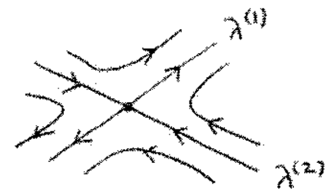
MAT 244

Phase portraits for linear 2×2 systems $\vec{x}' = A\vec{x}$ (A real)

I) Distinct real eigenvalues

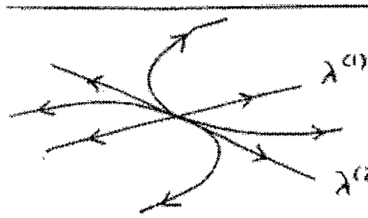
Ia) $\lambda^{(1)} > 0 > \lambda^{(2)}$



Saddle

Ib) $\lambda^{(1)} > \lambda^{(2)} > 0$

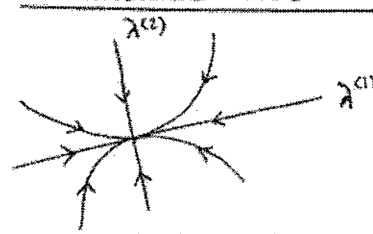
Note that $\lambda^{(2)}$ dominates for $t \to -\infty$ while $\lambda^{(1)}$ dominates for $t \to \infty$.



Unstable node

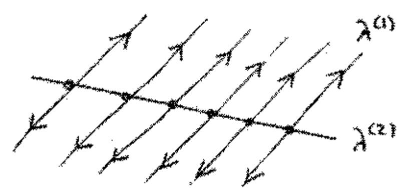
Ic) $0 > \lambda^{(1)} > \lambda^{(2)}$

Here $\lambda^{(2)}$ dominates for $t \to -\infty$ while $\lambda^{(1)}$ dominates for $t \to \infty$



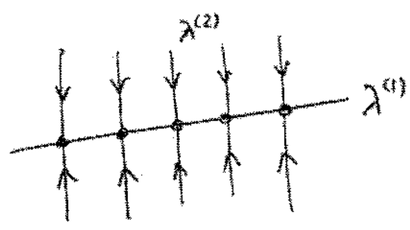
Stable node

Id) $\lambda^{(1)} > \lambda^{(2)} = 0$



Ie) $\lambda^{(1)} = 0 > \lambda^{(2)}$

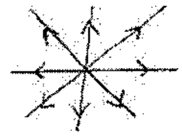
Similar to Id), arrow reversed:



II) Repeated real eigenvalue λ

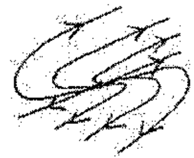
IIa) $\lambda > 0$

IIa1) A has two indep. eigenvectors with eigenvalue λ



unstable proper node

IIa2) A has only one eigenvector (up to factor)

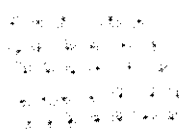


or

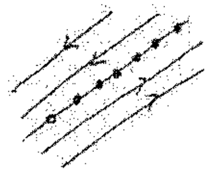
unstable improper node

IIb) $\lambda = 0$

IIb1) A has two independent eigenv. with eigenvalue 0: Happens only if $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Every sol. is constant



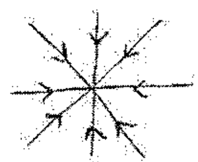
IIb2) A has only one eigenvector (up to factor)



or

IIc) $\lambda < 0$ (Like IIa, arrows reversed)

IIc1) A has two independent eigenvectors



stable proper node

IIc2) A has only one eigenvector (up to factor)

or

stable improper node

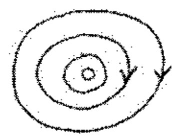
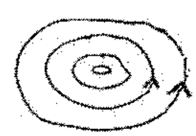
III) Complex eigenvalues $\lambda = a + ib$, $\bar{\lambda} = a - ib$ $(b \neq 0)$

III a) $a > 0$

 or 

unstable spiral

III b) $a = 0$ (purely imaginary eigenvalues)

 or 

Center

III c) $a < 0$

stable spiral

Remark: A simple way of deciding between the two possibilities in cases II a 2, II b 2, II c 2 and III a, III b, III c is to compute $\vec{x}\,'$ at a suitable point $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

For example, in III a) – III c) one can take $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$$\vec{x}\,' = A\vec{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$ showing that one has

counterclockwise motion for $a_{21} > 0$, clockwise motion for $a_{21} < 0$.