1. [6 points] Find the general solution of the following system of equations.

$$\begin{cases} x' = -3y + 7z \\ y' = -y + 5z \\ z' = -y + z \end{cases}$$

Solution. This is equivalent to the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 0 & -3 & 7 \\ 0 & -1 & 5 \\ 0 & -1 & 1 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\det(A - \lambda I) = -\lambda\left((-1-\lambda)(1-\lambda) + 5\right) = -\lambda(\lambda^2 + 4),$$

hence its eigenvalues are $\lambda = 0, \pm 2i$. To find the eigenvector $\mathbf{u}$ corresponding to $\lambda = 0$, we must sovle $A\mathbf{u} = 0$, and we find $\mathbf{u} = (1\ 0\ 0)^{\mathrm{T}}$.
To find the eigenvector $\mathbf{v}$ corresponding to $\lambda = 2i$, we must solve

$$\begin{pmatrix} -2i & -3 & 7 \\ 0 & -1-2i & 5 \\ 0 & -1 & 1-2i \end{pmatrix} \mathbf{v} = 0 \implies \begin{pmatrix} -2i & -3 & 7 \\ 0 & -1-2i & 5 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{v} = 0$$

If we denote the components of $\mathbf{v}$ by $\mathbf{v} = (v_1\ v_2\ v_3)^{\mathrm{T}}$, then the second equation is

$$(1+2i)v_2 = 5v_3,$$

which we can solve by taking $v_2 = 5$ and $v_3 = 1 + 2i$. Inserting these values into the first equation, we find

$$v_1 = \frac{1}{2i}\left(-3v_2 + 7v_3\right) = 7 + 4i.$$

Note that any non-zero complex multiple of $\mathbf{v}$ will also be an eigenvector, so there are many (equivalent) correct solutions. For our choice of $\mathbf{v}$, this yields a complex solution

$$\begin{aligned}
\begin{pmatrix} 7+4i \\ 5 \\ 1+2i \end{pmatrix} e^{2it} &= \begin{pmatrix} 7+4i \\ 5 \\ 1+2i \end{pmatrix} (\cos(2t) + i\sin(2t)) \\
&= \begin{pmatrix} 7\cos(2t) - 4\sin(2t) \\ 5\cos(2t) \\ \cos(2t) - 2\sin(2t) \end{pmatrix} + i \begin{pmatrix} 4\cos(2t) + 7\sin(2t) \\ 5\sin(2t) \\ 2\cos(2t) + \sin(t) \end{pmatrix} \\
&=: \mathbf{w}_1(t) + i\mathbf{w}_2(t).
\end{aligned}$$

Hence the general solution is

$$\mathbf{x}(t) = c_1\mathbf{w}_1(t) + c_2\mathbf{w}_2(t) + c_3\mathbf{u}.$$

2. [14 points] Consider the second order equation

$$x'' = x^4 - 5x^2 + 4$$

(a) [1 points] Reduce to the first order system in variables (x, y, t) with $y = x'$, i.e.

$$\begin{cases} x' = \dots \\ y' = \dots \end{cases}$$

(b) [3 points] Find solution in the form $H(x, y) = C$.

(c) [3 points] Find critical points and linearize system in these points.

(d) [5 points] Classify the linearizations at the critical points (i.e. specify whether they are nodes, saddles, etc., indicate stability and, if applicable, orientation) and sketch their phase portraits.

(e) [2 points] Sketch the phase portraits of the nonlinear system near each of the critical points.

(Bonus) [2 points] Sketch the solutions on (x, y) plane.

Solution. Setting $y = x'$, we find

$$y' = x'' = x^4 - 5x^2 + 4,$$

hence the original ODE is equivalent to the system

$$\begin{cases} x' &= y \\ y' &= x^4 - 5x^2 + 4 \end{cases}$$

Along solutions $(x(t), y(t))$ of this system, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{x^4 - 5x^2 + 4}{y},$$

hence

$$ydy = (x^4 - 5x^2 + 4)dx \implies \frac{1}{2}y^2 = \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x + C,$$

so we may take

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{5}x^5 + \frac{5}{3}x^3 - 4x.$$

The solutions are thus of the form

$$\frac{1}{2}y^2 + V(x) = E$$

where the potential $V(x)$ is given by

$$V(x) = -\frac{1}{5}x^3 + \frac{5}{3}x^3 - 4x.$$

To find the critical points of this system, we must solve $x' = y' = 0$. Since $x' = y$, we have $y = 0$. Now,

$$y' = x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4),$$

so we find $x = \pm 1, \pm 2$. Hence the critical points of the system are $(\pm 1, 0)$ and $(\pm 2, 0)$. To linearize the system, we compute the Jacobian matrix

$$J = \begin{pmatrix} \partial_x F & \partial_y F \\ \partial_x G & \partial_y G \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 4x^3 - 10x & 0 \end{pmatrix}.$$

At the points $(\pm 1, 0)$, the linearized system is given by the matrices

$$\begin{pmatrix} 0 & 1 \\ \mp 6 & 0 \end{pmatrix}.$$

The characteristic polynomial is $\lambda^2 \pm 6$, hence we find that $(-1, 0)$ is a saddle point (unstable), while $(1, 0)$ is a center (stable, but not asymptotically stable). The orientation of $(1, 0)$ is clockwise (negative).
At the points $(\pm 2, 0)$, the linearized system is given by the matrices

$$\begin{pmatrix} 0 & 1 \\ \pm 12 & 0 \end{pmatrix}.$$

The characteristic polynomial is $\lambda^2 \mp 12$, hence we find that $(2, 0)$ is a saddle point (unstable), while $(-2, 0)$ is a center (stable, but not asymptotically stable). The orientation of $(-2, 0)$ is clockwise (negative).

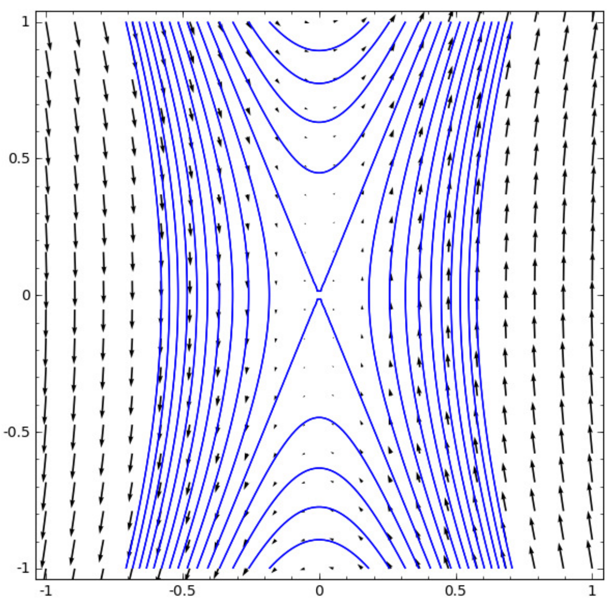
Remark 1. (i) Alternatively we can consider Hessian of function H in its stationary points (which are exactly equilibrium points we found earlier:

$$\operatorname{Hess} H = \begin{pmatrix} \mp 12 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{at} \quad (\pm 2, 0), \qquad \operatorname{Hess} H = \begin{pmatrix} \pm 6 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{at} \quad (\pm 1, 0),$$

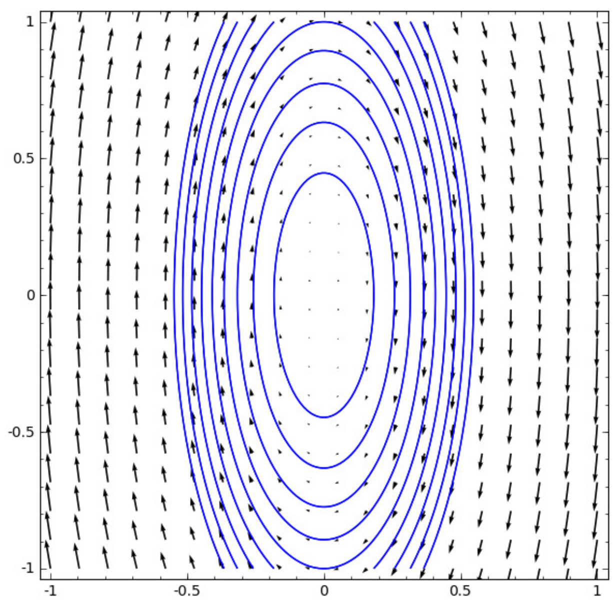
and we conclude that $(-2, 0)$ and $(1, 0)$ are minima (centres for trajectories) and $(-1, 0)$ and $(2, 0)$ are saddles (saddles for trajectories as well);

(ii) Since $x' = y$ we conclude that as $y > 0$ movement is to the right and all centres have clockwise orientation.

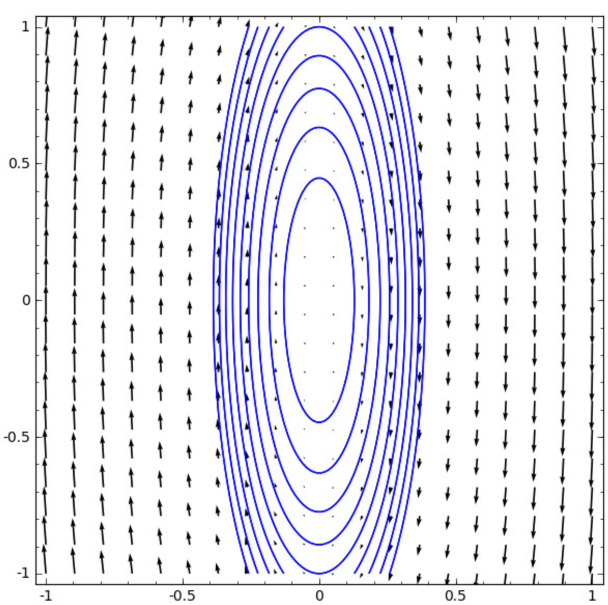
Note that by part (b), the system is integrable, and the points $(1, 0)$ and $(-2, 0)$ are true centers in the full non-linear system. The phase portraits are as follows:



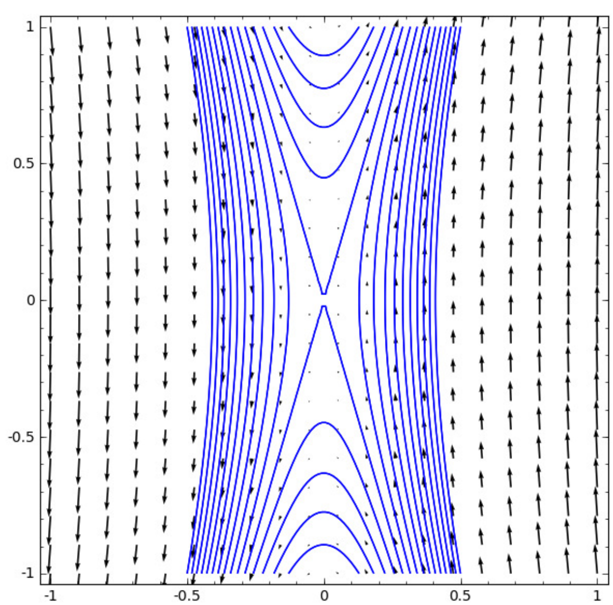
(a) $(-1, 0)$



(b) $(1, 0)$

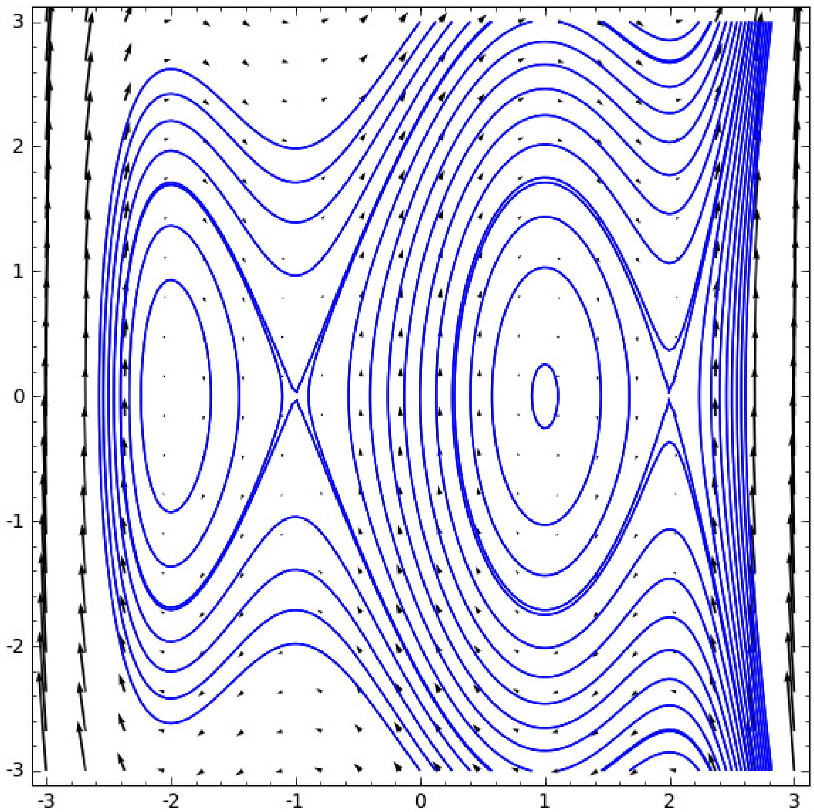


(c) $(-2, 0)$

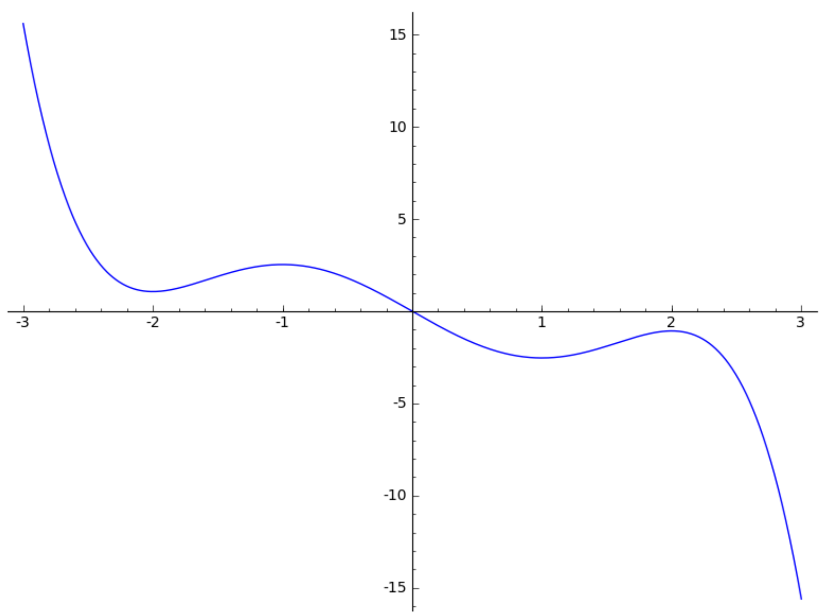


(d) $(2, 0)$

Trajectories near equilibrium points



(e) The phase portrait of the full non-linear system; note that separatrices passing through two saddle points do not coincide



(f) Plot of potential $V(x) = -\frac{1}{5}x^5 + \frac{5}{3}x^3 - 4x$; maxima of V correspond to saddles of H and minima of V correspond to minima of H