sin(2t), \\ y'''(t) &= (8 - 8C_4 + 8tC_2)\sin(2t) + (-8C_3 - 12C_2 - 8tC_4)\cos(2t). \end{aligned}$$

Then $y(0) = C_1 = 1$, $y'(0) = C_2 + 2C_3 = 0$, $y''(0) = -4C_1 + 4C_4 = 0$, $y'''(0) = -8C_3 - 12C_2 = 0$ and solving these equations yields $C_1 = C_4 = 1$ and $C_2 = C_3 = 0$.

ANSWER: $y(t) = \cos(2t) + t\sin(2t)$.