FINITE TYPE INVARIANTS AND n-EQUIVALENCE OF 3-MANIFOLDS

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ABSTRACT. We announce a new approach to a definition of finite type invariants and introduce a notion of n-equivalence for 3-manifolds with boundary. For the $\mathbb{Z}HS$ we state equivalence of different definitions of finite type invariants. The theory for 3-manifolds is made completely parallel to the corresponding theory for links. We show how to reduce general classification problems of arbitrary 3-manifolds up to n-equivalences to the study of finite type invariants of string links and integer homology spheres.

INVARIANTS DE TYPE FINI ET n-ÉQUIVALENCE DE VARIÉTÉS DE DIMENSION TROIS

RÉSUMÉ: Nous annonçons une nouvelle approche de la définition des invariants de type fini et introduisons la notion de n-équivalence de variétés de dimension trois. Nous montrons que pour les sphères d'homologie entière, les différentes définitions des invariants de type fini sont équivalentes. Nous rendons la théorie pour les variétés de dimension trois complètement similaire à celle pour les entrelacs. Nous montrons comment réduire les problèmes de classification des variétés de dimension trois à n-équivalence près, à l'étude des invariants de type finis des enchevêtrements purs et des sphères d'homologie entière.

Version francaise abrégée. Toute variété sera lisse, orientée, connexe, compacte, à bord et à coins. Soit L l'entrelacs framé dans la boule à anse V, de framing 0, présenté dans la Figure 1.b. La chirurgie d'une variété de dimension trois M le long de l'image de L par une immersion $V \to M$ est appelée une *Y-chirurgie.* Elle est déterminée par l'image dans M du graphe Γ de la Figure 1, ce que nous appelerons un Y-graphe. La classe des variétés s'obtenant les unes les autres par des Y-chirurgies est appelée une classe de *0-équivalence.*

Une collection de Y-graphes disjoints dans M, divisée en r sous-collections $S^{(1)}, \dots, S^{(r)}$ est appelée un *schéma de variation de dimension r sur M*. Pour $\sigma = (\sigma_1, \dots, \sigma_r) \in \{0,1\}^r$, posons $S(\sigma) = \sqcup_{i:\sigma_i=1} S^{(i)}$ et notons $[S(\sigma)]$ le résultat de la Y-chirurgie sur M le long de $S(\sigma)$. Soit ν un invariant de variétés de dimension trois, à valeur dans un groupe abélien. Pour un schéma de variation S sur M, posons $\nu(S) = \sum_\sigma (-1)^{|\sigma|}\nu([S(\sigma)])$. L'invariant ν est de *type fini*, s'il existe d, tel que $\nu(S) = 0$ pour tout schéma S de dimension $> d$. Le *degré* de ν désignera le minimum parmi ces entiers d et sera noté $d(\nu)$.

Théorème 1. *Pour des invariants de sphères d'homologie entière à valeur complexe, la définition ci-dessus est équivalente à celles d'Ohtsuki et de Garoufalidis. Les degrés vérifient : $2d^{Oh} = 3d = 6d^G$.*

Un *cylindre d'homologie* sur une variété de dimension deux F est une variété qui est 0-équivalente à $F \times I$. Soit D_m un disque avec m trous et $HC(m)$ l'ensemble des cylindres d'homologie sur D_m. Un schéma de variation S de dimension r sur $M \in HC(m)$ sera dit *destructif,* si $[S(\sigma)]$ coïncide avec $F \times I$ pour tout $\sigma \in \{0,1\}^r$ non trivial. Si $M \in HC(m)$ admet un schéma de variation destructif de dimension strictement supérieur à n, alors M sera dite *n-triviale.*

Une *n-variation* d'une variété M est le remplacement de $D_m \times I$, plongé dans M, par un cylindre d'homologie $(n-1)$-trivial sur D_m *via* l'homéomorphisme identité de $\partial(D_m \times I)$. Les variétés M_1

et M_2 sont *n-équivalentes,* si M_2 peut être obtenu à partir de M_1 par une $(n+1)$-variation. Un invariant de degré au plus n est constant sur une classe de n-équivalence.

Théorème 2. *Les classes de n-équivalence de cylindres d'homologie sur une variété de dimension deux forment un groupe nilpotent de type fini.*

Théorème 3. *Les invariants de degré au plus n classifient les classes de n-équivalence de cylindres d'homologie sur une variété de dimension deux.*

Les classes de n-équivalence d'enchevêtrements purs à m brins forment un groupe, voir [3], [4]. Le théorème suivant est un analogue complet du théorème *4* pour les enchevêtrements purs.

Théorème 4. *Les invariants de degré au plus n classifient les classes de n-équivalence d'enchevêtrements purs.*

Notons String(m,n) le groupe des classes de n-équivalence d'enchevêtrements purs à m brins qui sont 1-triviaux. Notons $HC(m,n)$ le groupe des classes de n-équivalence des cylindres d'homologie sur D_m. Soit Co : String$(m,n+1) \to HC(m,n)$ l'homomorphisme qui à chaque enchevêtrement pur associe son complémentaire. Soit In : $HC(m,n) \to HC(0,n)$ l'homomorphisme défini par l'attachement d'anses d'indice 2 sur D_m, ce qui donne D^2. Soit Ad : $HC(0,n) \to HC(m,n)$ l'homomorphisme défini par l'attachement de $D_{m+1} \times I$ le long de leur bord $S^1 \times I$.

Théorème 5. *Ces applications forment une suite exacte scindée*

$$String(m,n+1) \xrightarrow{Co} HC(m,n) \underset{Ad}{\overset{In}{\rightleftarrows}} HC(0,n) \to 0$$

Les théorèmes *3 - 5* réduisent un calcul des classes de n-équivalence d'anses pleines d'homologie, à l'évaluation d'invariants de types finis d'enchevêtrements purs et de boules d'homologie entière.

Pour des variétés quelconques, la structure des classes de n-équivalence est plus complexe. Le dernier théorème de ce papier identifie ces classes avec une certain classe double dans un groupe de classes de n-équivalence de cylindres d'homologie. Cette identification est fondée sur une construction comparable aux scindements de Heegaard.

Throughout this paper by a manifold we mean a smooth oriented compact connected manifold (may be with boundary and corners).

1. Y-Surgery

We call the graph $\Gamma \subset \mathbb{R}^3$ shown in Figure 1.a by a *standard Y-graph.* The edges of Γ are framed with a vector field normal to the plane of the picture. A framed graph C in a 3-manifold M is called a *Y-graph* if it is the image of Γ under a smooth embedding $\phi_C : V \to M$ of a neighborhood V of Γ. The embedding ϕ_C can be recovered from C up to isotopy. Let L be a 6-component link in V shown in Figure 1.b. All components of L are 0-framed. The surgery of M along the framed link $\phi_C(L)$ is called a *surgery along C,* or a *Y-surgery.* The resulting manifold is defined up to a diffeomorphism, which is the identity outside of a small neighborhoood of the Y-graph. Denote it by $Y(M,C)$. Equivalent surgeries, but with different initial data, were used by several authors (Borromean surgeries [9], blinks [5]).

One can realize a Y-surgery by removing a handlebody of genus 3 followed by attaching it back in another way. This does not change the boundary of the manifold. The groups $H_i(M,k)$, where $k \subset \partial M$, are naturally identified with the corresponding groups for $Y(M,C)$. The identification preserves all the natural homomorphisms relating these groups to each other and to the homology of the boundary, as well as the intersection numbers, and the linking numbers in the torsion subgroup of 1-homology. If M is a Spin-manifold, $Y(M,C)$ is naturally provided with a Spin-structure. Note that Y-surgeries can change the triple cup product.

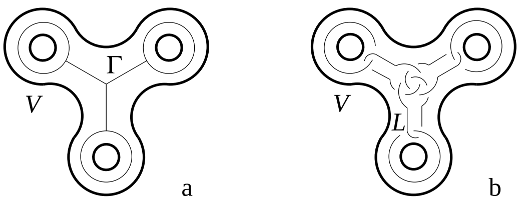

FIGURE 1

Vice versa, let the boundaries of manifolds M_1 and M_2 be identified: $\partial M_1 = \partial M_2 = \partial$. Let $P \subset \partial$ be a set, which intersects each component of ∂ in a single point. Suppose there are isomorphisms $H_1(M_1, k) \to H_1(M_2, k)$, for k being either ∂, or all subsets of P. Assume that these isomorphisms preserve all structures mentioned above. In the case of closed manifolds, it is sufficient just to fix an isomorphism in the first homology group preserving the linking form. In the case of homology handlebodies, it suffices just to identify the boundaries. Let M_1, M_2 be equipped with Spin-structures, which agree on the boundary. Then one can prove (cf. [9]) that by a sequence of Y-surgeries M_1 can be transformed into M_2, in the following sense. The sequence of Y-surgeries applied to M_1 gives rise to a manifold M_3, whose homology is identified with the homology of M_1. There is a Spin-diffeomorphism between M_3 and M_2, which is identical on the boundary and induces the desired isomorphisms in homology. The class of manifolds which can be transformed to each other by Y-surgeries is called a *0-equivalence class.*

2. **Finite Type Invariants**

Let E be a finite collection of pairwise disjoint Y-graphs in M. Then one can make subsequent Y-surgeries along the graphs of E. The result does not depend on the order of Y-surgeries and will be denoted by $Y(M, E)$. Let such a collection be subdivided into r disjoint subcollections $S^{(1)}$, … , $S^{(r)}$. The ordered set $S = (S^{(1)}, \dots, S^{(r)})$ is called a *variation scheme of dimension r on M.* For $\sigma = (\sigma_1, \dots, \sigma_r) \in \{0,1\}^r$, put $S(\sigma) = \cup_{i:\sigma_i=1} S^{(i)}$ and $[S(\sigma)] = Y(M, S(\sigma))$.

Let ν be an invariant of 3-manifolds with values in an abelian group. This can be an invariant with respect to all diffeomorphisms, but we would rather fix a narrower class of diffeomorphisms preserving certain additional structures, which are preserved by Y-surgeries. In particular, one can fix the boundary and consider the diffeomorphisms which are trivial on the boundary. One can also consider the diffeomorphisms which respect fixed isomorphisms of the homology groups to some abstract groups. Additionally, one can consider Spin-diffeomorphisms. For a variation scheme of dimension r on M, put $\nu(S) = \sum_{\sigma\in\{0,1\}^r} (-1)^{|\sigma|} \nu([S(\sigma)])$, where $|\sigma| = \sum_{i=1}^r \sigma_i$. Here we assume that $\nu(M)$ (and hence each of $\nu([S(\sigma)])$) is defined.

The invariant ν is called an *invariant of finite type*, if there exists d, such that $\nu(S) = 0$ for any variation scheme of dimension greater than d. The minimal such d is called the *degree* of ν and is denoted by $d(\nu)$.

Let us compare this notion of finite type invariants with the other ones. For $\mathbb{C}$-valued invariants of integer homology spheres denote by d^{Oh} the Ohtsuki degree [10], and by d^G the Garoufalidis degree [2].

Theorem 1. *The above definition of finite type invariants is equivalent to the definitions by Ohtsuki and Garoufalidis. The degrees are related as follows:* $2d^{Oh} = 3d = 6d^G$.

This answers an old question [2] about the equivalence of the notions of finite type invariants due to Ohtsuki [10] and Garoufalidis [2]. Recently, Cochran and Melvin [1] extended the definition of Ohtsuki to arbitrary 3-manifolds. Their notion of finite type invariants is not equivalent to the one introduced in this paper. The definition of Cochran and Melvin imposes restrictions only on the values of an invariant on manifolds with the same cohomology ring, contrary to the definition above. The approach of this paper is equivalent to the approaches of Garoufalidis and Levine [5] and Habiro [6].

3. The n-Equivalence

Let F be an oriented smooth compact connected 2-manifold. A *homology cylinder over F* is a manifold, which is 0-equivalent to $F \times I$. The boundary of a homology cylinder over F is identified with $\partial(F \times I)$. Two homology cylinders can be composed by identifying the copy of $F \times 1$ in the first of them with the copy of $F \times 0$ in the second.

Let D_m be a disk with m holes. Denote by $HC(m)$ the set of homology cylinders over D_m considered up to diffeomorphisms, which are identity on the boundary. A variation scheme S of dimension r on $M \in HC(m)$ is called *destroying,* if $[S(\sigma)]$ coincides with $F \times I$ for each nontrivial $\sigma \in \{0,1\}^r$. If $M \in HC(m)$ admits a destroying variation scheme of dimension greater than n, then M is called *n-trivial.*

By an *n-variation* of a manifold M we will mean a replacement of $D_m \times I$, embedded into M, by an $(n-1)$-trivial homology cylinder over D_m via the identity homeomorphism of $\partial(D_m \times I)$.

Some obvious properties of n-variations are:

1. A 1-variation is the same as an arbitrary sequence of Y-surgeries.
2. An n-variation is a k-variation for any $k < n$.
3. A composition of several n-variations is an n-variation.

The following proposition is a key property of n-variations.

Proposition 2. *After any n-variation, one can apply another n-variation, such that the final result is the initial manifold.*

Manifolds M_1 and M_2 are called *n-equivalent,* if M_2 can be obtained from M_1 by an $(n+1)$-variation. All invariants of degree at most n take the same values on n-equivalent manifolds.

4. The Main Theorems

Theorem 3. *n-equivalence classes of homology cylinders over an oriented smooth compact connected 2-manifold form a finitely generated nilpotent group.*[1]

Theorem 4. *Finite type invariants of degree at most n classify homology cylinders over an oriented smooth compact connected 2-manifold up to n-equivalence.*

Homology cylinders over a compact 2-manifold with a non-empty boundary are homology handlebodies. Various decompositions of the handlebody into products $F \times I$ define various group structures in the set of n-equivalence classes of homology handlebodies. The structure corresponding to $F = D_m$ admits the simplest description. Denote the corresponding group (which consists of n-equivalence classes of homology cylinders over D_m) by $HC(m,n)$.

Classes of n-equivalence of m-string links form a group, see [3], [4].[2] The following theorem is a complete analogue of Theorem *4* for string links.

Theorem 5. *Finite type invariants of degree at most n classify string links up to n-equivalence.*

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[1]An equivalent result was also announced by Habiro [6].

[2]In [3] string links are called tangles due to translator's mistake.

Denote by $String(m,n)$ the group of n-equivalence classes of 1-trivial m-string links. Let $Co : String(m,n+1) \to HC(m,n)$ be the homomorphism which corresponds to the transition from a string link to the complement of its neighborhood. Let $In : HC(m,n) \to HC(0,n)$ be the homomorphism defined by attaching 2-handles, which turn D_m into D^2. Let $Ad : HC(0,n) \to HC(m,n)$ be the homomorphism defined by attaching $D_m \times I$ along the boundary.

Theorem 6. *There is an exact splitting sequence*

$$String(m,n+1) \overset{Co}{\to} HC(m,n) \underset{Ad}{\overset{In}{\rightleftarrows}} HC(0,n) \to 0$$

The kernel of Co can be explicitly described by a list of its generators. We do not go into details because of complexity of this description.

Theorems *4*, *6* and *5* reduce a calculation of n-equivalence classes of homology handlebodies to an evaluation of finite type invariants of string links and integer homology balls. For string links there is the universal Kontsevich invariant [7]. Furthermore, n-equivalence classes of homology balls and homology spheres are in one-to-one correspondence and for the homology spheres there is the univrsal Le-Murakami-Ohtsuki invariant [8]. However these universal invariants take values in $\mathbb{C}$, and it would be desirable to have universal invariants with values in other domains. For a complete combinatorial description of n-equivalence classes of homology handlebodies we miss similar invariants with values in finite groups.

For arbitrary manifolds, the structure of n-equivalence classes is more complicated. One can assume that the boundary is not empty, since removing a ball does not change the structure of these classes. Consider Spin-manifolds with fixed boundary and homology. Fix isomorphisms of the homology groups onto some abstract groups. Consider the 0-equivalence class U of manifolds with these structures.

Represent some manifold M from U by a sort of Heegaard splitting. Namely, let $M = A \cup (F \times I) \cup B$, where F is a surface with non-empty boundary, A is attached to $F \times 0$ and B to $F \times 1$. It is assumed that $C = A \cup (F \times I)$ and $D = B \cup (F \times I)$ are handlebodies and $H_1(C, F \times I) = H_1(D, F \times I) = 0$. Any manifold can be obtained by a replacement of $F \times I$ by a homology cylinder over F.

Denote by $\mathcal{F}$ the group of n-equivalence classes of homology cylinders over F. Denote by $\mathcal{A}$ and $\mathcal{B}$ the subgroups of $\mathcal{F}$ such that $x \in \mathcal{A}$ (respectively, $x \in \mathcal{B}$), if $x \cup A$ (respectively, $x \cup B$) is n-equivalent to $F \times I \cup A$ (respectively, $F \times I \cup B$).

Theorem 7. *The construction above defines a map of $\mathcal{F}$ onto the set $\mathcal{U}$ of n-equivalence classes of manifolds from U. The preimage of each element of $\mathcal{U}$ is a double coset of the form $\mathcal{A}x\mathcal{B}$ with $x \in \mathcal{F}$.*

Notice that x, $x \cup A$ and $x \cup B$ are homology handlebodies.

Remark 8. Similarly to Theorem *7*, one can relate n-equivalence classes of knotted graphs to double cosets in the groups of string links.

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