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VARIATIONS OF KNOTTED GRAPHS. GEOMETRIC TECHNIQUES OF n-EQUIVALENCE

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Dedicated to the memory of V. A. Rokhlin

ABSTRACT. The theory of variations of knotted graphs is a geometric calculus intended for the study of 3-manifolds and 1-dimensional objects embedded in them in the context of invariants of finite type and related theories. The paper presents a self-contained exposition of this theory with complete proofs.

The paper systematically presents a theory that I developed in 1995 in a series of talks at the topology seminar of the Uppsala University. The text was written at the same time. However, when I was working on the text, some circumstances came to light that made me to delay the publication. In particular, it turned out that the notion of variation of a knotted graph can be generalized slightly, after which it can be described by external properties. This description does not use the construction that is employed in the present exposition and seems to be chosen quite arbitrarily.

Naturally, I have attempted to modify the presentation with regard to the new approach. As a result, I discovered that all external results can be presented even in a simpler way, without employing the specific constructions of the present paper (see [G2]). Thus, all powerful techniques developed here turned out to miss worthy applications.

Two reasons have made me to return to the old text. First, the study of n-equivalence classes of links and 3-manifolds (the same techniques work perfectly well for the latter) has reached the stage of specific computations. Here we cannot avoid constructive descriptions, and the techniques suggested turn out to be efficient.

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*Mikhail Nikolaevich Gusarov perished tragically on the 25th of June, 1999. He drowned in the sea during a visit to Tel Aviv. This extremely absurd death happened when he was at the very peak of his powers.

Gusarov has written a series of brilliant and deep papers on the theory of invariants of finite type. Strictly speaking, he also deserves the credit for discovering the very concept of an invariant of finite type. In 1987, Gusarov embarked on a study of the knot-theory invariants known nowadays as Vasil′ev invariants. Starting with absolutely different grounds, V. A. Vasil′ev rediscovered this class of invariants one year later.

Gusarov's paper brought to your notice is published without any substantial textual changes. Nevertheless, the preparation of publication has turned out to be not an easy task, because practically all the numerous pictures had to be made anew. Mostly, they existed only in the form of sketches, and some fragments had to be recovered from the text. Figures 1–12, 20, 21, and 23–32 were made by A. N. Shumakovich, and Figures 13–19 and 22 by O. Ya. Viro.

The subject of the paper is a geometric calculus of modifications of 3-manifolds. The axes of the modifications are graphs with vertices of valency 1 and 3. In other words, the modifications are performed along embedded Feynman diagrams. The more complicated is a graph, the more carefully does the modification along the graph treat the manifold: the higher is the degree of invariants whose values may happen to be changed under the action of the modification. Thus, Gusarov's calculus of modifications (*perekleĭkas*) turns out to be dual to the theory of invariants of finite type.

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Second, an unexpected circumstance makes one to surmise that the basic construction presented below is far from being as accidental as this seems at first glance. Absolutely independently, another researcher (see [H]) succeeded in following nearly the same route. Habiro's clasper theory is equivalent to the theory of n-variations. However, the method of description and, what is even more important, the specific method of drawing pictures are somewhat different. Thus, there are two description languages, which admit (for the present!) precise and easy translation into each other, but they may turn out to be far from being equally convenient.

What follows is my text of 1995. It seemed to me impossible to modify it after substantial change of the point of view without destroying the living tissue of the paper. Being self-contained, the text has not become out of date, though, certainly, now it brings up many new associations. In Remark 10.7, I have only omitted an announcement, which was not confirmed. In general, now I am much less enthusiastic about partly-defined invariants (§10) than before.

Introduction

In the paper, by a *knotted graph* K we mean a smooth embedding of a graph in a 3-manifold M; the embedding is considered up to a smooth isotopy that can be extended to a neighborhood of the image of K (which we also denote by K). Furthermore, we can require that the isotopy do not change the image of K in a neighborhood of a certain compact subset of M.

§1 is devoted to Y-modifications. This 3-dimensional construction is a technical basis for the entire paper. It may seem that everything else is a superstructure over this operation. This is not quite so. Actually, the Y-modifications appeared when nearly everything else already existed. However, they quite successfully formed the basis of the construction, allowing us to simplify both the definitions and the proofs quite substantially. A promising fact is that the Y-modifications have come from another world, the world of modifications of 3-manifolds. But even here, where they are trivial from the 3-dimensional point of view, they turn out to be a very powerful tool.

In §2 we describe the n-variations of knotted graphs, which are the main object of study in the paper. They are treated as sufficiently subtle operations, which do not change coarse properties of the embedding, in particular, the homotopy type and the invariants of low degrees. Nevertheless, Theorem 5.1 shows that in fact they can also be regarded as modifications. However, in the paper we do not develop the 3-dimensional direction any further.

In §3 we extend the definitions towards more coarse operations. The I-modifications and 1-variations constructed here may be not of much interest on their own, but they make natural the consideration of all what lies in the limits of a given homotopy type. Furthermore, they are a bridge to more usual methods of describing operations.

§4 is devoted to operations with modifications. Though, in principle, the statements are also suitable for the study of manifolds, the techniques are developed in the direction required for the study of embeddings of graphs; these techniques have been brought to the level of a formal calculus of pictures.

The questions treated in §5 are subtle and, as yet, unclear in many respects. That section is practically out of use if we deal with the Euclidean space or the sphere. Furthermore, since n-variations are local (including those occurring inside a ball), §5 is also not very useful for the study of isolated operations. Actually, §5 must be used when the operations interact globally in the manifold. In fact, §5 gives precise meaning to the subsequent formulations, which makes them suitable for such situations. The reader who is not going to employ the techniques in question in similar situations and does not care about detailed justifications will spare much effort if he contents himself with the first impression of what similar axes are, and does not look more closely into the subtleties related to fixing disks when performing operations defined up to isotopy.

§6, which is a logical continuation of §4, lifts the operations to the level of n-variations. The text here should be regarded as an explanation to the pictures, rather than an independent presentation illustrated by them. However, the degree of formalization here is lower than in §4, as evidenced by the dotted frames in the pictures. The frames hide echoes of the problems touched on in §5. The reader may give no attention to them during the first acquaintance, but we must not forget about them when putting the techniques in use. The problems disappear only if we act at the level of the first approximation. Then, practically, the n-variations become "diagrams of Chinese characters", being their 3-dimensional analog. Nine theorems of this section demonstrate the high plasticity of the new techniques, and, probably, many readers will have associations from adjacent fields when looking at the pictures. Certainly, the incompleteness of formalization hinders employing the techniques, but even in this form they can serve as an efficient tool.

§7 is devoted to the notion of n-equivalence, which is principal from the conceptual point of view. All the above gives this notion a powerful technical basis. From the visual point of view, it is more convenient to interpret n-equivalence in terms of a very simple operation of throwing a loop of order n (see Definition 7.4 and Theorem 7.5).

In §8, we deal with variation schemes. This is a certain evolution of a quite efficient notion introduced earlier in [G1]. To a certain extent, variation schemes allow us to algebraize the techniques developed here. They also serve as a bridge to truly algebraic techniques of n-equivalence, which, however, is beyond the scope of the paper.

In §9, we apply the techniques developed above to string links. In particular, we prove that the n-equivalence classes of string links constitute a solvable group. In principle, this statement has already been verified with the help of the previous definition of variation schemes, but the proof was considerably more complicated, and it was published only for knots [G1]. Theorem 9.3, which is also important on its own, shows that for knots the new version of the definition of n-equivalence is equivalent to the previous one.

§10 relates n-equivalence to invariants of finite degree, which are the most popular notion in this field. From this point of view, two embeddings are n-equivalent if and only if they cannot be distinguished by invariants of degree n. In §10 this statement is refined. In particular, we are to somewhat widen the class of invariants under consideration: we allow them to be partly defined.

§1. Y-MODIFICATIONS

Definition 1.1. Let M be a 3-manifold, and let the graph Γ depicted in Figure 1a) (a Y-graph) be smoothly embedded in M. The edges at the central vertex make angles of $120°$. The ends of the closed edges merge at the vertices into smooth curves orthogonal to the nonclosed edge. Each closed edge is framed by a normal vector field that is tangent to the nonclosed edge at the vertex. The nonclosed edges are framed by normal lines that are coplanar with the tangent lines of the edges at the vertices. The manifold M is oriented in a neighborhood of Γ.

We extend the embedding to a diffeomorphism mapping the neighborhood U of Γ depicted in Figure 1b) onto a regular neighborhood U' of the image of Γ and preserving the orientation and the framings. Such a diffeomorphism is determined uniquely up to an isotopy fixed on Γ. The diffeomorphism transfers the three curves depicted on the boundary of U into M. (The curves are obtained from one another by rotation, and one of them is separately depicted in Figure 1c).) We cut out U' from M, attach 2-handles along the indicated curves, and then paste in a 3-handle. This yields a new manifold.

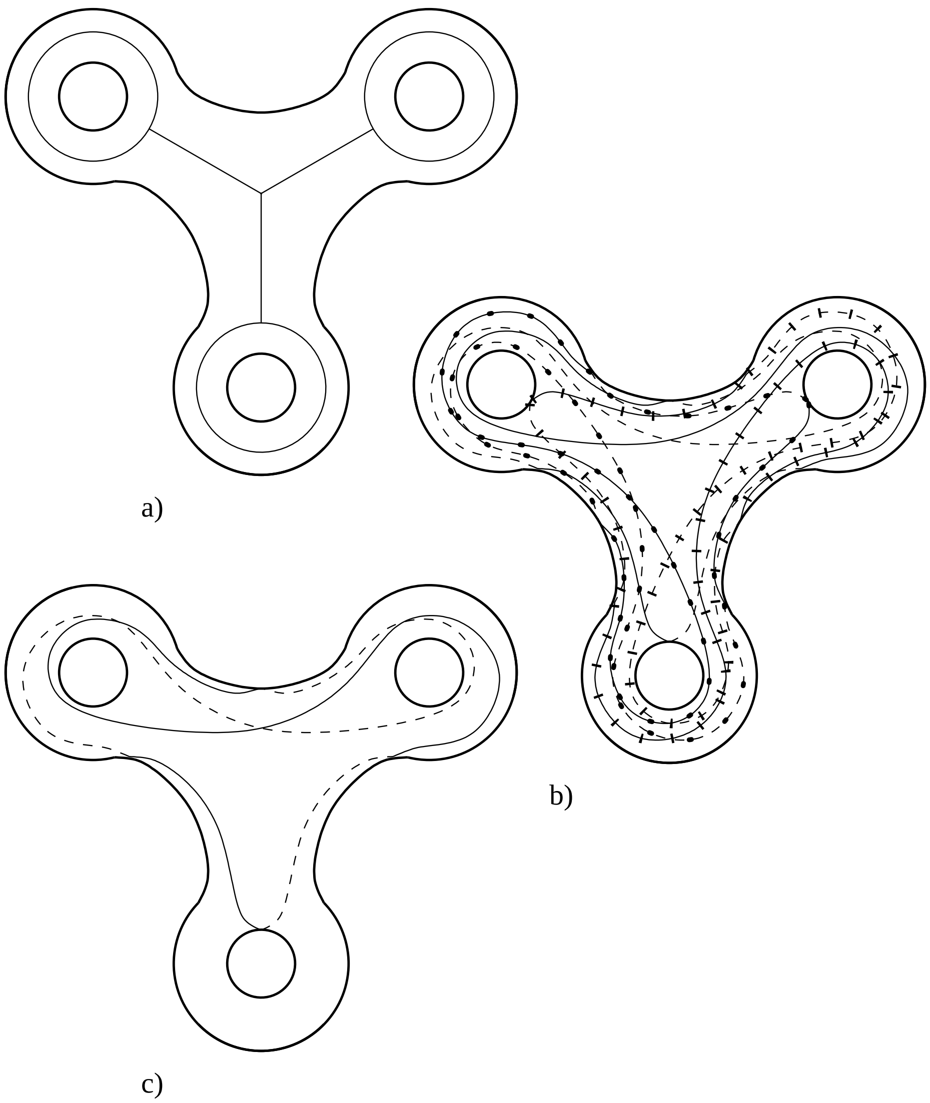


FIGURE 1

This procedure is the Y-*modification* of M in a neighborhood of Γ.

Theorem 1.2. *Suppose at least one of the loops of a* Y*-graph* Γ *bounds a smooth embedded disk* D *in* M *that is tangent to the framing and does not touch* Γ *by its inner points. Then the* Y*-modification in a neighborhood of* Γ *yields a manifold diffeomorphic to* M*. Furthermore, up to an isotopy fixed on* Γ*, there exists a unique diffeomorphism that is identical outside a regular neighborhood* V *of the set* $\Gamma \cup D$*.*

Proof. Let the disk D span the lower loop in Figure 1b). Then the complement of a regular neighborhood of Γ is obtained by attaching a 1-handle to $M \setminus V$. Obviously, the 1-handle cancels with the 2-handle attached along the curve depicted in Figure 1c). The remaining two 2-handles are attached along the curves depicted in Figure 2a); after filling in the lower hole, these curves have become isotopic to the meridians (Figures 2b)–c)). Now, it is obvious that we can choose the handles in M, and up to isotopy this can be

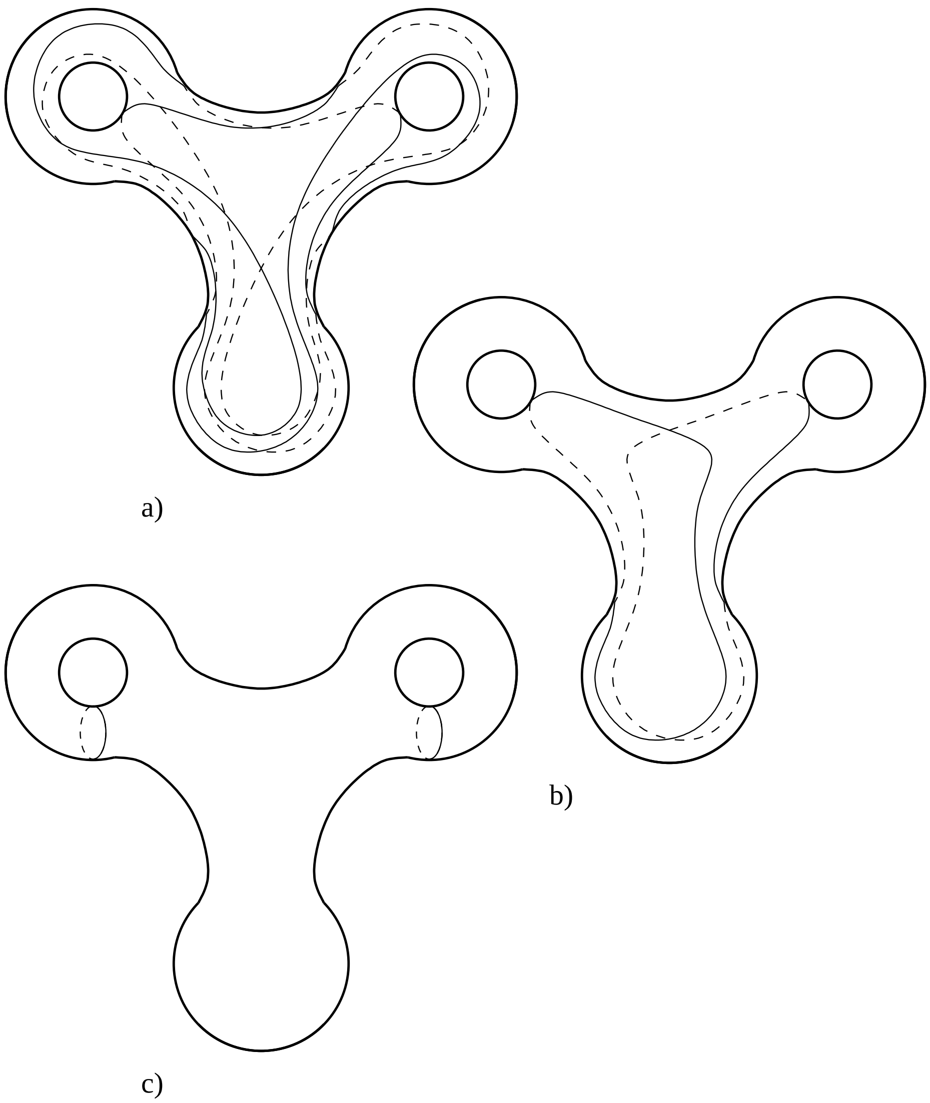


FIGURE 2

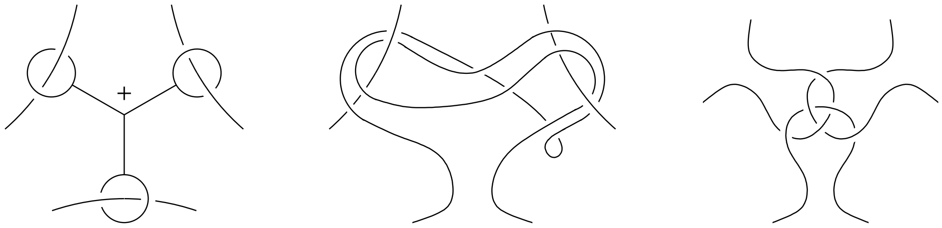
done in a unique way. □

Remark 1.3. Let all three loops of Γ be small and unknotted and have zero framings. Assume that there are three arcs of K linked with these loops (Figure 3a)). If we perform a Y-modification in a neighborhood of Γ and identify the result with M, then the arcs will be linked together (Figure 3b)). Certainly, all three arcs are equipollent, and we obtain the Borromeo link (Figure 3c)).

§2. n-VARIATIONS

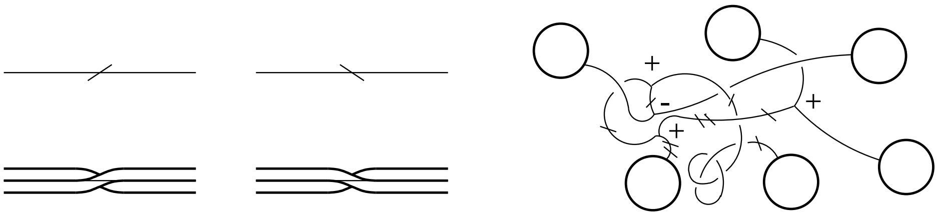
Definition 2.1. For $n > 1$, an *n-variation axis* (also an *axis of an n-variation*, a *variation axis of order n*, or simply an *n-axis*) is a graph T of a special form, smoothly embedded in a 3-manifold M and framed in a special way. More precisely, the following conditions must be fulfilled.

(1) The graph T has $2n$ vertices, each of which is 3-valent.



a) b) c)

FIGURE 3



a) b)

FIGURE 4

(2) If from T we delete all loops, i.e., the edges starting and ending at one and the same vertex, a tree remains. In what follows, the vertices incident with loops are said to be *terminal*, and the other vertices are said to be *inner*.
(3) All loops bound smoothly embedded mutually disjoint disks; the latter do not touch T by their inner points. The disks are given up to isotopy, so that it is unnecessary to fix them if, for instance, M is a sphere.
(4) The tangent lines of the edges at an inner vertex make angles of $120°$. At a terminal vertex, the nonclosed edge is orthogonal to the ends of the closed edge, which merge into a smooth curve.
(5) Each nonclosed edge is framed by pairs of normal mutually orthogonal lines ("crosses").
(6) At each of the vertices, one of the framing lines is orthogonal to the other edges.
(7) If the endpoints of an edge are inner vertices, then the corresponding two lines indicated in condition (6) belong to different families in the framing. For the edges containing a terminal vertex, such lines belong to one family.
(8) The manifold M is (locally) oriented at each of the inner vertices. If M is oriented, then it suffices simply to put a sign $+$ or $-$.

In the pictures, we assume that one of the framing lines remains in the viewing plane nearly all the time, and only near the oblique strokes does the framing obtain a quarter of a twist (Figure 4a)). An example of a 5-axis is depicted in Figure 4b).

Definition 2.2. Let M be a 3-manifold, K a knotted graph embedded in M, and T an n-axis in M disjoint with K. Then the *n-variation of K along T* is performed as follows.
(1) We fix some disks spanning the loops of the axis T, take the union of T and the disks, and choose a regular neighborhood B of the union. Obviously, this is a ball, and all subsequent constructions will be done inside B.
(2) We cut all edges joining the inner vertices of T; this splits T into four-vertex graphs. Attaching small loops to the newly formed vertices, we obtain $n+1$ Y-graphs. We link the loops together near the cutting points (Figure 5).

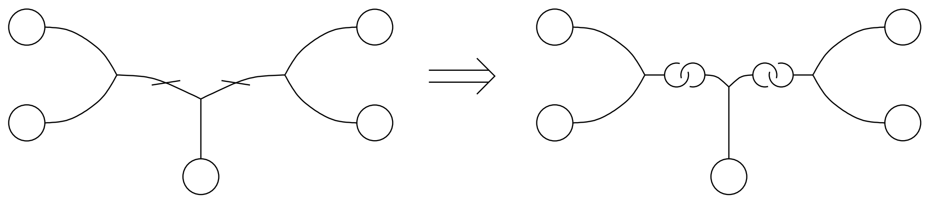


FIGURE 5

(3) The edges of the Y-graphs are framed by the corresponding lines framing the axis T. The choice of the lines is made by extending from the central vertex. Under a proper choice of the planes in which the loops are added, the possibility of such a framing is guaranteed by condition (7) of Definition 2.1. Each of the closed edges acquires a trivial framing (more precisely, trivial with respect to the ball B).
(4) We perform a Y-modification in a neighborhood of each of the $n+1$ Y-graphs obtained. The required orientations are marked at the corresponding vertices of the axis T. The neighborhoods to be modified do not touch the graph K.
(5) The ball B chosen at step (1) remains a ball. Therefore, we can identify the manifold M with the result of the modification in such a way that the identification is identical outside B.

After such an identification, we obtain a modified embedding of the graph K.

Proof of consistency. At each step we have some freedom, but, clearly, this changes nothing up to isotopy. Hence, it suffices to prove that, after all modifications, the ball B remains a ball (step (5)). We prove this by induction on the number of the inner vertices of the axis T. Take one of the Y-graphs containing a terminal vertex of T. By Theorem 1.2, we may assume that in a neighborhood of this Y-graph the modification does not change either the ball B or the other Y-graphs. Hence, the remaining modifications can be assembled into one or two variations, depending on how many, two or one, terminal vertices are contained in the Y-graph chosen. By the induction hypothesis, these variations preserve certain balls containing their axes. The balls can be taken mutually disjoint and lying in B. The latter condition is required in order that the variation performed first could not interfere with the second variation. □

Remark 2.3. Requirements of conditions (2) and (3) of Definition 2.1 are redundant for Definition 2.2. For its consistency, it would suffice that every connected component of T have at least one loop bounding a disk (cf. Theorem 2.5). This would yield modifications which have much in common with n-variations. However, operations with such modifications are described in a much more complicated way.

Remark 2.4. If we completely renounce condition (3) of Definition 2.1, then we obtain some operations that change the manifold M, but bear close similarity to n-variations. This approach leads to a definition of an n-equivalence of 3-manifolds with the properties that closely resemble those of the n-equivalence of knotted graphs. We note that here $(n+1)$-variations should be regarded as n-variations.

Theorem 2.5. *Let D be one of the disks spanning the loops of an n-axis T. Then we may assume that the variation with axis T changes the graph K only near $T \cup D$.*

Proof. We order the inner vertices of T so that the shortest path (along T) from any vertex to the disk D pass only through vertices with smaller numbers. After that, we perform the modifications in the order of numbers of their central vertices. Applying Theorem 1.2 every time, we obtain the required result. □

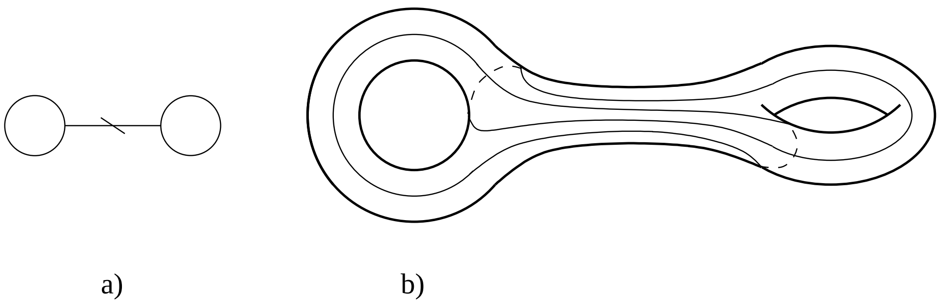


FIGURE 6

Corollary 2.6. *If one of the loops of the axis of an n-variation is not linked with a graph (i.e., the disk spanning the loop does not intersect the graph), then we may assume that the variation leaves the graph fixed.* □

In what follows, n-variations are also called *variations of order n*.

§3. 1-VARIATIONS

In the preceding section, we defined only variations of order greater than 1. It is easily seen that the definitions in §2 are not suitable for variations of order 1. Indeed, in this case we would be obliged to perform a zero number of modifications, which gives, at best, a trivial transformation. However, precisely the 1-variations are expected to allow us to change the embedding almost arbitrarily. In any case, a regular homotopy fixed in the vicinities of the vertices is expected to admit a transformation by a sequence of 1-variations into a regular isotopy with the same property. The choice of a specific form of 1-variations is dictated by the convenience of extension to variations of higher orders. This explains the order of presentation in the paper.

Definition 3.1. Let M be a 3-manifold, and let the graph Γ depicted in Figure 6a) (an I-graph) be smoothly embedded in M. At each vertex, the ends of the closed edge of Γ merge into a smooth curve orthogonal to the nonclosed edge. Every closed edge is framed by a normal vector field that is tangent to the nonclosed edge at the vertex. The nonclosed edge is framed by pairs of normal mutually orthogonal lines; at the vertices, one of the framing lines is tangent to the closed edge, and at the two endpoints the lines belong to different families. The manifold M is oriented in a neighborhood of Γ.

We extend the embedding of Γ to a diffeomorphism mapping the neighborhood U of Γ depicted in Figure 6b) to a regular neighborhood U' of the image of Γ and preserving the framings. Such a diffeomorphism is determined uniquely up to an isotopy fixed on Γ. The diffeomorphism transfers the two curves depicted on the boundary of U into M. We cut out U' from M, attach 2-handles along the indicated curves, and then paste in a 3-handle. This yields a new manifold. This procedure is the I-*modification* of M in a neighborhood of Γ.

Theorem 3.2. *Suppose that at least one of the loops of an* I*-graph Γ bounds a smooth embedded disk D that is tangent to the framing and does not touch Γ by its inner points. Then the* I*-modification in a neighborhood of Γ yields a manifold diffeomorphic to M. Furthermore, up to an isotopy fixed on Γ, there exists a unique diffeomorphism that is identical outside a regular neighborhood V of $\Gamma \cup D$.*

Proof. As in the proof of Theorem 1.2, the complement of a regular neighborhood of Γ is obtained from $M \setminus V$ by adding a 1-handle, which is a neighborhood of the disk D and

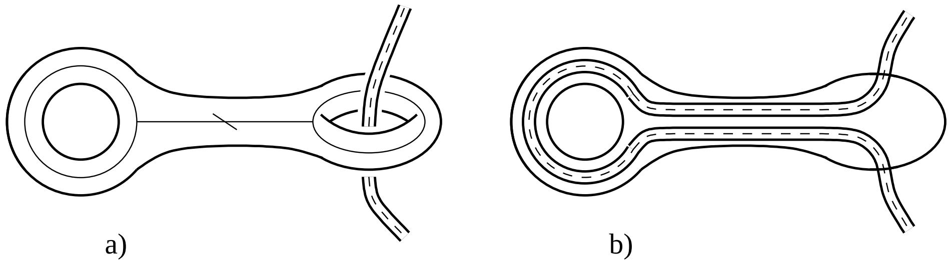


FIGURE 7

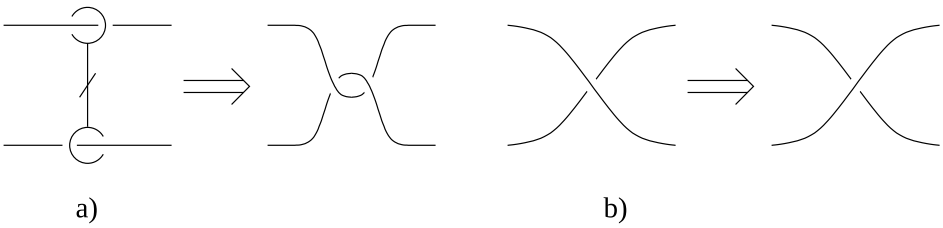


FIGURE 8

cancels with one of the 2-handles. The second 2-handle is a meridian of the handle of V that was not filled in. □

Remark 3.3. Let the graphs to be modified intersect a certain ball with a handle only along several parallel curves, as shown in Figure 7a). Then after an I-modification in a neighborhood of the I-graph shown there (by Theorem 3.2, this operation does not change the manifold), the curves pass as shown in Figure 7b).

Definition 3.4. Let the two loops of an I-graph Γ bound two smoothly embedded disjoint disks that do not touch Γ by their inner points, are tangent to the framings on their boundaries, and are determined up to homotopy. Then Γ is called a 1-*axis*, and the I-modification in a neighborhood U of Γ is called a 1-*variation* with respect to any graph K that does not intersect U.

Remark 3.5. Suppose the disks of a 1-axis intersect a graph K transversally, at one point each. Then, up to immaterial replacement of local objects by infinitesimal objects, the 1-axis is none other than what was called a *simple variation direction* in [G]. The modification (*perekleĭka*) itself coincides with the modification in this direction (in the sense of [G]). The action of a 1-variation is depicted in Figure 8a). If the axis of the 1-variation is orthogonal to the viewing plane, then an under-crossing is simply replaced by an over-crossing (Figure 8b)).

Remark 3.6. Theorem 3.2 implies immediately that Theorem 2.5 and Corollary 2.6 hold true for 1-variations.

§4. GEOMETRIC OPERATIONS WITH MODIFICATIONS

Theorem 4.1 (on cutting modifications crosswise). *The modifications of balls with handles depicted in Figures* 9a) *and* 9b) *lead to the same results as the pairs of modifications depicted in Figures* 9c) *and* 9d), *respectively. This means that the bodies obtained are diffeomorphic, the diffeomorphism can be chosen to be identical on the boundary, and then it is unique up to an isotopy fixed on the graph.*

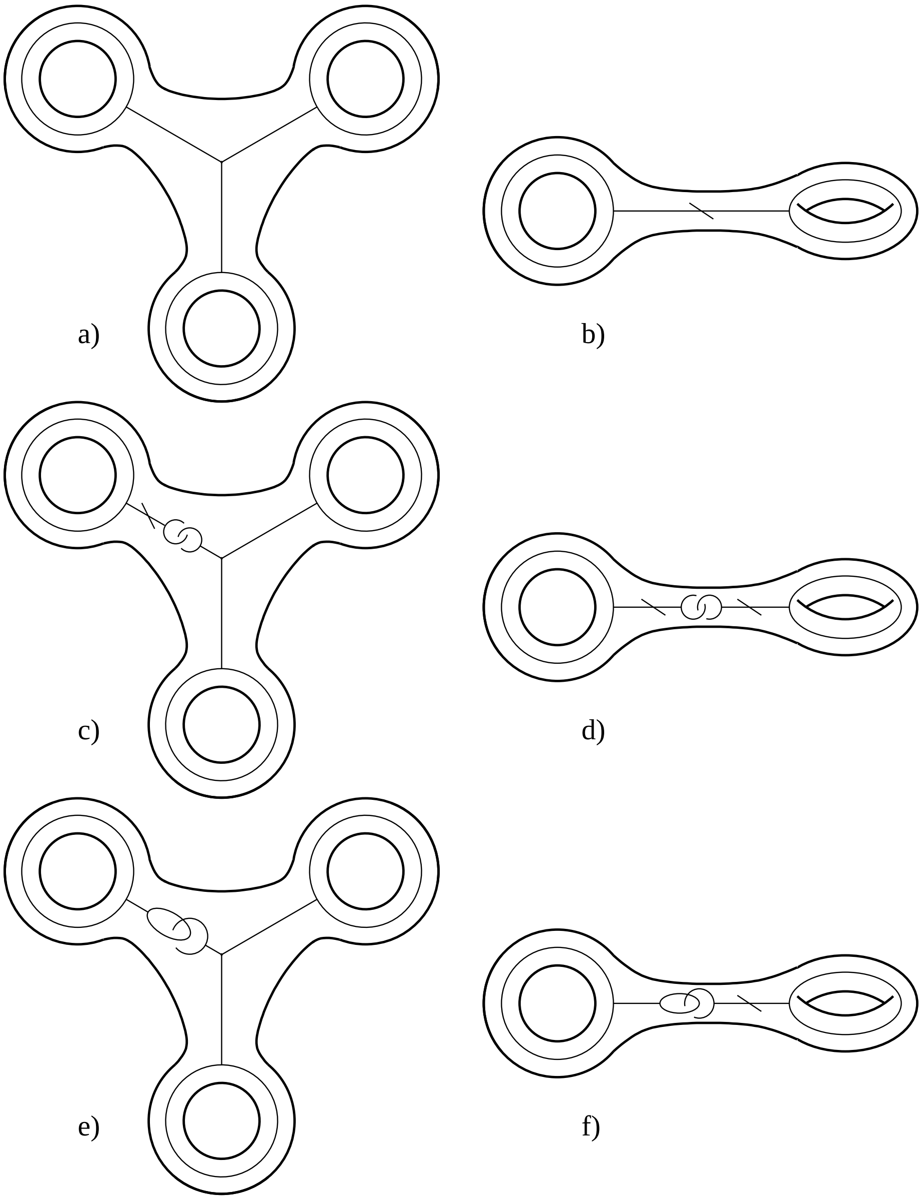


FIGURE 9

Proof. In the second pair of pictures, we apply Remark 3.3 to regular neighborhoods of the left-hand I-modifications with spanned right-hand disks, as depicted in Figures 9e) and 9f), respectively. The result is shown in the first pair of pictures. The required uniqueness follows from the fact that the bodies obtained are also balls with handles, though, certainly, their meridians are different. For the first pair of pictures, this is seen from the definition of a modification. □

Theorem 4.2 (on splitting an I-modification)**.** *In a ball with three handles, the modification depicted in Figure* 10a) *leads to the same result as the two modifications depicted in Figure* 10b)*.*

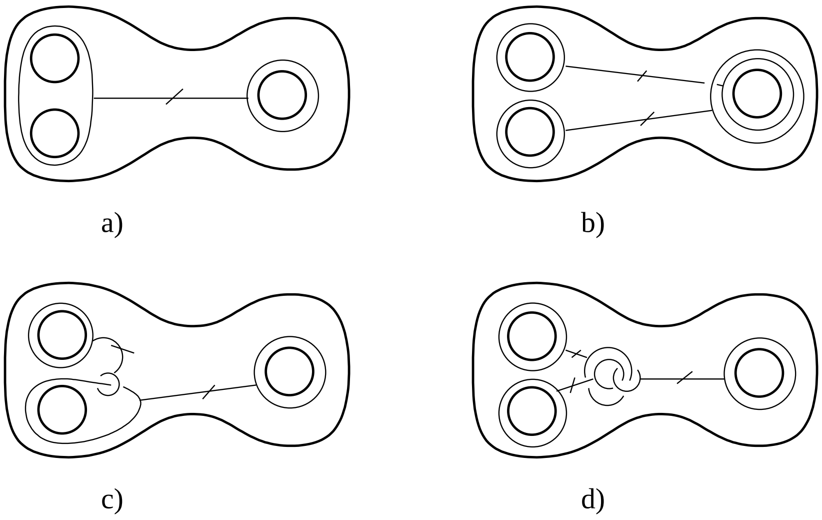


FIGURE 10

Proof. Successively applying Remark 3.3, we obtain Figures 10c)–d)–b) from Figure 10a). Figure 10c) shows that the resulting body H is obtained by attaching one handle to a ball with a handle, so that H is a ball with two handles, and no problems concerning the uniqueness of identification arise. □

Theorem 4.3 (on cutting a Y-modification into I-modifications). *The* Y*-modification depicted in Figure* 11a) *leads to the same result as the groups of* I*-modifications in the following pictures*:
11b) — *symmetric cutting into three* I*-modifications*;
11c) — *asymmetric cutting into three* I*-modifications*;
11d) — *cutting into two* I*-modifications.*

Proof. Thrice cutting up the Y-modification in Figure 11a), we obtain Figure 11e). Using Remark 1.3, we arrive at Figure 11b) and deform the loops as shown in Figure 11f). Now, we see that Figure 11g) is obtained from Figure 11f) by splitting two I-modifications and from Figure 11c) by cutting two I-modifications crosswise. Applying Remark 3.3 to Figure 11c), we obtain Figure 11d). □

Theorem 4.4 (on inversion of an I-modification). *Let U be a regular neighborhood of an* I*-graph. Then we can embed another* I*-graph in U in such a way that the* I*-modifications in neighborhoods of the two* I*-graphs* (*certainly, these neighborhoods are disjoint and contained in U*) *do not change U. Some variants of such an embedding are given in Figures* 12a)–c).

Proof. Figure 12d) shows an I-graph Γ with a contractible loop. A modification in a neighborhood of Γ does not change anything. Splitting Γ as shown in Figure 12e) or in Figure 12f), we obtain the variants depicted in Figures 12a) and b), respectively. Rotating Figure 12b) one-half revolution around the horizontal axis, we obtain Figure 12c). □

Theorem 4.5 (on splitting a Y-modification). *In a ball with four handles, the* Y*-modification depicted in Figure* 13a) *leads to the same result as the two* Y*-modifications depicted in Figures* 13b) *or* 13c).

Proof. We cut the Y-modification into two I-modifications (Figure 13d)), add two mutually inverse I-modifications (Figure 13e), cf. Figure 12c)), and assemble the pairs of

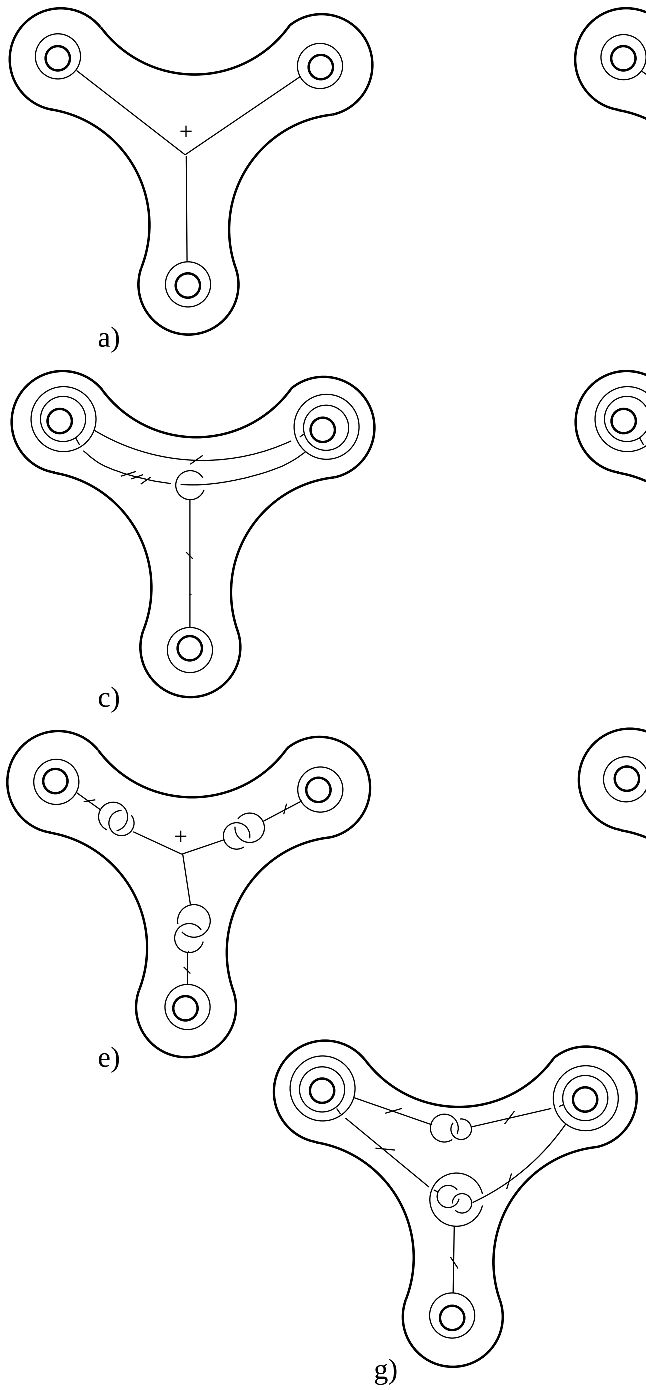


Figure 11

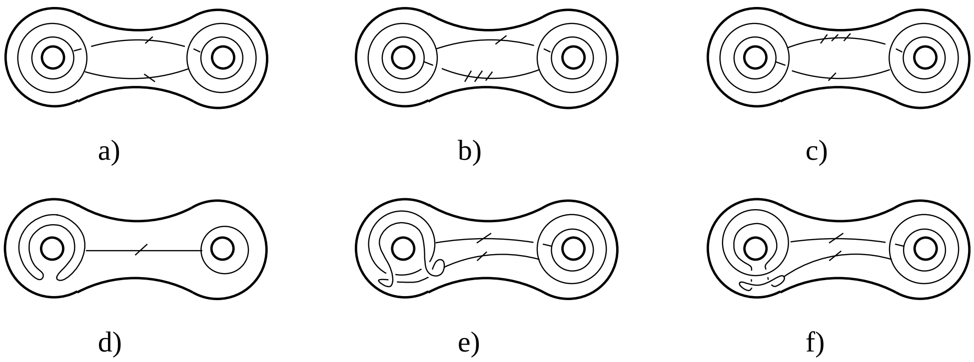


FIGURE 12

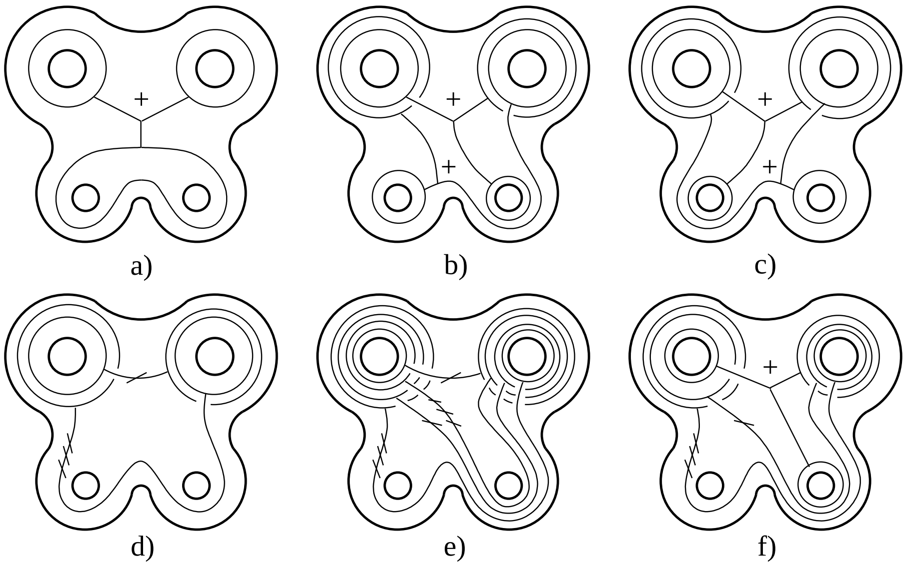


FIGURE 13

I-modifications into Y-modifications (Figures 13f), b)). If, moving counter-clockwise, we interchange the lower holes, then nothing changes in Figure 13a), and Figure 13b) is transformed into Figure 13c). ☐

Remark 4.6. The two halves obtained by cutting the Y-modification are not equipollent. For one of them (the left-hand one in Figure 13b) and the right-hand one in Figure 13c)), the pair of edges connecting the upper loops is linked with one of the lower holes. This half will be called *bad*, and the other will be called *good*.

Theorem 4.7 (on inversion of a Y-modification)**.** *Let U be a regular neighborhood of a Y-graph. Then we can embed another* Y*-graph in U in such a way that the* Y*-modifications in neighborhoods of both graphs do not change U. An example of such an embedding is depicted in Figure* 14a)*.*

Proof. The inverse modification in Figure 14a) is the bad half formed by splitting the Y-modification depicted in Figure 14b). The Y-modification does not change U because its lower loop bounds a disk. ☐

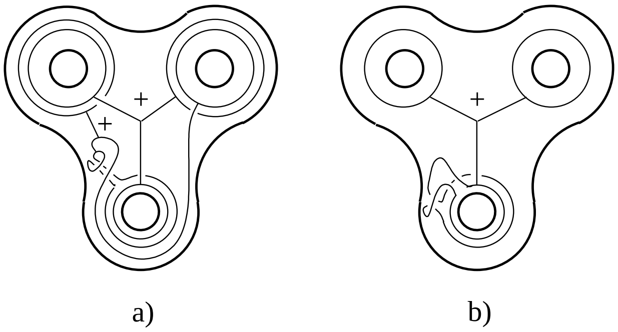


Figure 14

§5. Similar variations

Theorem 5.1 (on the meridians of an n-variation). *Let T be an n-axis. Then a regular neighborhood H of T is a ball with $n+1$ handles (Figure 15a)). On its boundary ∂H, we have two families consisting of $n+1$ simple closed curves each. The curves in one of them (called parallels), are the curves along which ∂H intersects the disks spanning the loops of the axis T. The curves belonging to the other family (called old meridians), bound small disks intersecting the corresponding loops of T. We claim that ∂H contains yet another family of $n+1$ mutually disjoint simple closed curves (called new meridians) such that the following conditions are fulfilled:*

(1) *a new meridian intersects only one parallel at exactly one point and does not intersect the corresponding old meridian;*
(2) *after performing the modifications corresponding to the n-variation in H, we obtain a body H' in which the new meridians bound smoothly embedded mutually disjoint disks;*
(3) *cutting H' along the disks indicated in condition (2), we obtain a ball.*

Thus, an n-variation can be described as follows: we cut out a regular neighborhood of a variation axis and, instead of it, paste in another ball with $n+1$ handles.

Proof. For the 1- and 2-variations, everything is as required by definition. We proceed further by induction. We cut one of the inner edges of the axis T as required in condition (2) of Definition 2.2. Then T is subdivided into two smaller axes, and it is obvious that we must perform precisely the modifications corresponding to these two axes. Up to smoothing, a regular neighborhood of the axis T can be cut into regular neighborhoods of the small axes (Figure 15b)). Cutting is performed along a torus S with a hole, which is depicted in Figure 15c) in a different projection. For the convenience of representation, four large holes are made in S. The dotted line u is a parallel in the near body and a meridian in the far body. Vice versa, the dotted line v is a meridian in the near body and a parallel in the far body. By the induction hypothesis, both the near body N and the far body F have systems of new meridians. Each of the new meridians does not intersect either the parallel or the meridian of the new handle, i.e., either u or v. Hence, they can enter the separating surface S by arcs isotopic either to AB, or to CD, or to a segment of the boundary. The arcs of the latter (third) type can be moved away from S. The remaining arcs can be straightened out and made parallel to the curves indicated. Now, the meridian of N corresponding to the new handle passes along CD only once. The remaining meridians of the near body N can pass along AB several times. For the far body F, the situation is opposite: the meridian of the new handle passes once along AB, and the other pass several times along CD. We draw the meridians of N by solid

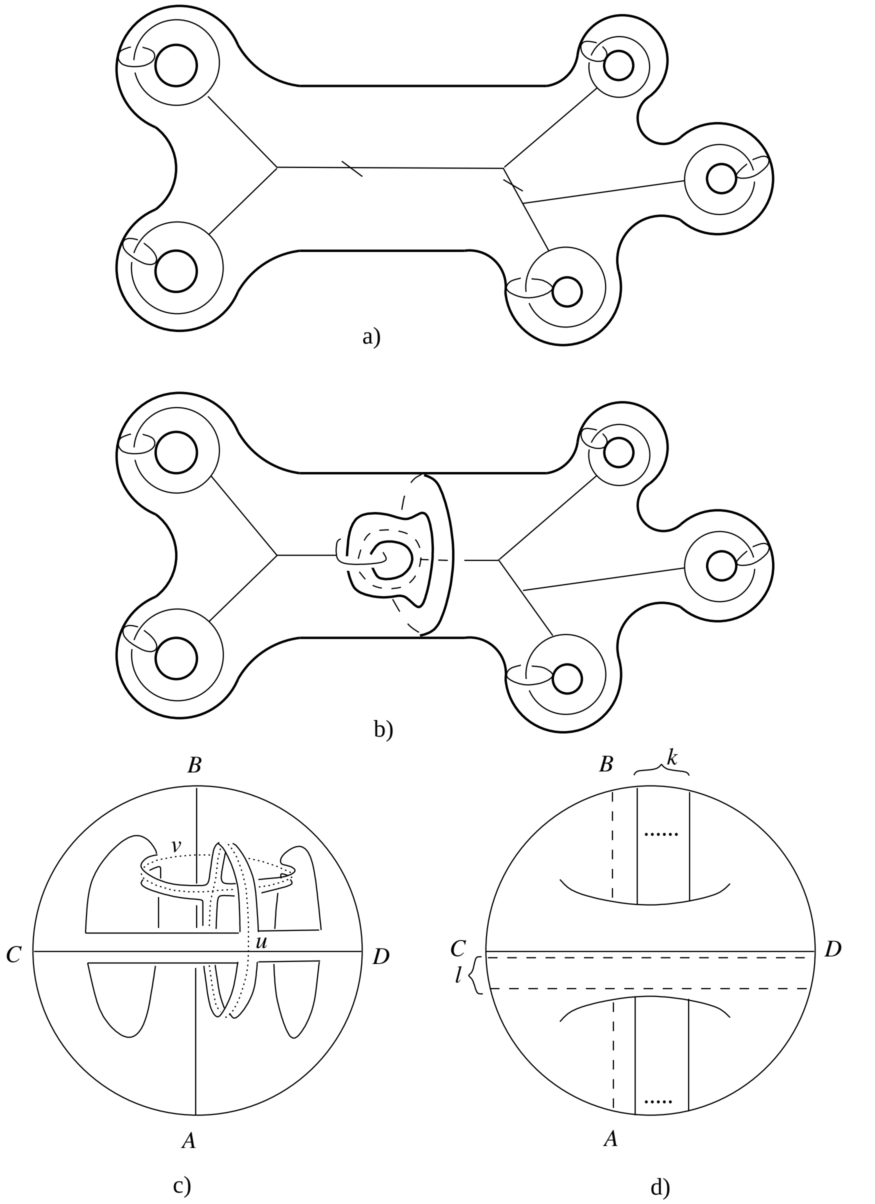


FIGURE 15

lines and those of F by dotted lines (Figure 15d)).

Consider the segments of meridians of the old handles of N that are parallel to AB. Denoting by k the number of such segments, we take k copies of a disk D spanning the meridian of F corresponding to the new handle, make these disks mutually parallel so that their boundaries meet the separating surface S along the above segments, and then attach the above disks to the disks spanning the meridians of N. Similarly, we attach l parallel copies of the disk spanning the meridian of the new handle of N to the disks spanning the meridians of the old handles of F. Now, on each handle of the initial body H, the new meridian bounds the corresponding disk. Conditions (1) and (2) are obviously fulfilled.

Let us see how the complement of the disks is glued together. Doubling one of the disks, we subdivide the far body F into k balls. The near body N is divided similarly into l balls. Gluing together proceeds along $l-1$ horizontal and $k-1$ vertical strips and along an exterior region, which is also homeomorphic to a disk. Gluing together $k+l$ balls along $k+l-1$ disks can give only a ball. Note that the resulting body is necessarily connected because the new meridians do not divide its boundary. The nonrealizable case, where k or l is equal to 0, is easy to exclude by making one of the meridians pass to and fro along S. □

Definition 5.2. The *action domain* of an n-variation $\mathcal{T}$ is a regular neighborhood U of the axis of $\mathcal{T}$ in the initial position, and the ball with handles pasted in instead of U in the modified position.

Remark 5.3. The action domain of an n-variation is determined up to isotopy. While before the variation the domain has a visible axis, the position of which determines the domain, there is not such an axis after the variation. Certainly, we can mark *some* axis determining the domain, but Theorem 5.1 gives no specific form of such an axis. We should remember that during an isotopy we must keep track of the embedding of the body, and not only of the image of the embedding. If the action domain of an n-variation is fixed, then any object that does not intersect this domain in the position after the variation can be put in a one-to-one correspondence with the same object in the position before the variation. This correspondence is inverse to the n-variation in question.

Definition 5.4. Several n-variations are said to be *independent* if the unions of their axes with some representatives of the disks spanning the loops of the axes can be enclosed in mutually disjoint balls. The axes themselves are also said to be *independent*.

Motivation. Since we may assume that all changes produced by n-variation are localized in the corresponding ball, it follows that performing some independent variations can neither hinder performing the others (for instance, by entangling their loops), nor affect the character of the identifications occurring. The process of modification is even more stable: it suffices that the axes be mutually disjoint. The result of performing independent variations does not depend on their order, and we may assume that they are done simultaneously. For a collection of variations, independence in the sense of the above definition is more than pairwise independence.

Definition 5.5. Two n-axes with fixed disks transversally intersecting a graph K at the same points are *similar* (with respect to K) if, by an isotopy fixed at the initial points of intersection with K, one of the axes (together with the disks) is taken to the second axis completely, including the disks and the framings. It is assumed that in the process of isotopy the axis and the disks may intersect the graph K arbitrarily, but the tangent planes of the disks remain fixed (or at least transversal to K) at the initial intersection points.

Remark 5.6. Instead of considering a homotopy of the union of an axis with the disks, which is a 2-dimensional object, we can proceed as follows. In the disks, the points of their intersection with Γ can be joined with the points of attachment of the disks to the axis by smooth mutually disjoint curves; these curves for the first axis must be coordinated with those for the second. We frame the curves by normals to the disks. Now, it suffices to trace only the isotopy of the resulting framed graphs, because we can first pull a disk into a regular neighborhood of the curves added (Figure 16), then perform the isotopy together with the neighborhood, and then unfold the disk.

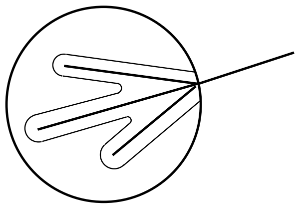


FIGURE 16

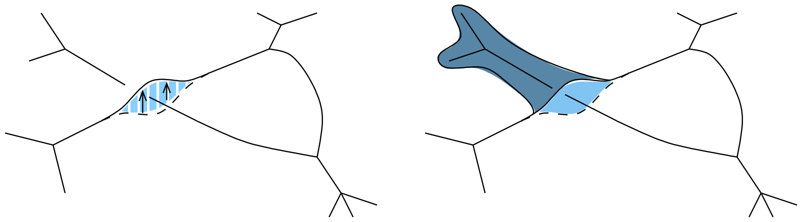


FIGURE 17

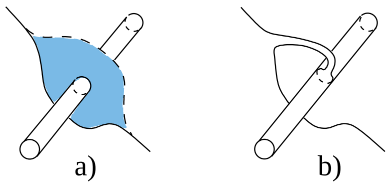


FIGURE 18

Remark 5.7. The framed graphs the isotopy of which must be traced in accordance with the preceding remark are trees, so that instead of an isotopy it suffices to have a regular homotopy. Indeed, in a generic situation, such a homotopy has only the following singularities: one of the branches passes to the other side of the other branch. Since a regular neighborhood of a tree is a ball, we can avoid passing through the tree if we go along the boundary of the ball. We must only bypass the neighborhood from the proper side, i.e., from that opposite to the side where the bypassing arc lies (Figure 17).

Theorem 5.8 (on the invariance of similarity)**.** *Suppose we are given two k-axes with fixed disks, and also an n-variation axis disjoint with them. Then these k-axes are similar after the n-variation in question if and only if they are similar before it.*

Proof. By Remark 5.7, it suffices to trace the homotopy of certain framed graphs. Until a graph, say Γ, enters the action domain U of the n-variation, no problems arise. Since U can be regarded as a regular neighborhood of a certain axis T, it suffices to keep track of what happens when an edge of Γ intersects T. Clearly, a loop ℓ will be attached to the edge of Γ such that ℓ lies on the boundary ∂U of U, is contractible there, and has trivial framing (Figure 18). After the n-variation or, conversely, before the n-variation, the action domain U is changed, but the loop ℓ remains contractible because U is enclosed in a ball. It suffices to keep track of the framing of ℓ, and only its parity is of importance, because in the process of regular homotopy it is easy to twist the framing an even number of times. Since the framing may be assumed to be normal in ∂U, it is required that the homology class represented by ℓ in ∂U have even self-linking coefficient.

The loop ℓ is homologous to a linear combination of mutually disjoint meridians.

Hence, the self-linking coefficient of ℓ is even if the meridians (*old* in the new ball, or *new* in the old ball) have even self-linking coefficients. However, by Theorem 5.1, the homology class of a new meridian can differ from that of the old meridian only by the classes of parallels that do not intersect both meridians (in fact, for $n > 1$ the two meridians coincide). Since the parallels bound outside the action domain of the n-variation and, certainly, are mutually disjoint, it follows that all their combinations have even self-linking coefficients. Consequently, the old meridian and the new meridian also have self-linking coefficients of equal parity. Therefore, these coefficients are even, since one of the meridians bounds. □

Remark 5.9. In what follows, we identify the similarity classes of the axes before and after the n-variation. Theorem 5.8 guarantees that this identification is one-to-one. To apply the theorem, it suffices to invoke Remarks 5.6 and 5.7 and push the graph K out of the action domain of the variation, which presents no difficulties because the domain is a ball with handles. It should be noted that, generally speaking, even if after several variations the graph K returns back in some sense, we cannot identify the old similarity classes with the new ones, since nontrivial holonomy may occur.

Theorem 5.10 (on the independence of variations). *Suppose several variations of arbitrary orders are applied successively to a graph K. We fix the disks of the axes of these variations. Then there are axes that are similar to the initial axes and determine independent variations leading to the same result as the initial variations. The axis of the first variation can be preserved unchanged.*

Proof. We proceed step by step, starting with the last variation. At each step, we have several independent variations of the result of a certain variation. We must find their preimages satisfying the requirements of the theorem.

The disks of the current variation intersect the boundary of a regular neighborhood U of the variation axis along parallels. After the variation, the parts of these disks that do not intersect U remain disks spanning parallels of U. We perturb these parts so that they contain neither the vertices of K nor the points of intersection of K with the disks of the forthcoming operations, and also transversally intersect the edges of K. Adding regular neighborhoods of the above disks to U, we obtain a ball B, and we may assume that the neighborhoods in question form parallel tubes in B. We may also assume that the edges of K pass through these neighborhoods by parallel segments (Figure 19).

Clearly, the axes of the forthcoming variations can be pushed out of B via an isotopy in the course of which they do not meet the graph K. Then the disks of the axes must also be pushed out of B, which is very easy to do, since we need not keep track of their position relative to K. Every axis has two sets of disks: one for independence, and the other for similarity. We can handle both sets together or separately. The fixed points in the second set lie outside of B and, consequently, do not interfere. The isotopy has changed neither the result of the forthcoming variations, nor the similarity classes of their axes, but now the axes can be enclosed in mutually disjoint balls that are also disjoint with B. The preimages of the axes satisfy the requirements of the theorem, and the preimages of the balls prove the independence of the variations. We should only observe that though the action domain of the variation has changed, the preimage of the ball B enclosing it is a ball. Indeed, this is a ball with 1-handles to the parallels of which 2-handles are attached. □

Corollary 5.11. *Let several mutually disjoint variation axes be intended for sequential application, i.e., the axes of the forthcoming variations are changed by the preceding variations. We fix the disks of the axes and assume that the axes of the preceding variations do not intersect the disks of the forthcoming variations. Then there are axes similar to*

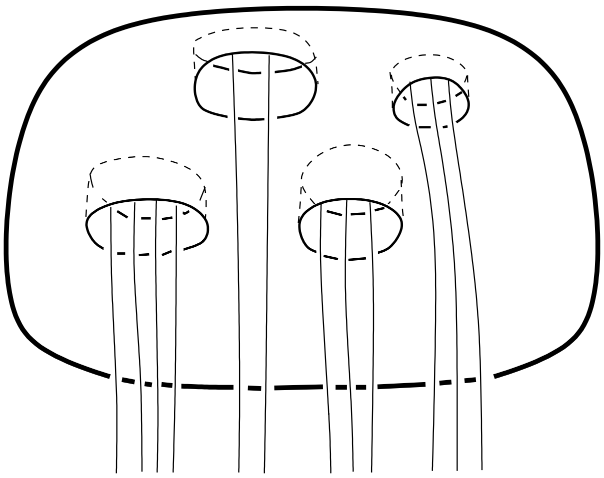
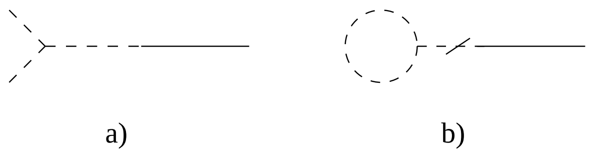


FIGURE 19



a) b)

FIGURE 20

the initial axes and such that the corresponding variations are independent, but lead to the same result as the initial variations. We may assume that each of the axes coincides with the corresponding new axis outside the union of regular neighborhoods of the axes of the preceding variations (together with the disks).

Proof. The new axes are obtained by applying Theorem 5.10 to the images of the initial axes. Similarity is ensured by Theorem 5.8. □

§6. GEOMETRIC OPERATIONS WITH VARIATIONS

In the pictures of this section, a dotted inner vertex (Figure 20a)) denotes the continuation of the axis beyond the rim of the picture, and if the dotted part of an edge indeed leads to an inner vertex, then it is marked with an even number of strokes. The edge may lead to a terminal vertex instead, but then it is marked with an odd number of strokes. In particular, Figure 20a) may also denote Figure 20b).

If a series of pictures describes an operation, then it is assumed that in a certain ball that we depict everything is as drawn (ambiguity is possible at the places enclosed in a dotted frame). In regular neighborhoods of the dotted parts of the axes of the first picture there may be some ambiguity, and at the other places simply nothing changes. Ambiguity usually consists in the possibility of passing over the axes determined up to similarity.

Theorem 6.1 (on cutting a variation crosswise). *If we cut the edge of the axis of an n-variation $\mathcal{T}$ into two parts, and attach two mutually linked loops to their new vertices (Figure 21), then we obtain the axes of two dependent variations. The sum of their orders is equal to $n+1$. Their combined application (in any order) is equivalent to the initial variation $\mathcal{T}$. Applied separately, the variations do not change the initial graph K.*

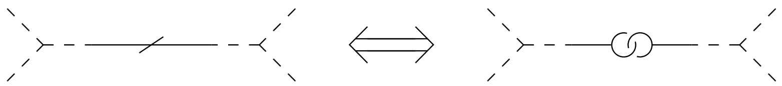


FIGURE 21

Proof. In order to verify that we have obtained two variation axes, it suffices to check the number of strokes on the new edges: it is odd for 1-variations and even for the other variations. Since two new vertices were added, the sum of the orders of the variations has increased by 1. Each of the new variation axes has a loop that is not linked with the graph K; hence, by Corollary 2.6, we may assume that the variation determined by the axis leaves K fixed. If an inner edge is cut, then we may assume that this is one of the cuttings of the second stage of the variation $\mathcal{T}$ (condition (2) of Definition 2.2), and, to perform $\mathcal{T}$, we do precisely the same modifications as for both parts together. For a terminal edge, we additionally use Theorem 4.1.

We need not keep track of specific identifications, assuming that everything happens inside a ball; this ball is a regular neighborhood of the union of the initial axis with the disks. The situation in the other theorems of this section is similar. □

Theorem 6.2 (on reuniting a variation)**.** *Let a variation $\mathcal{T}$ with axis T be cut crosswise into two variations $\mathcal{T}_1$ and $\mathcal{T}_2$ with axes T_1 and T_2. Suppose we are given several new additional independent axes with fixed disks and disjoint with T_1 and T_2. Then there are variation axes similar to the additional axes that are mutually independent and independent with the axis T, and are such that applying all the corresponding variations is equivalent to applying the additional independent variations and the variations $\mathcal{T}_1$ and $\mathcal{T}_2$.*

Proof. We perturb the disks of the axis T in order that they contain no points of intersection of the disks of the additional axes with the graph. The regular neighborhood of T (united with the new disks) is a ball B not containing these points. We may assume that the variations $\mathcal{T}$, $\mathcal{T}_1$, and $\mathcal{T}_2$ do not affect the complement of B. Since $\mathcal{T}$ is equivalent to applying $\mathcal{T}_1$ and $\mathcal{T}_2$ simultaneously, we may assume that the images of the additional axes under the combined application of $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined on the result of $\mathcal{T}$. We find the preimages of these axes in accordance with Theorem 5.10. We must only prove that they are similar to the additional axes.

We take one of the additional axes (say, T_3). Let T_4 be an axis similar to T_3 and not intersecting the ball B. The (pre)image of T_4 under the variations $\mathcal{T}_1$ and $\mathcal{T}_2$ is similar to the (pre)image of T_3. But both the image and the preimage of T_4 coincide with T_4. Therefore, T_4 is similar both to the axis T_3 and to the axis obtained from T_3, whence we see that the two latter axes are similar to each other. □

Theorem 6.3 (on splitting a variation)**.** *Let be T an n-axis (with fixed disks), and let one of the disks of T be cut into two parts by a curve emanating from a vertex and disjoint with the graph K (Figure 22a)). Let T_1 and T_2 be the axes obtained from T by removing one of the parts and smoothing the other part (Figure 22b)). Then there is an axis T_3 similar to T_2 and such that the variations $\mathcal{T}_1$ and $\mathcal{T}_3$ with axes T_1 and T_3 are independent and their combined application is equivalent to the variation $\mathcal{T}$ with axis T. The variation $\mathcal{T}_1$ is called the* good half, *and the variation $\mathcal{T}_3$ the* bad half *of the initial variation $\mathcal{T}$.*

We may assume that the axes T_2 and T_3 coincide outside a regular neighborhood of T_1.

Proof. For $n = 1$ or 2, we simply split an I- or a Y-modification, respectively. As required by the definition, for a variation of higher order we construct a chain of Y-

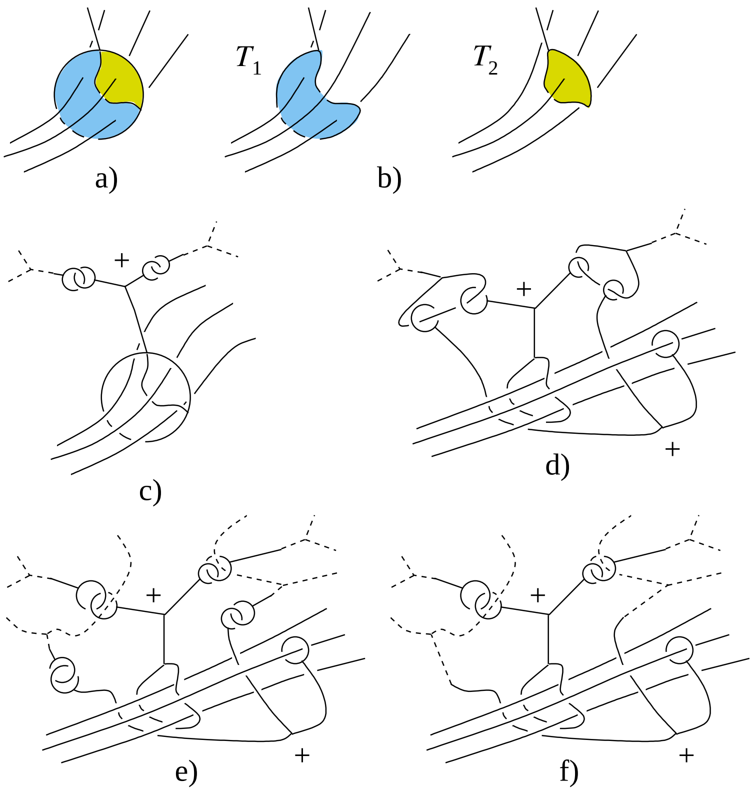


FIGURE 22

graphs. We successively split the modifications determined by these Y-graphs. We start with the modification involving the divided loop. The part corresponding to the axis T_1 is incorporated into the good half. Further, we shall proceed to the neighboring modifications, cutting them in such a way that the loop of the good (bad) half be linked with that of the good (bad) half (Figures 22c)–e)). Observe that the bad halves pass through the loops of the good halves, but not vice versa. It follows that the bad halves can be assembled into a single n-variation $\mathcal{T}_4$ with axis T_4, which can be chosen to be similar to T_2 (Figure 22f)).

The initial n-variation $\mathcal{T}$ is equivalent to applying first the n-variation $\mathcal{T}_4$, and then the good halves of the Y-modifications. We can change the order of the actions: first perform the Y-modifications, and then the n-variation along the axis T_5 obtained as a result of modification of T_4. Here, as before, we do not keep track of the identifications, since everything happens inside a ball. Viewing the Y-modifications as 2-variations, we can sequentially reunite them into a variation with axis T_1. After this, the axis T_4 is transformed into an axis, which can be taken as T_3. □

Theorem 6.4 (on pulling out an n-variation)**.** *Let $\mathcal{T}$ be an n-variation with axis T. Let an edge of T be linked with an edge e of the graph K to be modified (Figure 23a)). The n-variation $\mathcal{T}$ is equivalent to applying two variations with axes T_1 and T_2 such that T_1 differs from T only in that T_1 is unlinked from the edge e. The axis T_2, which has order $n+1$, is similar to the axis that is obtained if we join an edge of T_1 with the edge e by a new edge with a loop at the endpoint (Figure 23b)).*

Proof. We cut the variation $\mathcal{T}$ crosswise, on each side of the linking point (Figure 23c)), and add two mutually annihilating I-modifications (Figure 23d)), which we unite into a

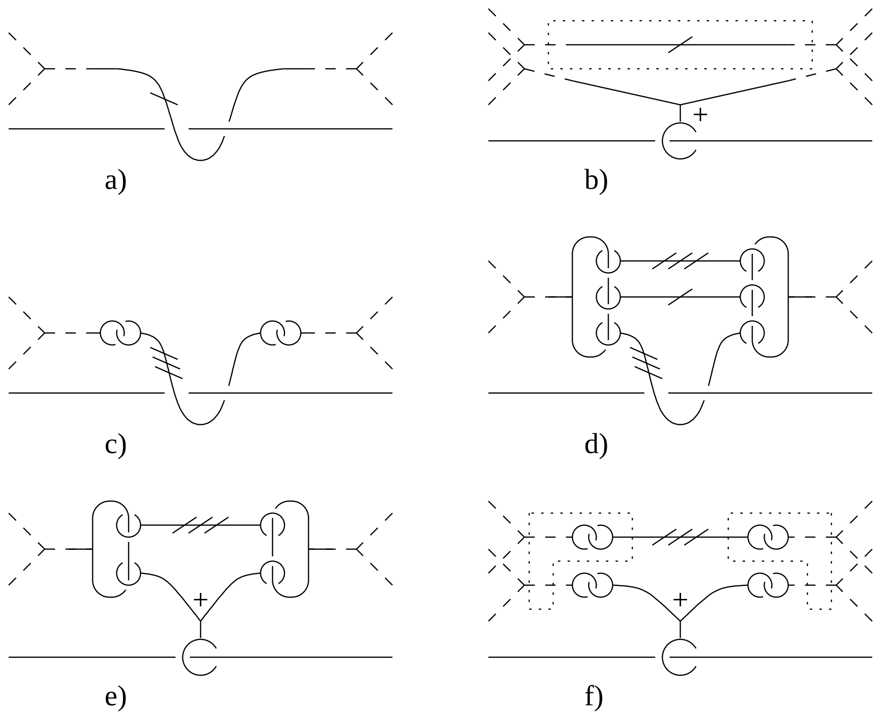


FIGURE 23

Y-modification (Figure 23e)). Then we split the pieces of T so that the good halves be linked with the I-modification, and the bad halves with the Y-modification (Figure 23f)). It remains to reunite the pieces of the variations cut crosswise. ☐

Theorem 6.5 (on the STU-transformation of variations)**.** *If two loops of the axes T_1 and T_2 of two independent variations are linked to one and the same edge of the graph K to be modified (Figure* 24a)*), then we can change the order of linking of the loops by adding another independent variation (Figure* 24b)*): the final result does not change. We must assume that T_1 and T_2 have not changed outside the picture. The axis T of the new variation is similar to the axis obtained by gluing together the axes T_1 and T_2, and the order of T is equal to the sum of the orders of T_1 and T_2.*

Proof. Making the loop of T_1 (Figure 24c)) to pass through that of T_2, we split the second variation (Figure 24d)). Now, by Theorem 6.4, we can pull out the first variation from the loop of the new variation (Figure 24e)). At this moment, the new axis T is assumed to be a part of the graph K. It remains to reunite the pieces of the new axis. We see that T has as many vertices as T_1 and T_2 taken together. ☐

Theorem 6.6 (on unlinking two variation axes)**.** *If two edges of the axes T_1 and T_2 of two independent variations are linked together (Figure* 25a)*), then we can unlink them by adding another independent variation with axis T (Figure* 25b)*): the final result does not change. It is assumed that T_1 and T_2 have not changed outside the picture. The axis T is similar to the axis obtained by joining T_1 and T_2 with a cross connection, and the order of T is greater by* 1 *than the sum of the orders of T_1 and T_2.*

Proof. We use Theorem 6.4 twice (Figure 25c), d)) and then reunite the pieces of the new variation. ☐

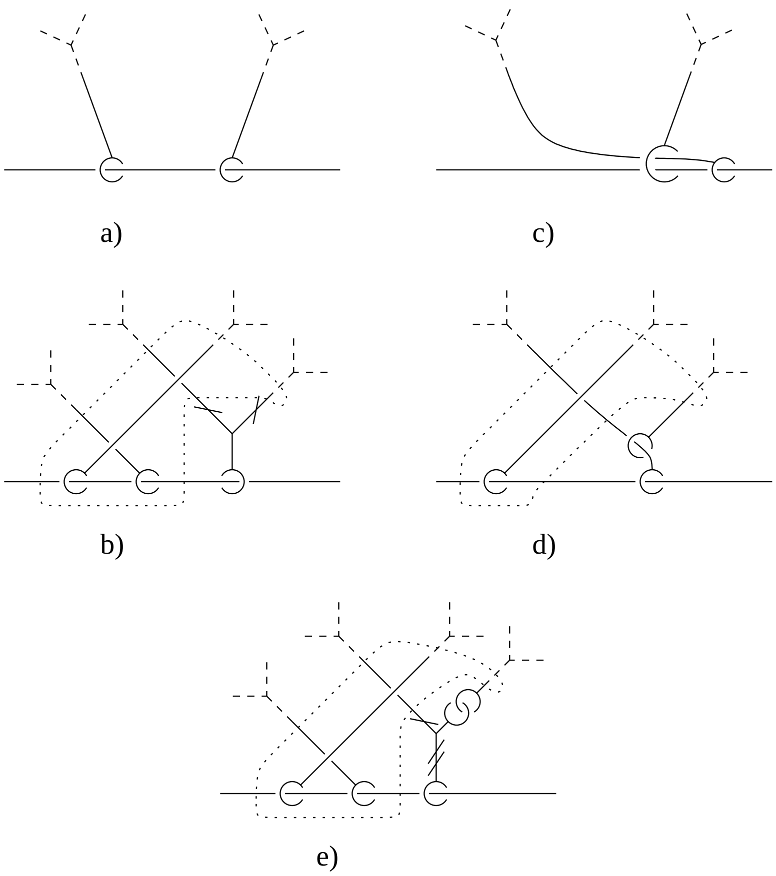


FIGURE 24

Theorem 6.7 (on the IHX-transformation for variations). *The n-variation in Figure* 26a) *is equivalent to two independent n-variations with axes similar to those depicted in Figure* 26b).

Proof. We cut the variation crosswise, thus singling out two Y-modifications, and cut the upper one into two I-modifications, and the lower one asymmetrically into three I-modifications (Figure 26c)). We cut one of the I-modifications crosswise and add to it two I-modifications annihilating each other (Figure 26d)), then unite the Y-modification cut asymmetrically (Figure 26e)). After that, we split the left-hand lower I-modification, cut one of the edges of the Y-modification crosswise, and add two more I-modifications annihilating each other (Figure 26f)). Once again we unite the Y-modification cut asymmetrically (Figure 26g)).

Now, we single out one of the required axes, T_1, starting with the left-hand Y-modification and splitting all mutually linked variations. Each time, we take bad halves. Theorem 6.2 and Corollary 5.11 ensure that the axis T_1 will be independent, since the other axes do not intersect the disks of T_1. The axis T_1 can be assembled from the parts depicted in Figure 26h) and can be deleted from the main picture (Figure 26i)). The second axis T_2 is singled out quite similarly (Figures 26j)–l)).

What remains is completely disconnected from the left-hand lower part of the axis. Now we can perform the operations in the reverse order, but this time leaving a free loop instead of linking with the indicated part of the axis. When we return to the beginning,

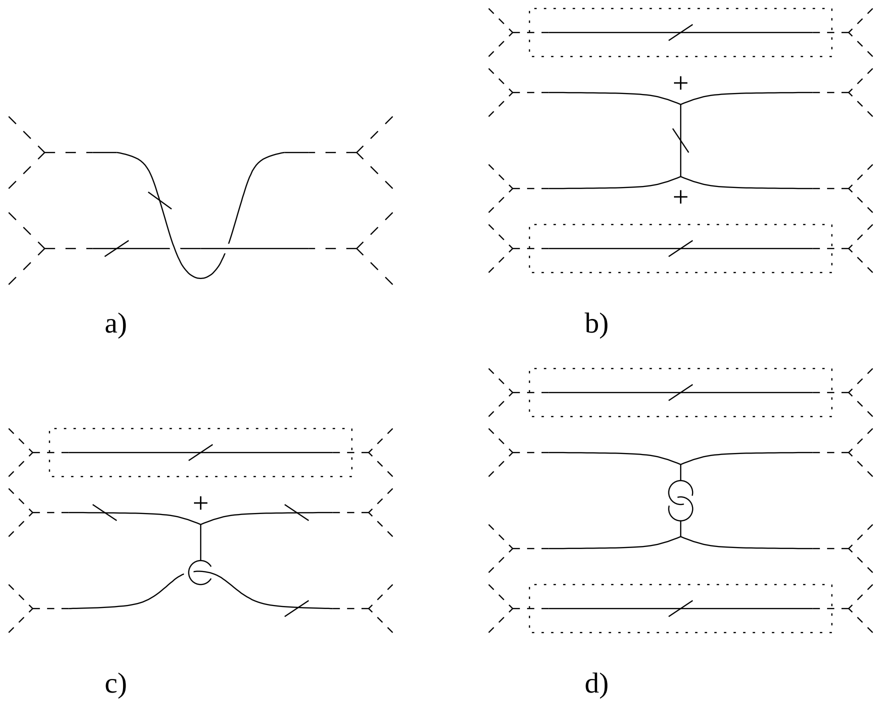


FIGURE 25

we obtain an axis with a free loop, which determines a trivial variation. The latter only modifies the axes T_1 and T_2 within their similarity classes. □

Theorem 6.8 (on everting a vertex). *An n-variation with axis T (Figure 27a)) is equivalent to another n-variation whose axis, up to similarity, differs from T by the orientation of one of the vertices and by the framings of two edges (Figure 27b)).*

Proof. We single out a Y-modification $\mathcal{Y}$ by splitting, cut it into two I-modifications (Figure 27c)), twist them, divide the loop of the lower part of the axis T into two parts (Figure 27d)), and then split this part of T (Figure 27e)). The right-hand half should be regarded as good: its application does not change anything up to similarity. We reassemble the Y-modification $\mathcal{Y}$, taking into account that the mirror symmetry changes the orientation of the vertex (Figure 27f)). After that, the axis is only subjected to reunification and isotopy. □

Theorem 6.9 (on the inverse variation). *In a regular neighborhood of any n-axis T, we can embed another variation axis T' independent with T and such that the combined application of the corresponding variations $\mathcal{T}$ and $\mathcal{T}'$ does not change the graph to be modified.*

Proof. Let T' be a variation axis differing from T only in that one of the loops of T' goes in both directions and not only in one direction (Figure 28). The result of the variation $\mathcal{T}'$ with axis T' is trivial by Corollary 2.6. We split the variation $\mathcal{T}'$ in such a way that the initial variation $\mathcal{T}$ be the good half. Then the bad half can serve as an inverse variation. □

§7. n-EQUIVALENCE OF KNOTTED GRAPHS

Definition 7.1. Two knotted graphs are *n-equivalent* if up to isotopy one of them

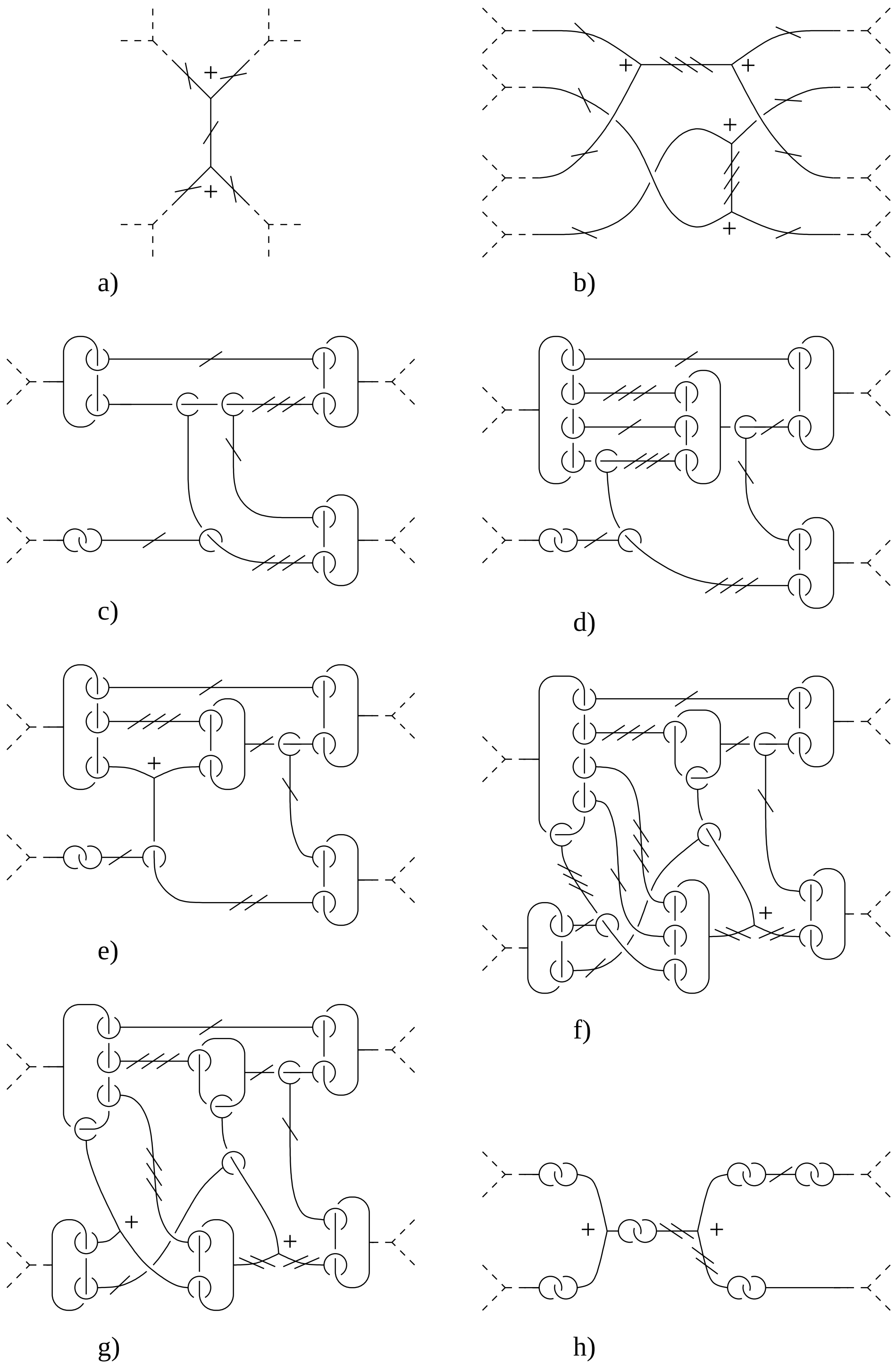


FIGURE 26

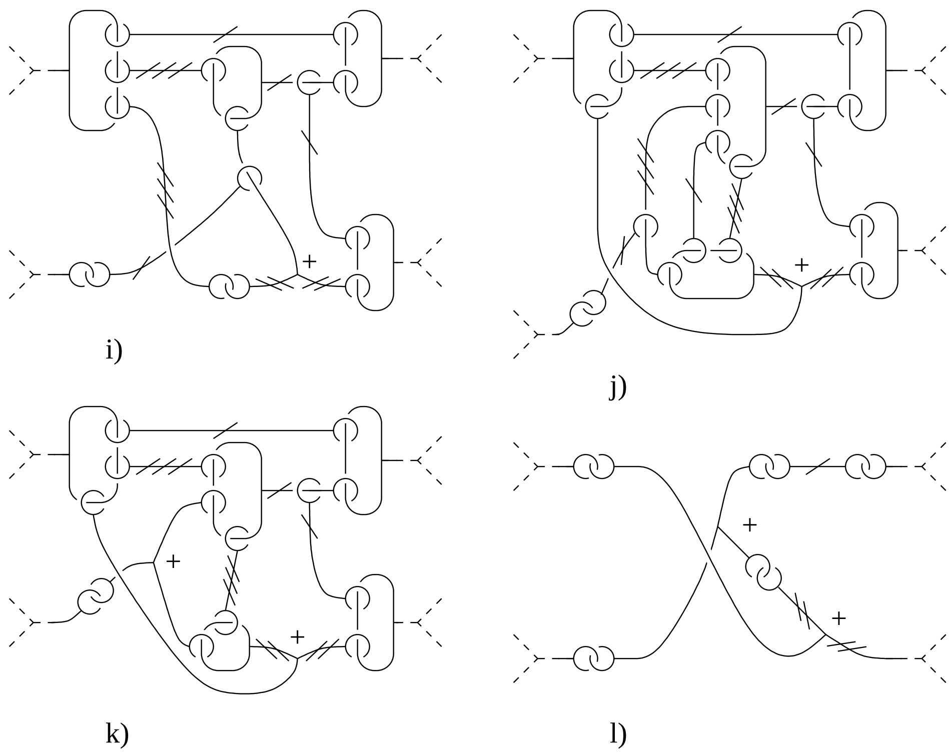


Figure 26. (continuation)

is obtained from the other by a sequence of variations of order greater than n each. (Theorem 6.9 guarantees that n-equivalence is well defined.)

Definition 7.2. An n-axis is *unramified* if each of its inner vertices is adjacent to at least one terminal vertex.

Theorem 7.3. *Up to isotopy, every knotted graph can be transformed into any graph $(n-1)$-equivalent to it by a sequence of n-variations with unramified axes.*

Proof. First, we verify that any variation can be replaced by a sequence of variations of the same order with unramified axes. Indeed, suppose the contrary. By the *distance* between two vertices of a graph we mean the minimal possible number of edges in a chain connecting the vertices, and by the *length* of an axis we mean the maximal of the distances between its vertices. We take one of the longest variation axes of the given order that are indecomposable into unramified axes and choose a chain of edges on which the length is realized. Since this axis is ramified, one of the vertices of the chain is adjacent to an inner vertex not belonging to the chain. We apply the IHX-transformation for these vertices (Figure 29). For both variations obtained, the corresponding chains are longer, contrary to our choice.

It remains to show that every variation $\mathcal{T}$ of order exceeding n with unramified axis can be replaced by an n-variation with unramified axis. For this purpose, we cut $\mathcal{T}$ crosswise in such a way that the order of one of the parts be equal to n, and perform the variation corresponding to the second part. By Theorem 6.1, this can be assumed to be an isotopy. Now, the first part is the required n-variation. $\square$

Definition 7.4. Let a knotted graph K meet a certain ball along $n+1$ loops linked as in Figure 30a). We replace this fragment by that depicted in Figure 30b). This operation

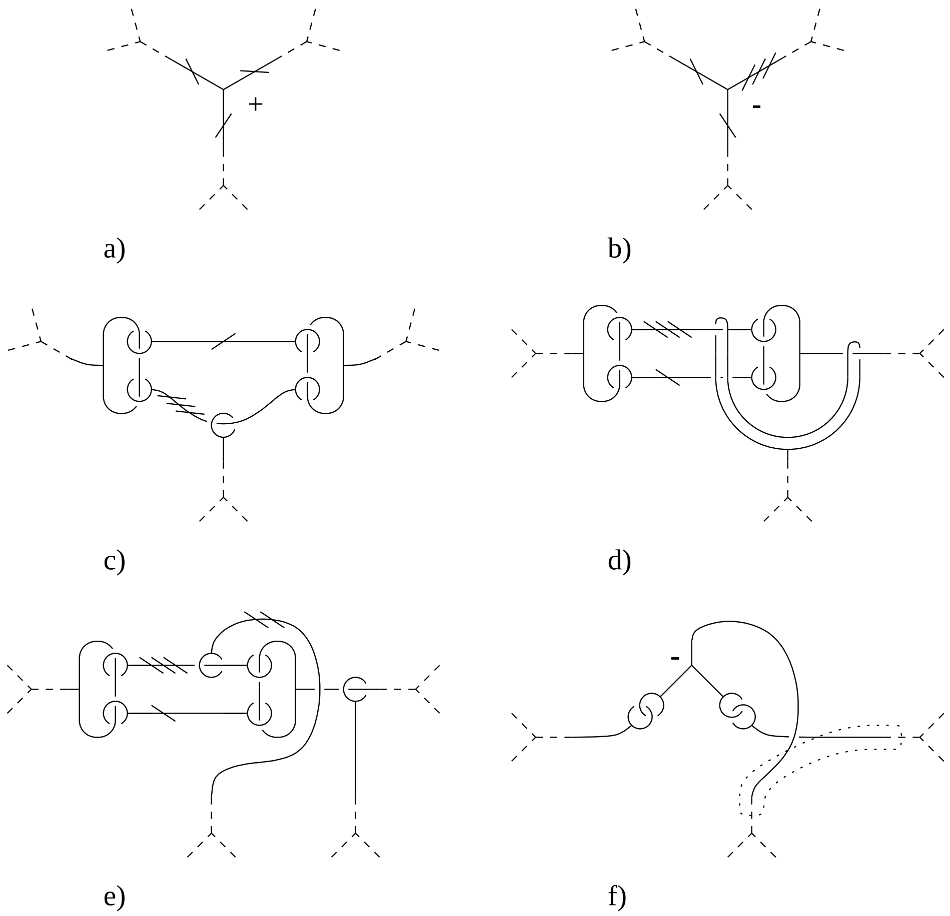


FIGURE 27

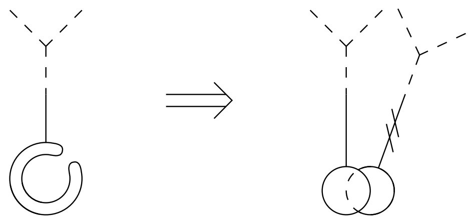


FIGURE 28

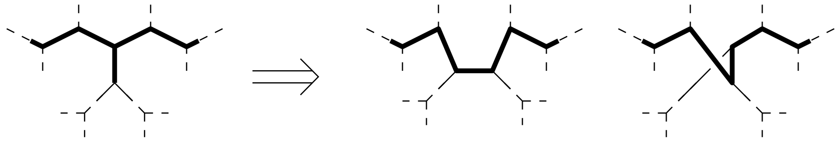


FIGURE 29

is called *throwing a loop of order n*.

Theorem 7.5. *By throwing loops of order* n*, a graph* K *can be transformed precisely to*

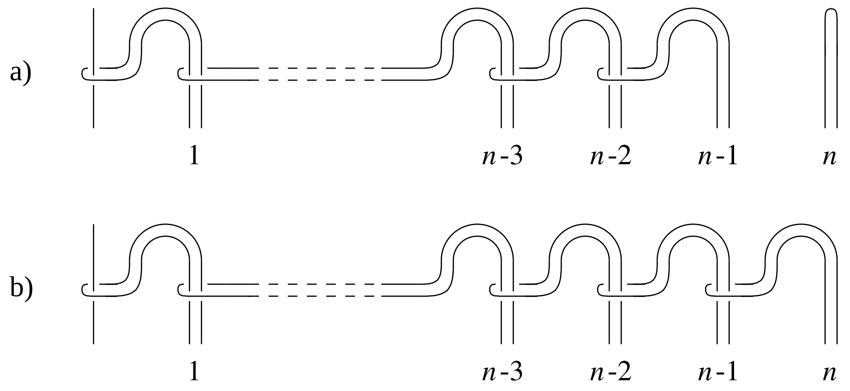


FIGURE 30

the graphs $(n-1)$*-equivalent to* K*.*

Proof. Every graph $(n-1)$-equivalent to K can be obtained from K by n-variations with unramified axes. If we disregard the framings, such an axis corresponds completely to Figure 31a). Scanning the inner vertices from left to right and everting them in the corresponding way an appropriate number of times, we can arrange that the orientations at them be consistent and that the framings of the inner edges correspond to Figure 31b). The framings of the edges leading to the terminal vertices are regulated simply by rotating the corresponding end. By splittings we make each of the loops be linked with exactly one arc.

We prove that an n-variation with such an axis is equivalent to throwing a loop of order n. We cut the n-variation into 2-variations and cut off a 1-variation from the right (Figure 31c)). We perform all 2-variations one by one, starting from the leftmost. Since each of them has a loop that is not linked with K, they can be realized by isotopies of K. By Remark 1.3, the shape of K can be depicted as in Figure 31d). The transformation under consideration reduces to a 1-variation. After an isotopy (Figure 31e)), if we look only at the upper part of the picture, we see simply throwing a loop.

Conversely, we can perform an isotopy (Figure 31f)) in the lower part of the fragment depicted in Figure 30a). Then in the upper part of the picture we shall see an n-variation equivalent to throwing a loop. □

Theorem 7.6. *If two* n*-variations of a given knotted graph* K *have similar axes, then the two graphs obtained by performing one of the variations are* n*-equivalent.*

Proof. First, we split the variations into parts so that each part correspond to choosing one point of intersection with the graph K on each of the disks. The pieces of similar axes will be similar. We choose a generic isotopy taking one of the similar axes into another, and keep track of the n-equivalence class of the graph obtained as a result of the variation the axis of which is changed. In the course of the isotopy, the axis may intersect the edges of K several times. Since a disk intersects K at a single point, Remark 5.6 shows that the disk is pulled in into a neighborhood of one segment, and the intersection of K with such a segment is equivalent to the intersection with an edge leading to this disk. By Theorem 6.4, intersection of K by an edge of the axis is equivalent to adding an $(n+1)$-variation, and, hence, this intersection does not change the n-equivalence class of K. Outside such intersections, the isotopy type of K does not change at all. Therefore, the n-equivalence class is the same at the beginning and at the end of the isotopy, as required. □

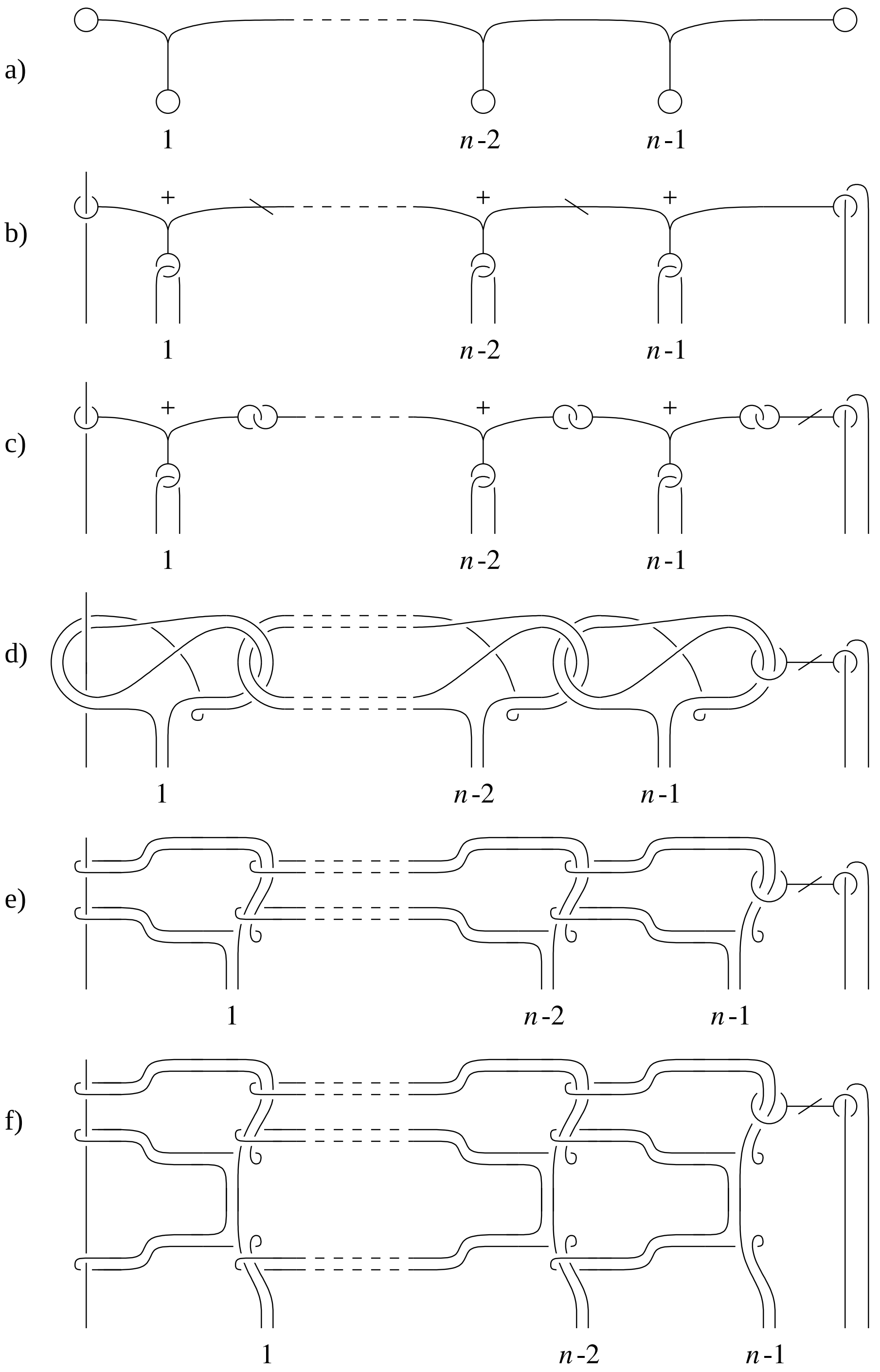


Figure 31

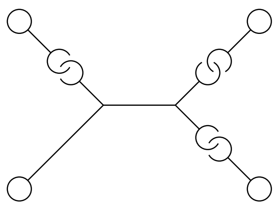


FIGURE 32

§8. VARIATION SCHEMES

Definition 8.1. A *variation direction* S of a knotted graph K is a set of the axes of several independent variations of K. Generally speaking, the variations have different orders. To *perform a variation in this direction* means to perform all these variations.

Definition 8.2. An *r-dimensional variation scheme* of a knotted graph K is an ordered collection $S = (S^{(1)}, \dots, S^{(r)})$ of variation directions of K that possesses the following two properties. First, any k-axis belongs to at most k of the directions $S^{(i)}$. Second, the union of all $S^{(i)}$ is a variation direction, i.e., all variations occurring in the scheme are mutually independent.

Remark 8.3. This is a generalization of the notion of a variation scheme as defined in [G1]. We obtain the old notion if we assume that the scheme contains only 1-axes with disks intersecting K at one point each, and then use Remark 3.5.

Definition 8.4. Let $S = (S^{(1)}, \dots, S^{(r)})$ be an r-dimensional variation scheme of a graph K. For every multiindex $\sigma = (\sigma_1, \dots, \sigma_r) \in \{0,1\}^r$, the variation direction $S(\sigma)$ of K is defined as follows. A variation axis belongs to $S(\sigma)$ if it belongs to at least one of $S^{(i)}$'s, and for every such $S^{(i)}$ we have $\sigma_i = 1$. The result of the variation in the direction $S(\sigma)$ is denoted by $[S(\sigma)]$.

Remark 8.5. It is not necessary that $S^{(i)}$ correspond to a certain multiindex. By a *direction* of the scheme S we mean $S(\sigma)$, not $S^{(i)}$.

Definition 8.6. Two variation schemes S and S' of equal dimension are *equivalent* if the graphs $[S(\sigma)]$ and $[S'(\sigma)]$ are isotopic for every multiindex σ.

Theorem 8.7 (on dissolving an n-variation)**.** *Every variation scheme $S = (S^{(1)}, \dots, S^{(r)})$ is equivalent to a scheme containing only 1-axes.*

Proof. We annihilate the variation axes of order greater than 1 independently of each other, by replacing them with 1-axes. To start with, we cut crosswise n terminal edges of any n-axis T, i.e., all edges but one (Figure 32). After that, we perform the remaining n-variation. By Theorem 6.1, we may assume that the variation does not change our graph K. Furthermore, K is not changed up to isotopy if we perform some of the variations that have been cut. Only when all of these 1-variations are performed, K is changed, and precisely in the same way as under the n-variation with axis T. One by one, we distribute the corresponding n 1-axes among those of the sets $S^{(i)}$ that contain T. If the number of such $S^{(i)}$ is smaller than n, several axes are put into some of them. This yields a scheme equivalent to S. □

Definition 8.8. Below we use the following notation for multiindices:
- $\sigma + \nu$: $(\sigma+\nu)_i$ is the remainder after division of $\sigma_i + \nu_i$ by 2;
- $\sigma\nu$: $(\sigma\nu)_i = \sigma_i\nu_i$;
- $\sigma \cup \nu$: $(\sigma \cup \nu)_i = \max(\sigma_i, \nu_i)$;

- $\bar{\sigma}$: $\bar{\sigma}_i = 1 - \sigma_i$;
- $|\sigma|$: $|\sigma| = \sigma_1 + \cdots + \sigma_r$ for $\sigma \in \{0,1\}^r$;
- θ: $\theta_i = 0$ for all i;
- ι: $\iota_i = 1$ for all i;
- $\sigma \preceq \nu$: $\sigma_i \leq \nu_i$ for all i;

Theorem 8.9 (on conjugate schemes)**.** *Let $S = (S^{(1)}, \dots, S^{(r)})$ be a variation scheme of dimension r, and ν a multiindex. Then there is a variation scheme S' such that for every multiindex σ the graphs $[S(\sigma)]$ and $[S'(\sigma + \nu)]$ are isotopic.*

Proof. First, we assume that the multiindex ν involves only one unity, for example, $\nu_m = 1$. We enclose the variation axes that belong to $S^{(m)}$ in mutually disjoint balls not touching the other axes. (This is possible because the variations are independent.) For each of these axes, we find the axis of an inverse variation inside the corresponding ball B. We choose the inverse variation in such a way that it is independent with the variation to be inverted and the result of their combined application is isotopic to the identical transformation via an isotopy fixed on the complement of B. Now, we perform a variation in the direction $S(\nu)$, and put the images of the axes of the inverse variations found into the sets $S^{(i)}$ containing the variations to be inverted. Leaving the images of the axes of the nonperformed variations in the sets that included them before transformation, except $S^{(m)}$, we obtain a scheme S'.

If ν involves more than one unity, then we perform such transformations several times, one for each unity. □

Theorem 8.10 (on corrections)**.** *Let S be an r-dimensional variation scheme, and let K_σ be a collection of knotted graphs corresponding to the multiindices $\sigma \in \{0,1\}^r$. We assume that for every σ the graphs $[S(\sigma)]$ and K_σ are $(r-1)$-equivalent. Then there is a scheme S' such that for every σ the graphs $[S'(\sigma)]$ and K_σ are isotopic. Furthermore, if for each $\sigma \preceq \nu$ we already have an isotopy, then we may assume that $S(\nu) = S'(\nu)$.*

Proof. Consider the multi-indices σ such that the graphs $[S(\sigma)]$ and K_σ are not isotopic, and also those that exceed some σ with this property. We arrange all these multiindices in nondecreasing order, and sequentially correct the corresponding variation directions. For this purpose, we choose a collection of axes of r-variations making $[S(\sigma)]$ isotopic to K_σ, ensure their mutual independence and independence of the axes already at hand, and then add them to all the sets $S^{(i)}$ with $\sigma_i = 1$. This does not violate $(r-1)$-equivalence, and the changes occur only for the multiindices that are yet to be met in the course of this procedure. □

§9. String links

Remark 9.1. By a *string link* we mean a collection of k smooth embeddings $f_i: \mathbb{R} \to \mathbb{R}^3$ such that their images are mutually disjoint and $f_i(x) = (x, i, 0)$ outside a certain compact set. A *shift* of a string link is a transformation of the form $f_i'(x) = f_i(x - t) + (t, 0, 0)$; this is a shift *to the right* if $t > 0$ and *to the left* if $t < 0$. When multiplying string links, we shift the second factor to the right sufficiently far, in order that the images of the perturbed portions of the second link lie to the right of the images of those of the first link; after this, we leave the first link in the left-hand part and the second link in the right-hand part.

Theorem 9.2 (on the group of n-equivalence classes of string links)**.** *The n-equivalence classes of string links with fixed number of strings constitute a group with respect to the operation of multiplication.*

Proof. It is obvious that the multiplication of classes is well defined. Automatically, this operation is associative and possesses an identity. We must only check the existence of inverse elements. This is done by induction on n. For $n = 0$, there is only one equivalence class, because all string links are homotopic to each other, and 1-variations allow two portions of the strings to pass through each other; thus, it is possible to transform a homotopy to an isotopy. Hence, we may assume that the inverse elements exist for the preceding value of n, i.e., for a given link L, there is a link L_{n-1} such that LL_{n-1} is $(n-1)$-equivalent to a trivial link.

The string link LL_{n-1} can be obtained from a trivial link by n-variations with loops linked only once with a single string. To each of these variations, we add an inverse variation. Such a collection of n-variations leads to a trivial string link. Now, we shift all axes of the inverse variations far to the right, and if they meet the fixed initial axes, we use Theorems 6.5 and 6.6. We obtain new variations, of order greater than n, and we need not keep track of the position of their axes. Performing all initial and all inverse variations, but not the newly formed variations, we obtain a string link, which consists of the link LL_{n-1} formed by the initial variations in the left-hand part, and a certain link L' formed by the shifts of the inverse variations in the right-hand part. In other words, this is the product $LL_{n-1}L'$, which is n-equivalent to a trivial link, because the trivial link is obtained from it as a result of the newly formed variations, of order exceeding n. Hence, $L_{n-1}L'$ is the required (right) inverse to L. We construct the left inverse absolutely similarly, and, since there are two inverse elements, they coincide, and each of them is a two-sided inverse. □

Theorem 9.3 (on crystallization of n-variations)**.** *Let S be an r-dimensional variation scheme of a string link L. Suppose that $[S(\sigma)]$ is isotopic to L whenever $|\sigma| < n$. Then there exists a variation scheme S_* equivalent to S and such that each variation axis of S_* belongs to at least n of the sets $S_*^{(i)}$.*

Proof. First, we consider the case where the initial string link L (i.e., $[S(\theta)]$) is trivial. We split the variations in S so that each loop of each axis be linked with only one string and only once.

We proceed by induction on n. For $n = 1$, the assertion is trivial. Assume that each variation axis in S belongs to at least $n-1$ of the sets $S^{(i)}$. We show that, using an equivalent transformation of S and preserving the above condition, we can reduce the number of the axes belonging to exactly $n-1$ of the sets $S^{(i)}$.

We take a multiindex σ such that $|\sigma| = n-1$ and $S(\sigma)$ is nonempty, and shift all axes in $S(\sigma)$ to the right. When doing so, we meet with obstructions of two types: a loop (alternatively, a nonclosed edge) of a moving axis T_1 collides with a loop (alternatively, a nonclosed edge) of a fixed axis T_2. Both types of obstructions are surmounted (see Theorems 6.5 and 6.6, respectively) by adding a new variation axis T the order of which is equal to or greater by 1 than the sum of the orders of the colliding axes T_1 and T_2. We add the axis T to all the sets $S^{(i)}$ containing at least one of the axes T_1 and T_2 (the orders of the latter allow us to do this). Then the scheme is replaced by an equivalent one. We notice that the new axis T belongs to all the sets $S^{(i)}$ with $\sigma_i = 1$, and at least to one more. We need not track the behavior of such axes: they "step aside" before the moving axes, and we only put the new axes in order, i.e., make them independent after surmounting every next obstruction. We continue shifting the axes in this way until all moving axes are completely to the right of all fixed axes (the newly formed axes may enter both domains).

Now, we have a variation scheme S_1 equivalent to S, and the axes of S_1 are subdivided into three classes. The axes that belong to none of the sets $S_1^{(i)}$ with $\sigma_i = 1$ lie on the left, while the axes belonging to precisely those of the sets $S_1^{(i)}$ for which $\sigma_i = 1$ lie on the right. Finally, the axes belonging to all sets $S_1^{(i)}$ with $\sigma_i = 1$ and to some other set can occupy an arbitrary position. We perform a variation in the direction $S_1(\sigma)$, delete the axes of the performed variations, and put the remaining (modified) axes into the sets $S_2^{(i)}$ having the same indices as the sets $S_1^{(i)}$ that contained them beforehand. We obtain a variation scheme S_2.

If $\nu \succeq \sigma$, then $[S_2(\nu)] = [S_1(\nu)] = [S(\nu)]$. If not, then none of the new variations has been performed (the third class). Hence, the result has the form of the product $[S_1(\nu)][S_1(\sigma)] = [S(\nu)][S(\sigma)]$. But $[S(\sigma)]$ is trivial because $|\sigma| < n$, whence here we also have $[S_2(\nu)] = [S(\nu)]$. Thus, S_2 is equivalent to S, but has fewer axes belonging to exactly $n-1$ sets. Repeating this procedure, we shall eventually get rid of such axes at all, as required.

Now, let $[S(\theta)]$ be nontrivial. Then we take a string link L inverse to $[S(\theta)]$ in the group of $(r-1)$-equivalence classes. Let S' be the variation scheme obtained by multiplying the scheme S by L from the right. Then $[S'(\sigma)]$ is $(r-1)$-equivalent to a trivial link for $|\sigma| < n$. By Theorem 8.10, there is a scheme S'' such that $[S''(\sigma)]$ is trivial if $|\sigma| < n$ and $[S''(\sigma)] = [S'(\sigma)]$ otherwise. As proved above, there is a scheme S''_* such that $S''_*(\sigma)$ is empty, $[S''_*(\sigma)]$ is trivial if $|\sigma| < n$, and $[S''_*(\sigma)] = [S(\sigma)]L$ otherwise. Multiplying S''_* from the right by $[S(\theta)]$, we obtain a scheme S'_*. As before, $S'_*(\sigma)$ is empty and $[S'_*(\sigma)] = [S(\theta)] = [S(\sigma)]$ if $|\sigma| < n$. Otherwise, $[S'_*(\sigma)] = [S(\sigma)]L[S(\theta)]$ is $(r-1)$-equivalent to $[S(\sigma)]$. Now, applying Theorem 8.10, we replace the $(r-1)$-equivalence by an isotopy, obtaining the required scheme S_*. □

Theorem 9.4. *Let $G(n)$ be the group of n-equivalence classes of string links with a certain fixed number of strings. For $0 \le k \le n$, let $G(n,k)$ be the kernel of the projection $\rho_{n,k}\colon G(n) \to G(k)$. Then:*

(1) $G(n) = G(n,0) \supset G(n,1) \supset \cdots \supset G(n,n) = 1$;
(2) $G(n,k-1)/G(n,k)$ *is an Abelian group depending on k, but not on n;*
(3) $[G(n,k), G(n,l)] \subset G(n, \min(k+l+1, n))$.

Proof. (1) The first equality expresses the simple fact that all string links with a given number of strings are 0-equivalent (see the proof of Theorem 9.2). The inclusions reflect the fact that $(k+1)$-equivalence implies k-equivalence.

(2) The possibility of factorization is doubtless, because $G(n,k)$ is the kernel of the homomorphism $\rho_{n,k}$. Since $\rho_{n,k}$ is an epimorphism, we have $G(k) = G(n)/G(n,k)$. Hence,

$$G(n,k-1)/G(n,k) = \rho_{n,k}(G(n,k-1)) = \operatorname{Ker} \rho_{k,k-1} = G(k,k-1).$$

Now, commutativity follows from assertion (3).

(3) Let a string link L_1 represent a class in $G(n,k)$, L_2 a class in $G(n,l)$, and let L'_1 and L'_2, respectively, represent the classes inverse to L_1 and L_2. Then L_1 and L'_1 (L_2 and L'_2) are k-equivalent (l-equivalent) to a trivial string link. Hence, they are obtained from a trivial link by sequences of $(k+1)$- or $(l+1)$-variations, respectively. We assume that $k+l+1 \le n$, since otherwise we can reduce k or l. We arrange these collections on a trivial string link separately from each other so that, from left to right, they form L_1, L_2, L'_1, and L'_2. We define a $(k+l+2)$-dimensional variation scheme S for a trivial string link by including in $S^{(i)}$ all variations forming L_1 and L'_1 for $i \le k+1$, and all variations forming L_2 and L'_2 for $i > k+1$. Then among $[S(\sigma)]$ there are string links of four types: a trivial one, $L_1L'_1$, $L_2L'_2$, and $L_1L_2L'_1L'_2$. The type $L_1L_2L'_1L'_2$ arises only for $\sigma = \iota$, and the first three types are n-equivalent to a trivial link. By Theorem 8.10, there is a $(k+l+2)$-dimensional variation scheme S' such that $[S'(\iota)] = L_1L_2L'_1L'_2$ and

the remaining $[S'(\sigma)]$ are trivial. Then, by Theorem 9.3, there is a similar scheme S_* each axis of which belongs to all $k+l+2$ of the sets $S_*^{(i)}$ and, consequently, has order at least $k+l+2$. Via variations with such axes, the string link $L_1L_2L_1'L_2' = [L_1, L_2]$ is obtained from (and, consequently, is $(k+l+1)$-equivalent to) a trivial link. Therefore, the class of the former link is in $G(n, k+l+1)$, as required. Obviously, this implies that the quotient groups are commutative. □

Remark 9.5. A string link is a crucial case of an embedded graph, to which we can completely reduce the study of framed embeddings of a connected graph in $\mathbb{R}^3$, and it is an important tool for handling more complicated situations.

§10. Invariants of finite degree

Definition 10.1. Let u be a mapping that takes values in an Abelian group G and is defined on a certain set of isotopy classes of embeddings of a graph in a 3-manifold. The mapping u is said to be an *invariant of degree n* if:

(1) along with each class, the domain of u includes all classes $(n-1)$-equivalent to it;
(2) if S is an $(n+1)$-dimensional variation scheme such that $u([S(\sigma)])$ is defined for every $\sigma \in \{0,1\}^{n+1}$, then

$$\sum_{\sigma\in\{0,1\}^{n+1}} u([S(\sigma)])(-1)^{|\sigma|} = 0. \tag{1}$$

Remark 10.2. Condition (2) of Definition 10.1 is absolutely standard if we take Theorem 8.7 and Remark 8.3 into account. Condition (1) is new. It has appeared for several reasons. First, for embeddings of a graph in a manifold (in contrast to knots in the sphere, for instance), the totality of situations for which the invariant should be defined is by no means obvious. Second, if we attempt to work in too wide a class, then we cannot regard, e.g., the Milnor numbers of (nonstring) links as invariants of finite degree, while they certainly have finite degree in the sense of our definition. Third, the mappings described in (1) are simply more adequate to the nature of the object studied. In this sense, everything is similar to obstruction theory.

Theorem 10.3 (on the subtlety of n-equivalence)**.** *If two knotted graphs are n-equivalent, then all their invariants of degree at most n coincide.*

Proof. Since the graphs are n-equivalent, they belong to the domains of the same invariants of degree at most $n+1$. Let a graph K' be obtained from a graph K via an $(n+1)$-variation with axis T, and let u be an invariant of degree $k \le n$. Let S be a $(k+1)$-dimensional variation scheme for K such that each $S^{(i)}$ consists of one axis T. Then $[S(\iota)] = K'$, and $[S(\sigma)] = K$ for $\sigma \ne \iota$. Therefore,

$$\begin{aligned} 0 &= \sum_{\sigma\in\{0,1\}^{k+1}} u([S(\sigma)])(-1)^{|\sigma|} \\ &= \sum_{\sigma\in\{0,1\}^{k+1}} u(K)(-1)^{|\sigma|} + [u(K') - u(K)](-1)^{|\iota|} \\ &= [u(K') - u(K)](-1)^{|\iota|}, \end{aligned}$$

i.e., $u(K') = u(K)$. By Theorem 7.3, the invariants coincide for any two n-equivalent graphs. □

Theorem 10.4 (on distinguishing classes for string links). *Any two string links L_1 and L_2 (with equal number of strings) that are not n-equivalent are distinguishable by invariants of degree n.*

Proof. Let k be the smallest number such that the string links L_1 and L_2 are not k-equivalent. Then $0 < k \leq n$. Let A be the $(k-1)$-equivalence class containing L_1 and L_2. We take a string link L in the class inverse to the class A and define a mapping $\rho\colon A \to G(k, k-1)$ in the notation of Theorem 9.4 by assuming that a string link X is taken to the k-equivalence class of the product XL. The choice of L guarantees that the latter class is indeed in the kernel of the homomorphism $\rho_{k,k-1}$. We notice that $G(k, k-1) = G(k, k-1)/\{1\} = G(k, k-1)/G(k, k)$ is an Abelian group. Since L_1 and L_2 are not k-equivalent, we have $\rho(L_1) \neq \rho(L_2)$.

We prove that ρ is an invariant of degree k. Let S be a $(k+1)$-dimensional variation scheme such that $[S(\sigma)] \in A$ for every σ. Multiplying it from the right by L, we obtain a scheme S_1 such that $[S_1(\sigma)] \in \rho([S(\sigma)])$. Let

$$u(\sigma) = -(-1)^{|\sigma|} \sum_{\tau \preceq \sigma} \rho([S(\tau)])(-1)^{|\tau|}.$$

Since the string links in these classes are $(k-1)$-equivalent to a trivial link, they can be formed by k-variations. Fixing such a collection of variations for each multiindex except ι, we arrange the axes of the collections along a trivial link so that the axes of every next collection are completely to the right of the axes of the preceding collection (the order of the collections is of no importance). We compose a variation scheme S_2 by putting the axes from the collections corresponding to the multiindices with $\sigma_i = 1$ into the sets $S_2^{(i)}$. Then for $\upsilon \neq \iota$ we have

$$\begin{aligned}
[S_2(\upsilon)] \in \sum_{\sigma \preceq \upsilon} u(\sigma) &= -\sum_{\sigma \preceq \upsilon} (-1)^{|\sigma|} \sum_{\tau \preceq \sigma} \rho([S(\tau)])(-1)^{|\tau|} \\
&= -\sum_{\tau \preceq \upsilon} \rho([S(\tau)])(-1)^{|\tau|} \sum_{\sigma : \tau \preceq \sigma \preceq \upsilon} (-1)^{|\sigma|} \\
&= -\rho([S(\upsilon)]).
\end{aligned}$$

The latter relation expresses the fact that for $\tau \neq \upsilon$ the inner sum contains the same number of $+1$'s and -1's. Similarly, for $\upsilon = \iota$ we have $[S_2(\iota)] = -\rho([S(\iota)]) - u(\iota)$. Hence,

$$\begin{aligned}
\sum_{\sigma \in \{0,1\}^{k+1}} [S_2(\sigma)]_k (-1)^{|\sigma|} &= -\sum_{\sigma \in \{0,1\}^{k+1}} \rho([S(\sigma)])(-1)^{|\sigma|} - (-1)^{|\iota|} u(\iota) \\
&= -\sum_{\sigma \in \{0,1\}^{k+1}} \rho([S(\sigma)])(-1)^{|\sigma|} + \sum_{\tau \preceq \iota} \rho([S(\tau)])(-1)^{|\tau|} \\
&= 0.
\end{aligned}$$

Here, $[\,]_k$ denotes the k-equivalence class of $[\,]$.

To compose a variation scheme S_3, we place S_1 next to S_2 and assume that $S_3^{(i)} = S_1^{(i)} \cup S_2^{(i)}$. For $\sigma \neq \iota$, the string link $[S_3(\sigma)]$ is k-equivalent to a trivial link. By Theorem 8.10, there is a scheme S_4 such that $[S_4(\sigma)]$ is trivial for $\sigma \neq \iota$, and $[S_4(\iota)] = [S_3(\iota)]$. By Theorem 9.3, there is a scheme S_* equivalent to S_4 and such that each axis of S_* belongs to all $k+1$ of the sets $S_*^{(i)}$ and, hence, has order at least $k+1$. Therefore, $[S_*(\iota)]_k = 0$.

Now, we check that condition (1) is fulfilled for ρ:

$$\begin{aligned}\sum_{\sigma\in\{0,1\}^{k+1}} \rho([S(\sigma)])(-1)^{|\sigma|} &= \sum_{\sigma\in\{0,1\}^{k+1}} [S_1(\sigma)]_k(-1)^{|\sigma|} \\ &= \sum_{\sigma\in\{0,1\}^{k+1}} [S_1(\sigma)]_k(-1)^{|\sigma|} + \sum_{\sigma\in\{0,1\}^{k+1}} [S_2(\sigma)]_k(-1)^{|\sigma|} \\ &= \sum_{\sigma\in\{0,1\}^{k+1}} ([S_1(\sigma)]_k + [S_2(\sigma)]_k)(-1)^{|\sigma|} \\ &= \sum_{\sigma\in\{0,1\}^{k+1}} [S_3(\sigma)]_k(-1)^{|\sigma|} \\ &= [S_3(\iota)](-1)^{|\iota|} = [S_4(\iota)](-1)^{|\iota|} = [S_*(\iota)](-1)^{|\iota|} \\ &= 0. \square\end{aligned}$$

Remark 10.5. In Theorem 10.4, an invariant of degree k is defined on a $(k-1)$-equivalence class. Using the same means, we can define such an invariant on an Entier$(k/2)$-equivalence class. In general, only the commutativity of the corresponding group is essential here. For the group of knots, i.e., of string links with one string, which is commutative, an invariant of degree k is simply the k-equivalence class. However, this case was considered earlier in [G1, G2].

Remark 10.6. Under an appropriate definition of an invariant of finite degree with values in an arbitrary (not necessarily Abelian) group, the n-equivalence class turns out to be such an invariant (of degree n), which is universal for string links.

Remark 10.7. Using special algebraic techniques that will be described elsewhere, we can show that the invariants of finite degree (in the usual sense) defined for all string links distinguish their n-equivalence classes; probably, even the invariants of degree at most n suffice for this.

Remark 10.8. However, in the general case, we cannot distinguish the n-equivalence classes only by the invariants of degree n (in the usual sense) that are defined on too wide a set (for instance, on the 0-equivalence class). The simplest example is the Arf-invariant for two-component links. This invariant has degree 2 for the components with even linking coefficient, but it can be extended to components with odd linking coefficient only up to an invariant of degree 3. The classes distinguished by the latter invariant cannot be distinguished by the invariants of degree 2 that are also defined for such links.

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