Classical Lie Algebra Weight Systems of Arrow Diagrams

by

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A thesis submitted in conformity with the requirements
for the degree of Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto

Abstract

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2010

The notion of finite type invariants of virtual knots introduced in [GPV] leads to the study of $\vec{\mathcal{A}}_n$, the space of diagrams with n directed chords mod 6T (also known as the space of arrow diagrams), and weight systems on it. It is well known that given a Manin triple together with a representation V we can construct a weight system.

In the first part of this thesis we develop combinatorial formulae for weight systems coming from standard Manin triple structures on the classical Lie algebras and these structures' defining representations. These formulae reduce the problem of finding weight systems in the defining representations to certain counting problems. We then use these formulae to verify that such weight systems, composed with the averaging map, give us the weight systems found by Bar-Natan on (undirected) chord diagrams mod 4T ([BN1]).

In the second half of the thesis we present results from computations done jointly with Bar-Natan. We compute, up to degree 4, the dimensions of the spaces of arrow diagrams whose skeleton is a line, and the ranks of all classical Lie algebra weight systems in all representations. The computations give us a measure of how well classical Lie algebras capture the spaces $\vec{\mathcal{A}}_n$ for $n \leq 4$, and our results suggest that in $\vec{\mathcal{A}}_4$ there are already weight systems which do not come from the standard Manin triple structures on classical Lie algebras.

Acknowledgements

At the conclusion of my U of T years, I would like to thank the following people:

Dror Bar-Natan, my thesis advisor, for his guidance and generosity with his time. His ideas made this thesis possible.

Misha Polyak, the external reader, for reading the thesis and giving valuable comments and suggestions.

The Department of Mathematics, for financial support. Especially I would like to thank Professor George Elliott for his donation to the OGSST, through which I was financially supported for four years.

Ida Bulat, the graduate administrator, for making every departmental matter seem deceptively easy.

The Knot at Lunch group (Jana Archibald, Hernando Burgos Soto, Karene Chu, Zsuzsanna Dancso, Peter Lee), for many interesting discussions.

The Gang of The Castle (Fernando Espinosa, Pinaki Mondal, Masrour Zoghi, Tony Huynh), for maintaining in me a certain level of sanity.

My family (my parents Thomas and Kit Ching, my brother Sammy), for their care and support which I can always count on, and

Silian, for making life in general more bearable.

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Chapter 1

Introduction

This thesis is about the space of directed chord diagrams modulo $6T$ (from now on the space is referred to as the space of arrow diagrams) and functions mapping it to Lie algebra-related spaces. Such functions are called weight systems, which have roots in the study of finite type invariants of virtual knots ([GPV]). The first chapter is a review of the notions of arrow diagrams and weight systems. Topics include arrow diagrams, acyclic directed trivalent diagrams, finite type invariants of virtual knot diagrams, weight systems coming from Manin triples and r-matrices, and construction of Manin triples from simple Lie algebras. In chapter 2 we present combinatorial formulae of weight systems coming from Manin triples constructed from classical Lie algebras ($\mathfrak{gl}$, $\mathfrak{so}$ and $\mathfrak{sp}$), and their defining representations. In Chapter 3 we present results of computations done jointly with Dror Bar-Natan. The results tell us how well these classical Manin triples capture the space of arrow diagrams when the skeleton is an oriented line and when the degree is low. We cover all representations in our computations by working with the universal enveloping algebras of the Manin triples.

This thesis is intended for an audience with background in finite type invariants. Standard references include [BN2] on finite type invariants of classical knots and [GPV] on finite type invariants of virtual knots.

1.1 Directed chord diagrams modulo 6T

A directed chord diagram with skeleton Γ (which is usually a disjoint union of oriented circles and lines) is a diagram with oriented chords joining distinct points of Γ. The space we study in this thesis is the space of directed chord diagrams modulo 6T. The 6T relation is as shown in figure 1.1. The solid lines are parts of the skeleton while the dotted arrows are the directed chords. (The solid line segments may come from the same connected component of the skeleton.) The pictures only show the part of the diagrams where they are different. The parts which are not shown are the same for each diagram. We make the following definition:

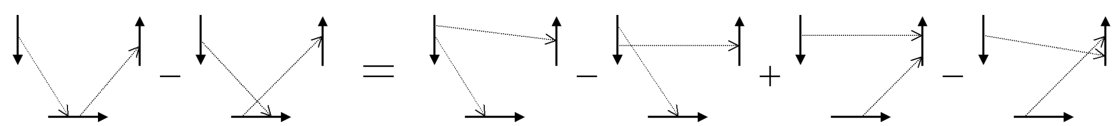

Figure 1.1. The 6T relation.

Definition 1.1.1. $\vec{\mathcal{A}}(\Gamma)$ *is the vector space which is the span of directed chord diagrams with skeleton* Γ *modulo 6T relations. From now on we will use "arrow diagrams" to refer to equivalent classes in* $\vec{\mathcal{A}}(\Gamma)$*. A linear functional on* $\vec{\mathcal{A}}(\Gamma)$ *is called a weight system. In this thesis we will also refer to functions from* $\vec{\mathcal{A}}(\Gamma)$ *to Lie algebra-related spaces (the universal enveloping algebra of a Lie algebra* $\mathfrak{g}$*, or the tensor product of multiple copies of* $\mathfrak{g}$*, its dual* $\mathfrak{g}^*$ *and a representation) as weight systems.*

In this thesis Γ is either a circle or a straight line. $\vec{\mathcal{A}}(\Gamma)$ is isomorphic to a space with a different presentation. We define a "directed Jacobi diagram" with skeleton Γ to be a directed graph whose vertices are either univalent or trivalent so that all its univalent vertices are attached to distinct points on Γ, and each trivalent vertex comes with an orientation (i.e., a cyclic order of the three edges incident at the vertex). A directed Jacobi diagram with skeleton Γ is called acyclic if the underlying directed graph (i.e., the diagram without Γ) does not contain any cycle.

Definition 1.1.2. *Let $\overrightarrow{NS}$ and $\overrightarrow{STU}$ be the relations as shown in figures 1.2 and 1.3. $\vec{\mathcal{A}}_{AJ}(\Gamma)$ ("AJ" stands for "acyclic Jacobi") is the span of acyclic directed trivalent graphs on Γ modulo $\overrightarrow{NS}$ and $\overrightarrow{STU}$.*

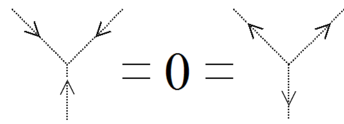

Figure 1.2. The $\overrightarrow{NS}$ relation. We quotient out by any diagram which contains one of the pictures above as a subdiagram.

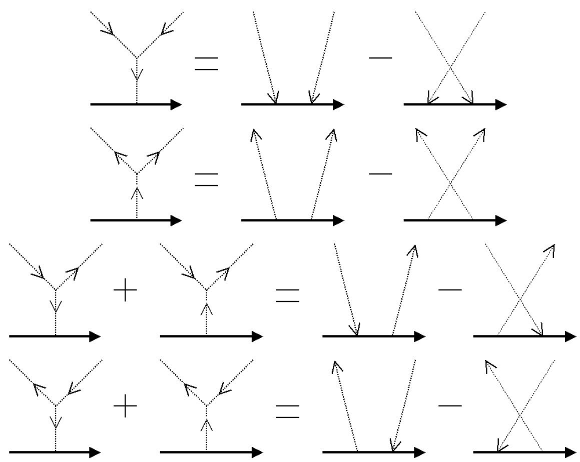

Figure 1.3. The $\overrightarrow{STU}$ relation. Each equation only shows the parts of the diagrams which are different.

$\vec{\mathcal{A}}(\Gamma)$ and $\vec{\mathcal{A}}_{AJ}(\Gamma)$ are related by the following theorem:

Theorem 1.1.1. *(See Polyak's [Po1], Theorem 4.7, Proposition 4.8 and Theorem 4.9) The inclusion map $\iota : \vec{\mathcal{A}}(\Gamma) \to \vec{\mathcal{A}}_{AJ}(\Gamma)$ induces an isomorphism between $\vec{\mathcal{A}}(\Gamma)$ and $\vec{\mathcal{A}}_{AJ}(\Gamma)$. The relations $\overrightarrow{AS}$ and $\overrightarrow{IHX}$ also hold in $\vec{\mathcal{A}}_{AJ}(\Gamma)$.*

The reason we introduce $\vec{\mathcal{A}}_{AJ}(\Gamma)$ is because of their similarity diagramatically to Lie bialgebras. More details will be presented in section 1.4.

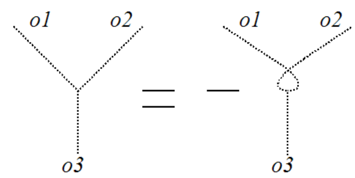

Figure 1.4. The $\overrightarrow{AS}$ relation. The arrows may be oriented anyway so long as they match at $o1$, $o2$ and $o3$. This corresponds to the reversal of a cylic order of the incident edges.

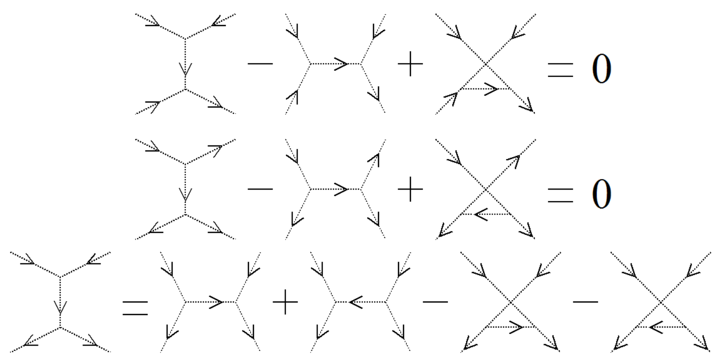

Figure 1.5. The $\overrightarrow{IHX}$ relation.

Let $\vec{\mathcal{A}}(\uparrow)$ denote the space $\vec{\mathcal{A}}(\Gamma)$ where Γ is an oriented line. There is a coproduct structure $\Delta : \vec{\mathcal{A}}(\uparrow) \to (\vec{\mathcal{A}}(\uparrow))^{\otimes 2}$ which we are going to use in chaper 3. It is given by the following formula.

Definition 1.1.3. *Let D be an arrow diagram whose skeleton is a line. $\Delta(D)$ is the sum $\sum D_1 \otimes D_2$, where the sum is over all ways to decompse D into two subdiagrams D_1 and D_2. (See fig 1.6 for an example.) We extend D to all of $\vec{\mathcal{A}}(\uparrow)$ by linearity.*

Proposition 1.1.1. *Δ is well defined on $\vec{\mathcal{A}}(\uparrow)$.*

Proof of Proposition 1.1.1. Consider the $6T$ relation as given in figure 1.7. We have to show Δ maps the left hand side to 0. This can be done by direct computation. In

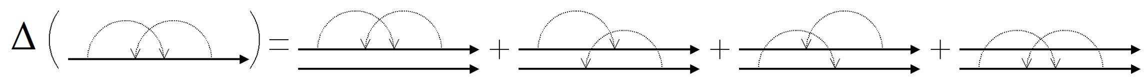

Figure 1.6. An example of a coproduct. The upper strand represents the first component while the lower strand represents the second component.

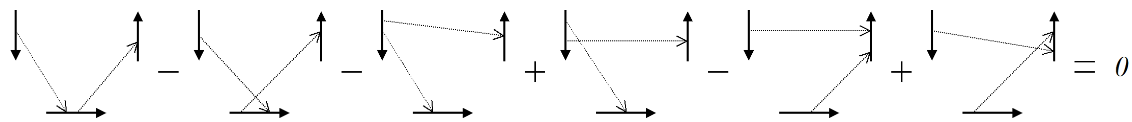

Figure 1.7. The 6T relation with all non-zero terms on one side.

particular given any $(n-2)$-tuple $(i_1, \ldots, i_{n-2})$ so that each $i_j = 1$ or 2, we consider all summands in the image under Δ of the left hand side such that each arrow a_j which does not participate in $6T$ appears in the $(i_j)^{th}$ component. Since we have four ways to decide where to put the two arrows which participate in $6T$, we have $6 \times 4 = 24$ such summands. Out of these 24, consider those in which the arrows which participate in $6T$ appear in different components. There are 12 of them and each term in figure 1.7 is responsible for two. We notice that the two coming from the first term cancel the two from the second term. Similarly the two coming from the third term cancel the two from the fourth, and the two from the fifth term cancel the two from the sixth. What we are left with, therefore, are those summands where the two arrows which participate in $6T$ appear in the same component, but this means we have a sum of a $6T$ relation in the first component and a $6T$ relation in the second component, which is 0 in $(\vec{\mathcal{A}}(\uparrow))^{\otimes 2}$. $\square$

1.2 Relations between finite type invariants and weight systems

This section is a review of the notion of finite type invariants of virtual knots and corresponding weight systems introduced in [GPV]. Also we consider the relation between weight systems and finite type invariants of oriented virtual knots modulo "braid-like" Reidemeister moves (see [BHLR] and below), which are Reidemeister moves where the part of the knot involved is locally a braid. We say an invariant of virtual knots is of type n if it vanishes on all virtual knot diagrams with more than n semi-virtual crossings. (The smallest such n is called the degree of the invariant.) A semi-virtual crossing is the difference between a real crossing and a virtual crossing. On the level of Gauss diagrams we use solid arrows to represent real crossings and dotted arrows to represent semi-virtual crossings (figure 1.8). An invariant is said to be of finite type if it is of type n for some n.

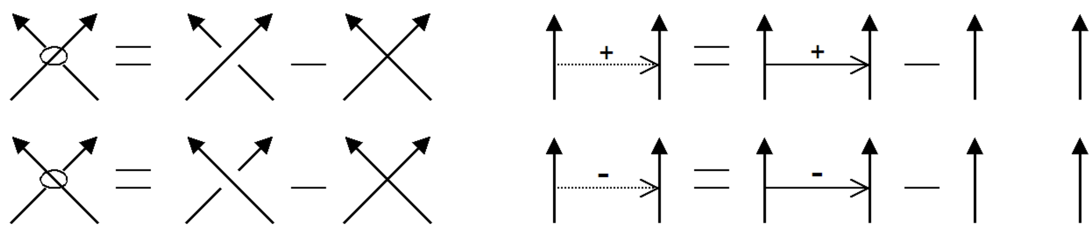

Figure 1.8. The semivirtual crossing. On the right are representations of semivirtual crossings in Gauss diagrams. The sign above each arrow is the sign of the corresponding crossing.

We may also think of figure 1.8 as representing a change of basis, so given a virtual knot diagram we can always express it as a linear combination of virtual knot diagrams with only semi-virtual and virtual crossings. Equivalently, given a Gauss diagram with solid arrows we can always turn it into a linear combination of Gauss diagrams with only dotted arrows. Reidemeister II and III expressed in this new basis can be seen in figures 1.9 and 1.10. On the level of Gauss diagrams the moves are given as relations in figures 1.11 and 1.12. A type n invariant can therefore be considered as a function of Gauss diagrams which respects these relations and vanishes on all diagrams with more than n dotted arrows.

Reidemeister II and III moves come in two types, the braid-like ones and the non-braid-like ones (figures 1.9 and 1.10). Braid-like Reidemeister moves are ones in which the orientations of the participating strands define a consistent ordering on the two (or three) crossings involved, i.e., the move takes place on a part of a braid.

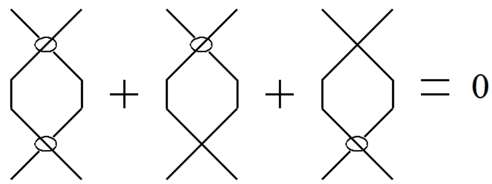

Figure 1.9. Reidemeister II in terms of semivirtual crossings. The braid-like ones are the ones where both strands go up or both strands go down, so that both strands visit the crossings in the same order.

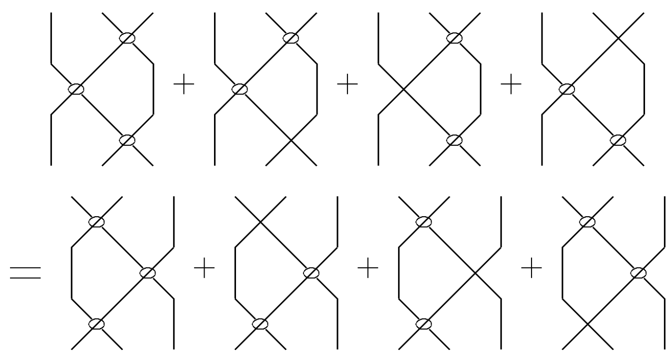

Figure 1.10. Reidemeister III in terms of semivirtual crossings. The non-braid-like ones are ones in which the middle strand goes up (respectively, down) while the other two strands go down (respectively, up). All the other ones are braid-like, i.e., the orientations on the strands order the crossings consistently. All braid-like Reidemeister III moves are consequences of braid-like Reidemeister II moves and the braid-like Reidemeister III move where all crossings have the same sign, i.e., where all the strands above go up or go down.

We make the following definition.

Definition 1.2.1. *The space of (long) braid-like virtual knots is the space of (long)*

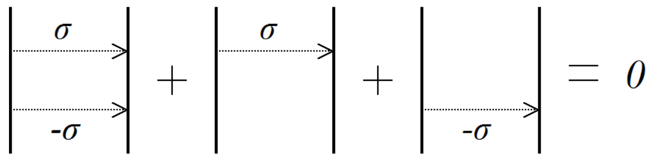

Figure 1.11. Reidemeister II in terms of Gauss diagrams for any sign σ. For braid-like Reidemeister II both strands go up or both strands go down.

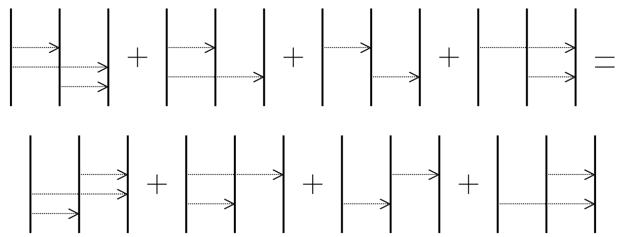

Figure 1.12. The 8T ("eight-term") relation, which is the Reidemeister III move represented with Gauss diagrams with only semivirtual crossings. The vertical strands may be oriented any way. The non-braid-like Reidemeister moves are ones in which the middle strand goes up (respectively, down) while the other two strands go down (respectively, up). The signs of the arrows are dictated by the orientation of the strands and figure 1.10. All other orientations give us braid-like Reidemeister III.

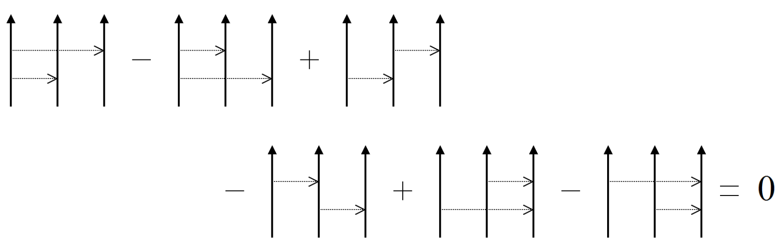

Figure 1.13. A different way of drawing 6T. The 6T relation can be obtained from the braid-like 8T relation where all arrows have the same signs by modding out by degree-$(n+1)$ diagrams.

virtual knot diagrams modulo braid-like Reidemeister II and braid-like Reidemeister III. Invariants of type n of braid-like virtual knots are those which vanish on diagrams with *more than n semi-virtual crossings.*

By the result of the computations presented in [BHLR], the space of braid-like virtual knots is not isomorphic to the space of virtual knots. (This also follows from Theorems 1.1 and 1.2 of Polyak's [Po2].) Let $\mathcal{V}_n$ be the space of all invariants of type n. If we consider an element in the space $\mathcal{V}_n/\mathcal{V}_{n-1}$, each equivalence class can be represented by a function on diagrams with exactly n arrows. Let ϕ be such a function, then ϕ vanishes on diagrams with $n+1$ semi-virtual crossings, so it vanishes on the term with two shown arrows in figure 1.11 if we assume the rest of the diagram contains $n-1$ arrows. This means that if D is a diagram with n arrows and D' is a diagram obtained from D by making a negative arrow (if one exists) positive, then $\phi(D) = (-1)\phi(D')$. Therefore, if D is a diagram with q negative arrows and D_+ is the diagram obtained from D by making all negative arrows positive, ϕ does not distinguish between D and $(-1)^q\phi(D_+)$, so signs of arrows are superfluous. Also ϕ must vanish on the two terms with three arrows shown in the braid-like 8T relation (figure 1.12) if we assume the rest of the diagram contains $n-2$ arrows. In particular, if all arrows carry the same sign, this gives us precisely the 6T relation above. Therefore any element of $\mathcal{V}_n/\mathcal{V}_{n-1}$ gives us a weight system on $\vec{\mathcal{A}}_n$. It remains open, however, if every weight system is the weight system induced by some element of $\mathcal{V}_n/\mathcal{V}_{n-1}$. That is, it remains open if all weight systems satisfy consequences of the $8T$ relations where all the diagrams involved have either degree n or $n-1$, and where all degree-$(n-1)$ diagrams cancel and only degree-n diagrams remain. Computational results presented in [BHLR] (up to degree 5) suggest that all weight systems on $\vec{\mathcal{A}}_n(\uparrow)$ integrate to finite type invariants of long braid-like virtual knots.

Note. We restrict ourselves to only the braid-like Reidemiester II and III here for the following reasons. If we were to introduce cyclic Reidemiester II, we will have to impose extra relations (called “XII” in [BHLR]) on the arrow diagrams. If we were to introduce cyclic Reidemeister III, we then lose the correspondence (suggested by computational results presented in [BHLR]) between weight systems and finite type invariants.

(Cyclic Reidemeister III moves generate the same 6T relations but, by Theorems 1.1 and 1.2 of Polyak's [Po2], are not consequences of braid-like Reidemeister II and III moves. Therefore they reduce the dimensions of the spaces of finite type invariants.)

1.3 Standard Manin triple structures on simple Lie algebras

In this section we review Manin triples and the closely related notion of Drinfeld doubles. We follow chapter 4 of [ES] to construct Manin triples from simple Lie algebras. One word about notation: throughout this thesis we use the Einstein summation notation, i.e., all indices which appear twice in an expression (once as an upper index and once as an lower index) are summed over. The reader may refer to chapter 3 of [ES] for background in Lie bialgebras, especially those which are cobounary, quasitriangular, or triangular.

Definition 1.3.1. *A Lie bialgebra* $(\mathfrak{g}, [,], \delta)$ *is a Lie algebra* $(\mathfrak{g}, [,])$ *with an antisymmetric cobracket map* $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ *satisfying the coJacobi identity*

$$(id + \tau + \tau^2)((\delta \otimes id)\delta(x)) = 0$$

and the cocycle condition

$$\delta([x, y]) = ad_x(\delta y) - ad_y(\delta x),$$

for any $x, y \in \mathfrak{g}$*, where* τ *is the cyclic permutation on* $\mathfrak{g}^{\otimes 3}$*.*

Definition 1.3.2. *A finite dimensional Manin triple is a triple of finite dimensional Lie algebras* $(\tilde{\mathfrak{g}}, \mathfrak{g}_+, \mathfrak{g}_-)$*, where* $\tilde{\mathfrak{g}}$ *is equipped with a metric (a symmetric nondegenerate invariant bilinear form)* $(.,.)$ *such that*

1. $\tilde{\mathfrak{g}} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ *as a vector space and* $\mathfrak{g}_+, \mathfrak{g}_-$ *are Lie subalgebras of* $\tilde{\mathfrak{g}}$*.*

2. $\mathfrak{g}_+, \mathfrak{g}_-$ *are isotropic with respect to* $(.,.)$*, i.e.,* $(\mathfrak{g}_+, \mathfrak{g}_+) = 0 = (\mathfrak{g}_-, \mathfrak{g}_-)$.

Given a Manin triple $(\tilde{\mathfrak{g}}, \mathfrak{g}_+, \mathfrak{g}_-)$*,* $\tilde{\mathfrak{g}}$ *is also called the Drinfeld double of* $\mathfrak{g}_+$ *and is denoted* $D\mathfrak{g}_+$.

As a consequence, $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are maximal isotropic subalgebras. Suppose $(\tilde{\mathfrak{g}}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. The metric then induces a nondegenerate pairing $\mathfrak{g}_+ \otimes \mathfrak{g}_- \to \mathbb{C}$, and hence a Lie algebra structure on $\mathfrak{g}_+^* \cong \mathfrak{g}_-$. Let δ be the induced coalgebra structure on $\mathfrak{g}_+$. We can check by direct computation (section 4.1, [ES]) that the cocycle condition is satisfied. $(\mathfrak{g}_+, [.,.], \delta)$ is therefore a Lie bialgebra.

In fact the process can be reversed. Given a Lie bialgebra $\mathfrak{g}$, we may define a symmetric nondegenerate bilinear form $(.,.)_{\mathfrak{g}\oplus\mathfrak{g}^*}$ on $\mathfrak{g} \oplus \mathfrak{g}^*$, by $((e,f),(e',f'))_{\mathfrak{g}\oplus\mathfrak{g}^*} = f(e') + f'(e)$. If $\{e_i\}$ is a basis of $\mathfrak{g}$ and $\{f^i\}$ is the corresponding dual basis of $\mathfrak{g}^*$, then we can define a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ by making $\mathfrak{g}$ and $\mathfrak{g}^*$ Lie subalgebras and setting, for any $f \in \mathfrak{g}^*$ and $e \in \mathfrak{g}$,

$$[f, e] = ad_e^* f - ad_f^* e.$$

The above definition is motivated by invariance. Since, for any $f, f' \in \mathfrak{g}^*$ and $e, e' \in \mathfrak{g}$, we must have $([f,e],e')_{\mathfrak{g}\oplus\mathfrak{g}^*} = (f,[e,e'])_{\mathfrak{g}\oplus\mathfrak{g}^*}$ and $([f,e],f')_{\mathfrak{g}\oplus\mathfrak{g}^*} = -(e,[f,f'])_{\mathfrak{g}\oplus\mathfrak{g}^*}$, the $\mathfrak{g}^*$ componenent and the $\mathfrak{g}$ component of $[f,e]$ must be $ad_e^* f$ and $-ad_f^* e$, respectively. In terms of the structure constants (with $[e_i, e_j] = c_{ij}^k e_k$ and $[f^r, f^s] = \gamma_t^{rs} f^t$) the relation above can be written as

$$[f^r, e_s] = c_{st}^r f^t - \gamma_s^{rt} e_t. \tag{1.1}$$

(See section 1.3 of [CP].) $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ is therefore a Manin triple.

There is a standard way to obtain Manin triples from simple Lie algebras, and those are the ones we are going to use in chapters 2 and 3. The construction below follows Chapter 4 of [ES]. Given a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ with metric $(.,.)$, we fix a Cartan subalgebra $\mathfrak{h}$ and consider a polarization of the roots $\Delta_+ \cup \Delta_-$ (with $\mathfrak{n}_+$ and $\mathfrak{n}_-$ the corresponding root spaces). For each root α we consider $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ (where $\mathfrak{g}_{\pm\alpha}$ are the root spaces corresponding to $\pm\alpha$) such that $(e_\alpha, f_\alpha) = 1$. Let $h_\alpha = [e_\alpha, f_\alpha]$. We consider the Lie algebra

$$\tilde{\mathfrak{g}} = \mathfrak{n}_+ \oplus \mathfrak{h}^{(1)} \oplus \mathfrak{h}^{(2)} \oplus \mathfrak{n}_-$$

where $\mathfrak{h}^{(1)} \cong \mathfrak{h} \cong \mathfrak{h}^{(2)}$ and with bracket defined by:

$$[\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)}] = 0, \qquad [\mathfrak{h}^{(i)}, e_\alpha] = \alpha(h)e_\alpha,$$
$$[\mathfrak{h}^{(i)}, f_\alpha] = -\alpha(h)f_\alpha, \quad \text{and} \quad [e_\alpha, f_\alpha] = \tfrac{1}{2}(h_\alpha^{(1)} + h_\alpha^{(2)}).$$

We define the following metric on $\tilde{\mathfrak{g}}$:

$$(x + h^{(1)} + h^{(2)}, x' + h'^{(1)} + h'^{(2)})_{\tilde{\mathfrak{g}}} = 2((h^{(1)}, h'^{(2)})_{\mathfrak{g}} + (h^{(2)}, h'^{(1)})_{\mathfrak{g}}) + (x, x')_{\mathfrak{g}}$$

We can check that $(\tilde{\mathfrak{g}}, \mathfrak{n}_+ \oplus \mathfrak{h}^{(1)}, \mathfrak{n}_- \oplus \mathfrak{h}^{(2)})$ is a Manin triple. In fact $\tilde{\mathfrak{g}}$ is a Lie bialgebra with r-matrix

$$\tilde{r} = \sum_{\alpha \in \Delta^+} e_\alpha \otimes f_\alpha + \frac{1}{2} \sum_i k_i^{(1)} \otimes k_i^{(2)},$$

i.e., $\delta(x) = ad_x(\tilde{r})$, where $\{k_i\}$ is an orthonormal basis of $\mathfrak{h}$ with respect to $(.,.)$. We define the projection $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ where

$$\pi|_{\mathfrak{n}_+ \oplus \mathfrak{n}_-} = Id \qquad \pi(h_\alpha^{(1)}) = h_\alpha = \pi(h_\alpha^{(2)})$$

This map endows $\mathfrak{g}$ with a quasitriangular Lie bialgebra structure with r matrix

$$r = \sum_{\alpha \in \Delta^+} e_\alpha \otimes f_\alpha + \frac{1}{2} \sum_i k_i \otimes k_i, \tag{1.2}$$

so $\delta(x) = ad_x(r)$. The Lie subalgebras $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$ and $\mathfrak{b}_- = \mathfrak{n}_- \oplus \mathfrak{h}$ are Lie subbialgebras.

Note that the map π is a Lie algebra homomorphism. In particular if $\mathfrak{g}$ is given as a matrix Lie algebra then π is a representation, and we call it the defining representation of $\tilde{\mathfrak{g}}$.

Definition 1.3.3. *Let $\mathfrak{g}$ be one of the classical Lie algebras and $(\tilde{\mathfrak{g}}, \mathfrak{g}_+, \mathfrak{g}_-)$ be the corresponding Manin triple constructed as above, we call the map π the defining representation of the Manin triple $(\tilde{\mathfrak{g}}, \mathfrak{g}_+, \mathfrak{g}_-)$.*

It is worth noting that we have an easier way to describe $\tilde{\mathfrak{g}}$. If we define

$$h_\alpha^C = \frac{1}{2}(h_\alpha^{(1)} + h_\alpha^{(2)}), h_\alpha^Z = \frac{1}{2}(h_\alpha^{(1)} - h_\alpha^{(2)}),$$

where C stands for "Cartan" and Z stands for "zentral" ("central"), and let $\mathfrak{h}^C$ be the space spanned by all h_α^C's, then we have

$$[\mathfrak{h}^C, f_\alpha] = -\alpha(h)f_\alpha, \qquad [\mathfrak{h}^C, e_\alpha] = \alpha(h)e_\alpha,$$
$$[\mathfrak{h}^C, \mathfrak{h}^C] = 0, \quad [\mathfrak{h}^Z, \tilde{\mathfrak{g}}] = 0, \quad \text{and} \quad [e_\alpha, f_\alpha] = h_\alpha^C.$$

Therefore we can write $\tilde{\mathfrak{g}} \cong \mathfrak{g} \oplus \mathfrak{h}^Z$, where the direct sum is a direct sum of Lie algebras. The map π is given by $\pi(e_\alpha) = e_\alpha$, $\pi(f_\alpha) = f_\alpha$, $\pi(h_\alpha^C) = h_\alpha$ and $\pi(h_\alpha^Z) = 0$.

1.4 Directed trivalent graphs and Lie tensors

In this section we follow section 3.2 of [Ha] to construct elements of tensors of Lie algebras out of diagrams from $\vec{\mathcal{A}}(\Gamma)$ and $\vec{\mathcal{A}}_{AJ}(\Gamma)$. First we notice that equation (1.1) immediately suggests a relation between Lie bialgebras and $\vec{\mathcal{A}}(\Gamma)$. Given a Lie bialgebra $\mathfrak{g}$ we consider its Drinfeld double $\mathfrak{g} \oplus \mathfrak{g}^*$. If $\{e_i\}$ is a basis of $\mathfrak{g}$ and $\{f^i\}$ is the corresponding dual basis of $\mathfrak{g}^*$, we follow [Ha] and put f^i at the end of an arrow and e_i at the head. (See figure 1.15.) We then move along fragments of the skeleton to get a tensor product of letters, picking up an f^i or an e_i whenever we encounter the tail of the head of an arrow. If we do the above to figure 1.1, we get figure 1.14, which is a diagrammatic representation of the following equation in $(D\mathfrak{g})^{\otimes 3}$:

$$f^i \otimes [e_i, f^j] \otimes e_j = [f^j, f^k] \otimes e_k \otimes e_j + f^i \otimes f^k \otimes [e_k, e_i].$$

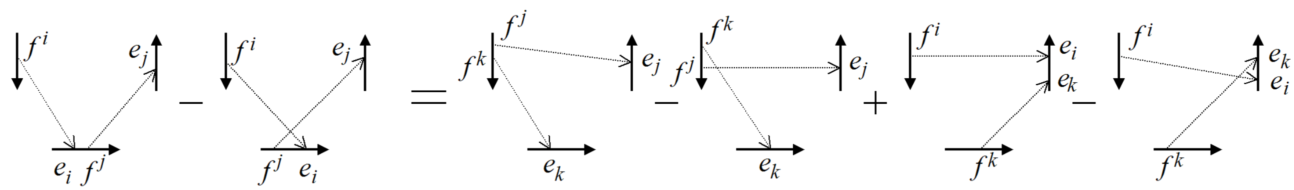

Figure 1.14. The 6T relation with each arrow labelled by $f^i \otimes e_i$.

(By equation (1.1) we can check that the equation holds as both sides are equal to $\gamma_i^{jk}(f^i \otimes e_k \otimes e_j) - c_{ik}^j(f^i \otimes f^k \otimes e_j)$.)

Given that $\vec{\mathcal{A}}(\Gamma)$ and $\vec{\mathcal{A}}_{AJ}(\Gamma)$ are isomorphic, we use the $\overrightarrow{STU}$ relations to interpret the trivalent vertices in elements of $\vec{\mathcal{A}}_{AJ}(\Gamma)$. Given $\overrightarrow{NS}$ we only have two types of vertices ("two in, one out" and "one in, two out"). The $\overrightarrow{STU}$ relation suggests that a vertex corresponds to a bracket in $D\mathfrak{g}$. The first two relations in figure 1.3 suggest that the "two in, one out" vertex should correspond to the bracket in $\mathfrak{g}$, while the "one in, two out" vertex should correspond to the bracket in $\mathfrak{g}^*$, or the cobracket in $\mathfrak{g}$. Once this correspondence is established, the last two relations in the same figure then just become a diagrammatic version of equation (1.1).

These are exactly the tensors Haviv introduced in his paper ([Ha]). To complete the picture we assign a representation $D\mathfrak{g} \to End(V)$ to each connected piece of the skeleton, so that the tail of a piece of the skeleton corresponds to a copy of V^* while the head of the skeleton corresponds to a copy of V.

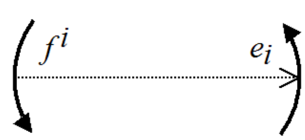

Figure 1.15. The arrow.

In more details the cobracket tensor is given by $f^i \otimes f^j \otimes [e_i, e_j] = c_{ij}^k(f^i \otimes f^j \otimes e_k)$, while the bracket tensor is given by $f^i \otimes \delta(e_i) = \gamma_i^{jk}(f^i \otimes e_j \otimes e_k)$, where c_{ij}^k and γ_i^{jk} are the structure constants for the bracket and the cobracket, respectively. (See figure 1.16.) It is worth noting that under this interpretation, the 3-term IHX relations become the Jacobi and coJacobi identities, and the 5-term IHX becomes the cocycle identity.

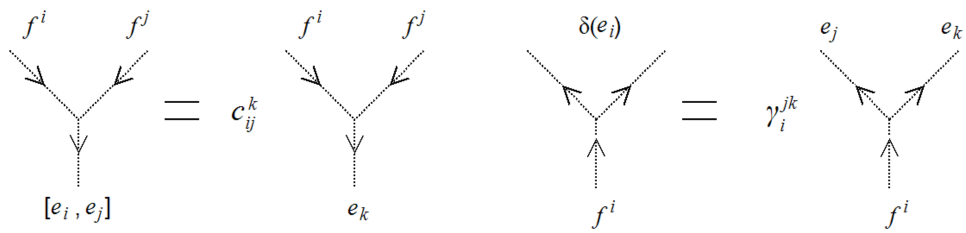

Figure 1.16. The bracket (two in, one out) tensor (left) and the cobracket (one in, two out) tensor (right).

To assign a tensor to a directed graph, we break it down into subgraphs with 0 or 1 vertex, assign tensors to these elementary pieces, and at the points of gluing along the dotted edges we contract these tensors using the metric. When we glue pieces of the skeleton we contract the corresponding pieces of V and V^*.

Given a representation $R : \tilde{\mathfrak{g}} \to End(V)$ where V is given a specified basis $b = \{v_1, ..., v_d\}$, if the skeleton is part of the picture, we assign Greek letters ranging over $\{1, ..., d\}$ to each section of the skeleton. For example in figure 1.17, we assign

$$R(e_i) \otimes f^i \in End(V) \otimes \tilde{\mathfrak{g}},$$

or

$$(v^\beta(R(e_i)(v_\alpha)))(v^\alpha \otimes v_\beta \otimes f^i) \in V^* \otimes V \otimes \tilde{\mathfrak{g}}$$

to the picture on the left. Also we assign

$$R(f^i) \otimes \delta(e_i) \in End(V) \otimes \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}},$$

or

$$v^\beta(R(f^i)(v_\alpha))(\delta e_i) = \gamma_i^{jk} v^\beta(R(f^i)(v_\alpha))(v^\alpha \otimes v_\beta \otimes e_j \otimes e_k) \in V^* \otimes V \otimes \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$$

to the picture on the right. Note, since $v^\beta = \langle v_\beta, .\rangle$ where $\langle ., .\rangle$ is the inner product of $\mathbb{C}^d$ with respect to the given basis, the same values can be written as $\langle v_\beta, R(e_i)(v_\alpha)\rangle f^i$ and $\langle v_\beta, R(f^i)(v_\alpha)\rangle \delta e_i$. Here we do not distinguish $R(f^i)$ or $R(e_i)$ from its matrix with respect to the basis b.

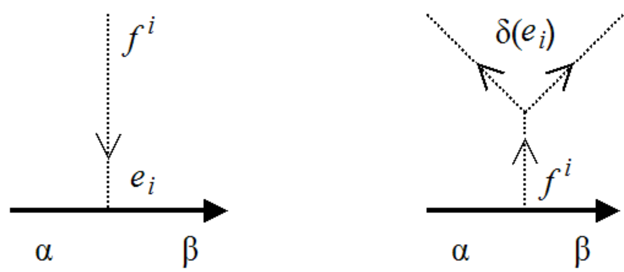

Figure 1.17. We assign a representation to the skeleton. The diagrams correspond to $(R(e_i))^\beta_\alpha f^i$ and $(R(f^i))^\beta_\alpha(\delta e_i)$, respectively. $(R(e_i))^\beta_\alpha$ ($(R(f^i))^\beta_\alpha$) is the entry in the β_{th} row and the α^{th} column of the matrix of $R(e_i)$ ($R(f^i)$) with respect to the given basis.

If we restrict ourselves to the case where the skeleton is a circle, we can see that the construction above gives us the trace of the tensor in the given representation. Note, however, if we don't specify a representation we have a weight system from arrow diagrams to the universal enveloping algebra $U(\tilde{\mathfrak{g}})$ (for $\vec{\mathcal{A}}(\uparrow)$) or $U(\tilde{\mathfrak{g}})/\{xw - wx : x \in \tilde{\mathfrak{g}}, w \in U(\tilde{\mathfrak{g}})\}$ (for $\vec{\mathcal{A}}(\bigcirc)$).

CYBE Weight Systems. Here we briefly introduce a different class of weight systems called CYBE weight systems. (CYBE stands for classical Yang-Baxter equation. See chapter 2 of [CP] and sections 3.2 and 3.3 of [Po1].) We recall figure 1.13, which is a different way of drawing $6T$. It is a diagrammatic way of writing the Classical Yang-Baxter Equation (CYBE)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

If we take an r-matrix, we get a map $\vec{\mathcal{A}}(\Gamma) \to U(\mathfrak{g})$ which is called a CYBE weight system. Note that a triangular Lie bialgebra $\mathfrak{g}$ (chapter 3, [ES]) gives us both a Drinfeld double weight system and a CYBE weight system. (For the Drinfeld double weight system we associate to each arrow the element in $\mathfrak{g} \otimes \mathfrak{g}^*$ which corresponds to the identity map on $\mathfrak{g}$, while for the CYBE weight system we associate to each arrow the r-matrix, an element in $\mathfrak{g} \otimes \mathfrak{g}$.) The two resulting weight systems, however, may have different ranks. One easy example is that given any non-trivial Lie algebra, we can define a triangular Lie bialgebraic structure on it by letting $r = 0$. In this case the CYBE weight system would be 0, but the Drinfeld double weight system would not be 0.

Chapter 2

Combinatorial Formulae in the Defining Representations

In this chapter we will present combinatorial formulae for weight systems coming from Manin triples constructed from classical Lie algebras, following chapter 1, and their defining representations. These formulae turn the problem of finding weight systems in the defining representation into certain counting problems. The metric in each of the Lie algebras is $(A, B) = tr(AB)$. Throughout this chapter m_{ij} or m^{ij} is the matrix whose ij-th entry is 1 and zero everywhere else. Given a matrix M, the term M_α^β is the entry in the β^{th} row and α^{th} column of M. Let π be the defining representation as given in section 1.3 (i.e., the map which identifies the two copies of the Cartan subalgebra), we let $x_i = \pi(e_i)$ and $\xi^i = \pi(f^i)$.

2.1 $\mathfrak{gl}(N)$

We begin with $\mathfrak{gl}(N)$, the Lie algebra of all $N \times N$ matrices. Note that $\mathfrak{gl}(N)$ is not simple, but our construction in section 1.3 can be applied to $\mathfrak{sl}(N)$. One decomposition of $\mathfrak{sl}(N)$ into positive and negative root spaces is

$$\begin{aligned}
\mathfrak{n}_+ &= \text{span}\{m_{ij} : i < j\} \\
\mathfrak{n}_- &= \text{span}\{m_{ji} : i < j\} \\
\mathfrak{h} &= \text{span}\{m_{ii} - m_{i+1,i+1} : 1 \leq i < n\}
\end{aligned}$$

Applying the procedure in section 1.3 results in the Manin triple $(\widetilde{\mathfrak{sl}}, \mathfrak{sl}_+, \mathfrak{sl}_-)$, where $\mathfrak{sl}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}^{(1)}$ and $\mathfrak{sl}_- = \mathfrak{n}_- \oplus \mathfrak{h}^{(2)}$. Let $\mathfrak{s}$ be the commutative Lie algebra of scalar matrices. We define

$$\mathfrak{gl}_+ = \mathfrak{sl}_+ \oplus \mathfrak{s}^{(1)}$$

and

$$\mathfrak{gl}_- = \mathfrak{sl}_- \oplus \mathfrak{s}^{(2)}$$

where $\mathfrak{s}^{(1)}$ and $\mathfrak{s}^{(2)}$ are two distinct copies of the (commutative) algebra of scalar matrices, and the direct sums above are direct sums of Lie algebras.

We define a Lie algebra $\widetilde{\mathfrak{gl}}$ which as a vector space is $\mathfrak{gl}_+ \oplus \mathfrak{gl}_-$, and whose metric and bracket are as follows. The metric is given by

$$(x + s^{(1)} + s^{(2)}, x' + s'^{(1)} + s'^{(2)})_{\widetilde{\mathfrak{gl}}} = 2((s^{(1)}, s'^{(2)}) + (s'^{(1)}, s^{(2)})) + (x, x')_{\widetilde{\mathfrak{sl}}}$$

where x, x' are arbitrary elements in $\widetilde{\mathfrak{sl}}$, while $s^{(1)}, s'^{(1)}$ are arbitrary elements in $\mathfrak{s}^{(1)}$ and $s^{(2)}, s'^{(2)}$ arbitrary elements in $\mathfrak{s}^{(2)}$. We define the bracket so that $\widetilde{\mathfrak{sl}}$ is a Lie subalgebra of $\widetilde{\mathfrak{gl}}$ (i.e., $[\widetilde{\mathfrak{sl}}, \widetilde{\mathfrak{sl}}]_{\widetilde{\mathfrak{gl}}} = [\widetilde{\mathfrak{sl}}, \widetilde{\mathfrak{sl}}]_{\widetilde{\mathfrak{sl}}}$) and

$$[\mathfrak{s}^{(i)}, x]_{\widetilde{\mathfrak{gl}}} = 0 \text{ for } i = 1, 2 \text{ and any } x \in \widetilde{\mathfrak{gl}},$$

where $(.,.)_{\widetilde{\mathfrak{sl}}}$ and $[.,.]_{\widetilde{\mathfrak{sl}}}$ are the metric and the bracket in $\widetilde{\mathfrak{sl}}$, respectively.

We consider the Manin triple $(\widetilde{\mathfrak{gl}}, \mathfrak{gl}_+, \mathfrak{gl}_-)$. In this thesis we use $\mathfrak{gl}(N)$ instead of $\mathfrak{sl}(N)$. Note that $\{e_{ij}\}_{i \leq j}$ forms a basis of $\mathfrak{n}^+ \oplus \mathfrak{s}^{(1)}$ and $\{f^{ij}\}_{i \leq j}$ forms the corresponding dual basis of $\mathfrak{g}^* \cong \mathfrak{n}^- \oplus \mathfrak{s}^{(2)}$. The map π is as given in section 1.3. We have $x_{ij} = \pi(e_{ij}) =$

m_{ij} and $\xi^{ij} = \pi(f^{ij}) = m^{ji}$. Following the previous chapter we put ξ^{ij} at the tail and x_{ij} at the head of each arrow. By equation (1.2) we multiply by a factor of $\frac{1}{2}$ when $i = j$.

We consider the tensor in figure 2.1. It corresponds to the map:

$$\sum_{i<j} \xi^{ij} \otimes x_{ij} + \frac{1}{2} \sum_{i} \xi^{ii} \otimes x_{ii} : V \otimes V \to V \otimes V.$$

In tensor form we can write it as $T^{\beta\nu}_{\alpha\mu} v^{\alpha} \otimes v^{\mu} \otimes v_{\beta} \otimes v_{\nu}$. What we mean by a **combinatorial formula** in this thesis is a combinatorial description of $T^{\beta\nu}_{\alpha\mu}$ in terms of its indices.

Figure 2.1. The simplest subdiagram in an arrow diagram.

We now find the coefficient $T^{\beta\nu}_{\alpha\mu}$. Since $x_{ij} = m_{ij}$ and $\xi^{ij} = m^{ji}$, the coefficient is given by

$$\begin{aligned} T^{\beta\nu}_{\alpha\mu} &= \sum_{i<j} (m^{ji})^{\beta}_{\alpha} (m_{ij})^{\nu}_{\mu} + \frac{1}{2} \sum_{i} (m^{ii})^{\beta}_{\alpha} (m_{ii})^{\nu}_{\mu} \\ &= \sum_{i<j} \delta^{j\beta} \delta^{i}_{\alpha} \delta^{\nu}_{i} \delta_{j\mu} + \frac{1}{2} \sum_{i} \delta^{i\beta} \delta^{i}_{\alpha} \delta^{\nu}_{i} \delta_{i\mu} \\ &= \begin{cases} \delta^{\beta}_{\mu} \delta^{\nu}_{\alpha} & \text{for } \alpha < \beta \\ \frac{1}{2} \delta^{\beta}_{\mu} \delta^{\nu}_{\alpha} & \text{for } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 1 & \text{for } \alpha = \nu < \beta = \mu \\ \frac{1}{2} & \text{for } \alpha = \nu = \beta = \mu \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The result can be expressed more clearly using a diagram as in figure 2.2, where we connect two Greek letters if they are required to be equal if $T^{\beta\nu}_{\alpha\mu}$ is to be non-zero. The double headed arrow is a shorthand for the relation above (i.e., the coefficient is 1 when the side from the tail of the double-headed arrow is bigger and $\frac{1}{2}$ when the two sides of the double-headed arrow are equal).

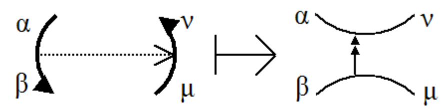

Figure 2.2. A diagrammatic representation of the $\mathfrak{gl}(N)$ tensor.

Direct verification that the above formula satisfies $6T$. Here we include a direct verification that the above formula satisfies the $6T$ relation.

Considering figure 1.1, let ζ_1 be the index corresponding to the free end of the skeleton in the top left corner, and let the indices corresponding to the other free ends be $\zeta_2, ..., \zeta_6$ counting counterclockwise (figure 2.3). By figure 1.1, the coefficient of $v^{\zeta_1} \otimes v_{\zeta_2} \otimes v^{\zeta_3} \otimes v_{\zeta_4} \otimes v^{\zeta_5} \otimes v_{\zeta_6}$ is possibly non-zero only if, for some α, β, γ, σ, μ and ν,

1. $\zeta_1 = \zeta_6 = \gamma$, $\zeta_2 = \zeta_3 = \alpha$ and $\zeta_4 = \zeta_5 = \beta$, or

2. $\zeta_1 = \zeta_4 = \sigma$, $\zeta_2 = \zeta_5 = \mu$ and $\zeta_3 = \zeta_6 = \nu$ for some $\alpha, \beta, \gamma, \mu, \nu, \sigma$.

The $\mathfrak{gl}$ tensor further dictates that $\alpha \geq \gamma$ and $\beta \geq \gamma$, and $\mu \geq \nu$ and $\mu \geq \sigma$.

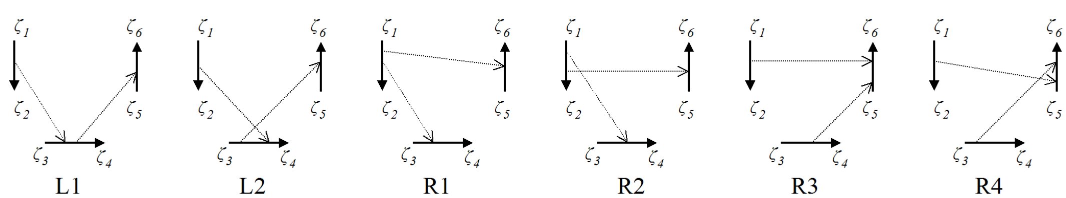

Figure 2.3. The terms of the 6T relation, with free ends of the skeleton labelled as described in the body of the text. Ln (respectively, Rn) is the n-th term on the left (respectively, right) in figure 1.1

Suppose only one of the conditions above are satisfied, we prove that the coefficients of $v^{\zeta_1} \otimes v_{\zeta_2} \otimes v^{\zeta_3} \otimes v_{\zeta_4} \otimes v^{\zeta_5} \otimes v_{\zeta_6}$ on both sides of $6T$ are equal. In the following table the

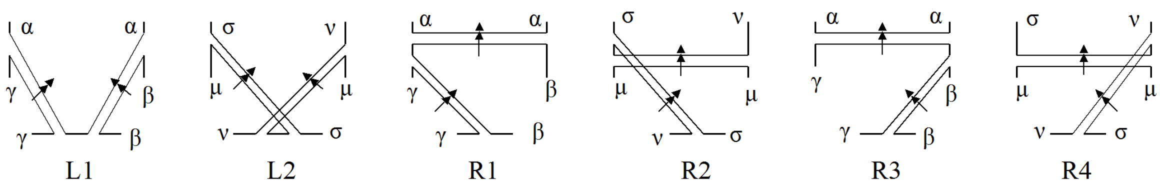

Figure 2.4. The terms of the 6T relation, with relations imposed on the indices by the $\mathfrak{gl}$ tensor. Ln (respectively, Rn) is the n-th term on the left (respectively, right) in figure 1.1

column Ln (Rn) represents the contribution of the n^{th} term on the left (right) hand side of $6T$ to the coefficient of $v^\gamma \otimes v_\alpha \otimes v^\alpha \otimes v_\beta \otimes v^\beta \otimes v_\gamma$ (if relation 1 above is satisfied) for different values of α, β, γ. If relation 2 is satisfied we give the contribution of each term to the coefficient of $v^\sigma \otimes v_\mu \otimes v^\nu \otimes v_\sigma \otimes v^\mu \otimes v_\nu$ for different values of μ, ν, σ. In all cases the total contribution from the left equals to the total contribution from the right.

Order	Relation satisfied	L1	L2	R1	R2	R3	R4
$\alpha > \beta > \gamma$	1 only	1	0	1	0	0	0
$\alpha > \beta = \gamma$	1 only	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0
$\alpha = \beta > \gamma$	1 only	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
$\beta > \alpha = \gamma$	1 only	1	0	0	0	1	0
$\beta > \alpha > \gamma$	1 only	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0
$\mu > \nu > \sigma$	2 only	0	-1	0	-1	0	0
$\mu = \nu > \sigma$	2 only	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0
$\mu > \nu = \sigma$	2 only	0	-1	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\mu = \sigma > \nu$	2 only	0	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$
$\mu > \sigma > \nu$	2 only	0	-1	0	0	0	-1
All ζ_i's equal	1 and 2	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$

Note that if both conditions 1 and 2 above are satisfied, we have $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_5 = \zeta_6$, hence the last row in the table.

2.2 $\mathfrak{so}(2N)$

One decomposition of $\mathfrak{so}(2N)$ into positive root spaces, negative root spaces, and Cartan subalgebra is as follows:

$$
\begin{aligned}
\mathfrak{n}_+ &= \operatorname{span}\{m_{ij} - m_{j+N,i+N} : 1 \le i < j \le N\} \cup \\
&\quad \operatorname{span}\{m_{i,j+N} - m_{j,i+N} : 1 \le i < j \le N\} \\
\mathfrak{n}_- &= \operatorname{span}\{m^{ji} - m^{i+N,j+N} : 1 \le i < j \le N\} \cup \\
&\quad \operatorname{span}\{m^{j+N,i} - m^{i+N,j} : 1 \le i < j \le N\} \\
\mathfrak{h} &= \operatorname{span}\{m_{ii} - m_{i+N,i+N} : 1 \le i \le N\}
\end{aligned}
$$

For $\widetilde{\mathfrak{so}}(2N)$ (with two copies $\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)}$ of the Cartan subalgebra, see section 1.3) we use the basis $\{e_{ijk}\}$ for $\mathfrak{n}_+ \oplus \mathfrak{h}^{(1)}$ and $\{f^{ijk}\}$ for $\mathfrak{n}_- \oplus \mathfrak{h}^{(2)}$, where $0 \le i \le j \le N$ and $k = 1, 2$, such that if we let $x_{ijk} = \pi(e_{ijk})$ and $\xi^{ijk} = \pi(f^{ijk})$, we get

$$
\begin{aligned}
x_{ij1} &= m_{ij} - m_{j+N,i+N}, \text{ where } i \le j \\
x_{ij2} &= m_{i,j+N} - m_{j,i+N}, \text{ where } i < j \\
\xi^{ij1} &= \frac{1}{2}(m^{ji} - m^{i+N,j+N}), \text{ where } i \le j \\
\xi^{ij2} &= \frac{1}{2}(m^{j+N,i} - m^{i+N,j}), \text{ where } i < j
\end{aligned}
$$

The arrow is, by equation (1.2), identified with the element

$$
\sum_{\substack{1 \le i < j \le N \\ k=1,2}} \xi^{ijk} \otimes x_{ijk} + \frac{1}{2} \sum_{1 \le i \le N} (m^{ii} - m^{i+N,i+N}) \otimes (m_{ii} - m_{i+N,i+N}).
$$

$T^{\beta\nu}_{\alpha\mu}$ in this case is given by

$$
\begin{aligned}
T^{\beta\nu}_{\alpha\mu} &= \sum_{\substack{i<j\\k=1,2}} (\xi^{ijk})^{\beta}_{\alpha}(x_{ijk})^{\nu}_{\mu} + \frac{1}{2}\sum_{i}(m^{ii} - m^{i+N,i+N})^{\beta}_{\alpha}(m_{ii} - m_{i+N,i+N})^{\nu}_{\mu} \\
&= \sum_{i\leq j}((\frac{1}{2})^{\delta^{ij}+1}(\delta^{\beta j}\delta^{i}_{\alpha} - \delta^{\beta,i+N}\delta^{j+N}_{\alpha})(\delta^{\nu}_{i}\delta_{j\mu} - \delta^{\nu}_{j+N}\delta_{i+N,\mu}) \\
&\quad +(\frac{1}{2})^{\delta^{ij}+1}(-\delta^{\beta,i+N}\delta^{j}_{\alpha} + \delta^{\beta,j+N}\delta^{i}_{\alpha})(\delta^{\nu}_{i}\delta_{j+N,\mu} - \delta^{\nu}_{j}\delta_{i+N,\mu}))
\end{aligned}
$$

In the expression above the first summand is given by the case when $k = 1$ while the second product corresponds to the case $k = 2$. We consider the first product. Letting A be the set $\{1, 2, ..., N\}$ and B the set $\{N + 1, N + 2, ..., 2N\}$, we can assign a tag A or B to each of α, β, μ and ν to indicate which set it belongs to. The four combinations that correspond to the four ways to expand the first summand in the last equation are as given in figure 2.5.

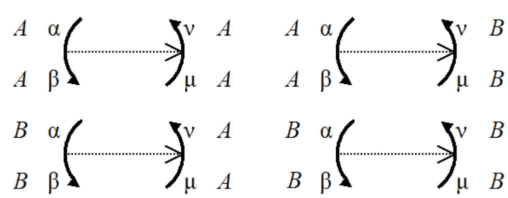

Figure 2.5. Four ways to assign A and B to α, β, μ and ν.

Starting at the top left corner and going clockwise the four pictures correspond to $\delta^{\beta,j}\delta^{i}_{\alpha}\delta^{\nu}_{i}\delta_{j\mu}$, $\delta^{\beta,j}\delta^{i}_{\alpha}\delta^{\nu}_{j+N}\delta_{i+N,\mu}$, $\delta^{\beta,i+N}\delta^{j+N}_{\alpha}\delta^{\nu}_{j+N}\delta_{i+N,\mu}$ and $\delta^{\beta,i+N}\delta^{j+N}_{\alpha}\delta^{\nu}_{i}\delta_{j\mu}$, respectively. If we remove the arrow and join two Greek letters to indicate that they are equal (or equal modulo N, if one comes from A and the other from B), then the first summand can be expressed diagramatically as in figure 2.6.

The double headed arrow here indicates that the value of the coefficient is 1 when the side from the tail of the double-headed arrow is bigger than the side from the head when we mod out by N, and $\frac{1}{2}$ when the two sides of the double-headed arrow are equal. Now we look at the product corresponding to the case $k = 2$. The only possible

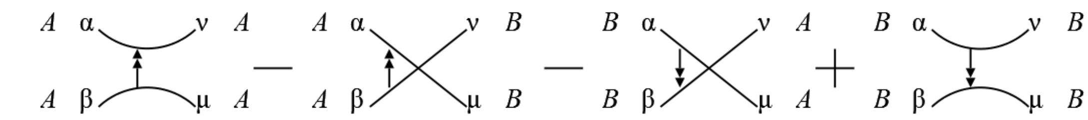

Figure 2.6. The $\mathfrak{so}(2N)$ tensor when $k = 1$.

combination of tags is given in figure 2.7. Similarly we can represent the second summand diagramatically. See figure 2.8.

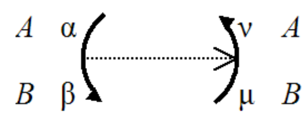

Figure 2.7. The only possible assignment of A and B when $k = 2$.

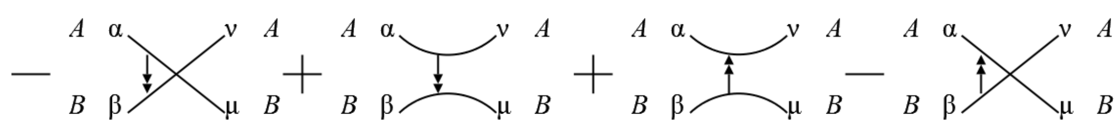

Figure 2.8. The $\mathfrak{so}(2N)$ tensor when $k = 2$.

In this case the equality may be weighted with any factor because when $\alpha = \beta = \mu = \nu$ we get zero. If we choose the factor to be $\frac{1}{2}$, as we did in the beginning of this subsection, diagramatically the coefficient $T_{\alpha\mu}^{\beta\nu}$ may be expressed as in figure 2.9. Note the last four terms in figure 2.9 can be simplified to figure 2.10, by the property of the double headed arrow. What this means is that for the particular combination of tags shown in the diagram, the coefficient is 1 whenever $\alpha = \nu$ and $\beta = \mu$, regardless of the relative order between α and β.

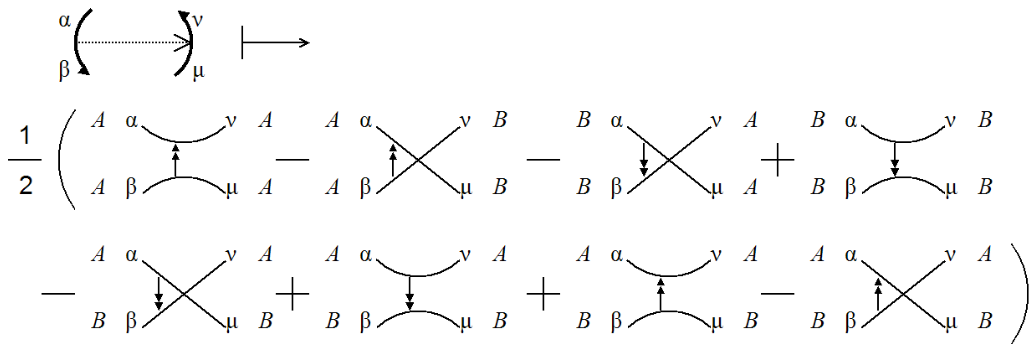

Figure 2.9. The $\mathfrak{so}(2N)$ tensor.

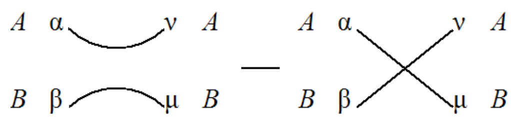

Figure 2.10. The last four terms from figure 2.9 can be simplified to this figure.

2.3 $\mathfrak{sp}(2N)$

One decomposition of $\mathfrak{sp}(2N)$ into positive root spaces, negative root spaces, and Cartan subalgebra is as follows:

$$\begin{aligned}
\mathfrak{n}_+ &= \text{span}\{m_{ij} - m_{j+N,i+N} : 1 \leq i < j \leq N\} \cup \\
&\quad \text{span}\{m_{i,j+N} + m_{j,i+N} : 1 \leq i \leq j \leq N\} \\
\mathfrak{n}_- &= \text{span}\{m^{ji} - m^{i+N,j+N} : 1 \leq i < j \leq N\} \cup \\
&\quad \text{span}\{m^{j+N,i} + m^{i+N,j} : 1 \leq i \leq j \leq N\} \\
\mathfrak{h} &= \text{span}\{m_{ii} - m_{i+N,i+N} : 1 \leq i \leq N\}
\end{aligned}$$

For $\widetilde{\mathfrak{sp}}(2N)$ (with two copies $\mathfrak{h}^{(1)}, \mathfrak{h}^{(2)}$ of the Cartan subalgebra, see section 1.3) we use the basis $\{e_{ijk}\}$ for $\mathfrak{n}_+ \oplus \mathfrak{h}^{(1)}$ and $\{f^{ijk}\}$ for $\mathfrak{n}_- \oplus \mathfrak{h}^{(2)}$, where $0 \leq i \leq j \leq N$ and

$k = 1, 2$, such that if we let $x_{ijk} = \pi(e_{ijk})$ and $\xi^{ijk} = \pi(f^{ijk})$ we get:

$$\begin{aligned}
x_{ij1} &= m_{ij} - m_{j+N,i+N}, \text{ where } i \leq j \\
x_{ij2} &= m_{i,j+N} + m_{j,i+N}, \text{ where } i \leq j \\
\xi^{ij1} &= \frac{1}{2}(m^{ji} - m^{i+N,j+N}), \text{ where } i \leq j \\
\xi^{ij2} &= (\frac{1}{2})^{\delta^{ij}+1}(m^{j+N,i} + m^{i+N,j}), \text{ where } i \leq j
\end{aligned}$$

The arrow is, by equation (1.2), identified with

$$\sum_{\substack{1 \leq i < j \leq N \\ k=1,2}} x_{ijk} \otimes \xi^{ijk} + \frac{1}{2} \sum_{1 \leq i \leq N} (m^{ii} - m^{i+N,i+N}) \otimes (m_{ii} - m_{i+N,i+N}).$$

If we look at the coefficient $T^{\beta\nu}_{\alpha\mu}$ we get:

$$\begin{aligned}
T^{\beta\nu}_{\alpha\mu} &= \sum_{1 \leq i < j \leq N} (\xi^{ij1})^{\beta}_{\alpha} (x_{ij1})^{\nu}_{\mu} + \sum_{1 \leq i \leq j \leq N} (\xi^{ij2})^{\beta}_{\alpha} (x_{ij2})^{\nu}_{\mu} \\
&\quad + \frac{1}{2} \sum_{1 \leq i \leq N} (m^{ii} - m^{i+N,i+N})^{\beta}_{\alpha} (m_{ii} - m_{i+N,i+N})^{\nu}_{\mu} \\
&= \sum_{i \leq j} ((\frac{1}{2})^{\delta^{ij}+1} (\delta^{\beta,j} \delta^{i}_{\alpha} - \delta^{\beta,i+N} \delta^{j+N}_{\alpha})(\delta^{\nu}_{i} \delta_{j\mu} - \delta^{\nu}_{j+N} \delta_{i+N,\mu}) \\
&\quad + (\frac{1}{2})^{\delta^{ij}+1} (\delta^{\beta,i+N} \delta^{j}_{\alpha} + \delta^{\beta,j+N} \delta^{i}_{\alpha})(\delta^{\nu}_{i} \delta_{j+N,\mu} + \delta^{\nu}_{j} \delta_{i+N,\mu}))
\end{aligned}$$

The extra factor of $\frac{1}{2}$ in the second product is due to the fact that $(x_{ii2}, \xi^{ii2}) = 4$. Like in the case of $\mathfrak{so}(2N)$ we assign tags A and B to the four corners to indicate which set each of α, β, μ and ν belongs to. The only possible labellings are the same as the ones for $\mathfrak{so}(2N)$. (See figures 2.5 and 2.7 for the cases $k = 1$ and $k = 2$, respectively.) Following the same steps as in the $\mathfrak{so}(2N)$ case we can see that the $\mathfrak{sp}(2N)$ tensor may be given as in figure 2.11. Note that it is almost the tensor for $\mathfrak{so}(2N)$ except for some sign difference. The last four terms in figure 2.11 can be simplified to figure 2.12, by the property of the double headed arrow.

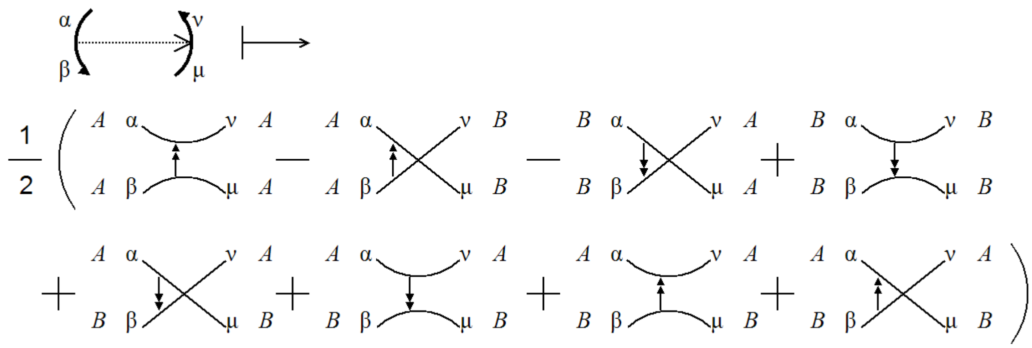

Figure 2.11. The $\mathfrak{sp}(2N)$ tensor.

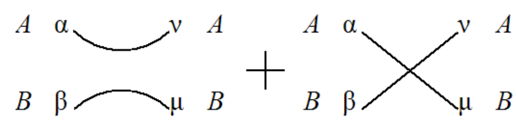

Figure 2.12. The last four terms from figure 2.11 can be simplified to this figure.

2.4 $\mathfrak{so}(2N+1)$

In this section, given a $(2N+1) \times (2N+1)$ matrix we count the rows and columns starting from 0.

$$
\begin{aligned}
\mathfrak{n}_+ &= \text{span}\{m_{i,0} - m_{0,i+N} : 1 \leq i \leq N\} \cup \\
&\quad \text{span}\{m_{i,j} - m_{j+N,i+N} : 1 \leq i < j \leq N\} \cup \\
&\quad \text{span}\{m_{i,j+N} - m_{j,i+N} 1 \leq i < j \leq N\} \\
\mathfrak{n}_- &= \text{span}\{m_{0,i} - m_{i+N,0} : 1 \leq i \leq N\} \cup \\
&\quad \text{span}\{m^{j,i} - m^{i+N,j+N} : 1 \leq i < j \leq N\} \cup \\
&\quad \text{span}\{m^{j+N,i} - m^{i+N,j} : 1 \leq i < j \leq N\} \\
\mathfrak{h} &= \text{span}\{m_{i,i} - m_{i+N,i+N} : 1 \leq i \leq N\}
\end{aligned}
$$

We fix a basis of $\widetilde{\mathfrak{so}}(2N+1)$: e_{i0}, f^{i0}, e_{ij1}, f^{ij1}, e_{ij2} and f^{ij2}, where $0 \leq i \leq j \leq N$. Letting $x = \pi(e)$ and $\xi = \pi(f)$ we have:

$$
\begin{aligned}
x_{i0} &= m_{i,0} - m_{0,i+N} \\
x_{ij1} &= m_{i,j} - m_{j+N,i+N}, \text{ where } i \le j \\
x_{ij2} &= m_{i,j+N} - m_{j,i+N}, \text{ where } i < j \\
\xi^{i0} &= \frac{1}{2}(m_{0,i} - m_{i+N,0}) \\
\xi^{ij1} &= (\frac{1}{2})^{\delta^{ij}+1}(m^{j,i} - m^{i+N,j+N}), \text{ where } i \le j \\
\xi^{ij2} &= (\frac{1}{2})^{\delta^{ij}+1}(m^{j+N,i} - m^{i+N,j}), \text{ where } i < j
\end{aligned}
$$

The coefficient $T^{\beta\nu}_{\alpha\mu}$ is then given as

$$
\begin{aligned}
T^{\beta\nu}_{\alpha\mu} &= \sum_{1\le i\le N} (\xi^{i0})^{\beta}_{\alpha}(x_{i0})^{\nu}_{\mu} + \sum_{\substack{i<j \\ k=1,2}} (\xi^{ijk})^{\beta}_{\alpha}(x_{ijk})^{\nu}_{\mu} \\
&\quad + \frac{1}{2}\sum_{i}(m^{ii} - m^{i+N,i+N})^{\beta}_{\alpha}(m_{ii} - m_{i+N,i+N})^{\nu}_{\mu} \\
&= \sum_{i\le j}(\frac{1}{2}(-\delta^{\beta,i+N}\delta^{0}_{\alpha} + \delta^{\beta,0}\delta^{i}_{\alpha})(\delta^{\nu}_{i}\delta_{\mu,0} - \delta^{\nu}_{0}\delta_{\mu,i+N}) \\
&\quad + (\frac{1}{2})^{\delta^{ij}+1}(\delta^{\beta,j}\delta^{i}_{\alpha} - \delta^{\beta,i+N}\delta^{j+N}_{\alpha})(\delta^{\nu}_{i}\delta_{j,\mu} - \delta^{\nu}_{j+N}\delta_{i+N,\mu}) \\
&\quad + (\frac{1}{2})^{\delta^{ij}+1}(-\delta^{\beta,i+N}\delta^{j}_{\alpha} + \delta^{\beta,j+N}\delta^{i}_{\alpha})(\delta^{\nu}_{i}\delta_{j+N,\mu} - \delta^{\nu}_{j}\delta_{i+N,\mu}))
\end{aligned}
$$

Using U, A and B to denote the sets $\{0\}$, $\{1, ..., N\}$ and $\{N+1, ..., 2N\}$ respectively, we can follow the same steps as in $\mathfrak{so}(2N)$ to get a diagrammatic representation of $T^{\beta\nu}_{\alpha\mu}$. The result is given in figure 2.13. Like for $\mathfrak{so}(2n)$ the last four terms can be simplified to the two terms in figure 2.10.

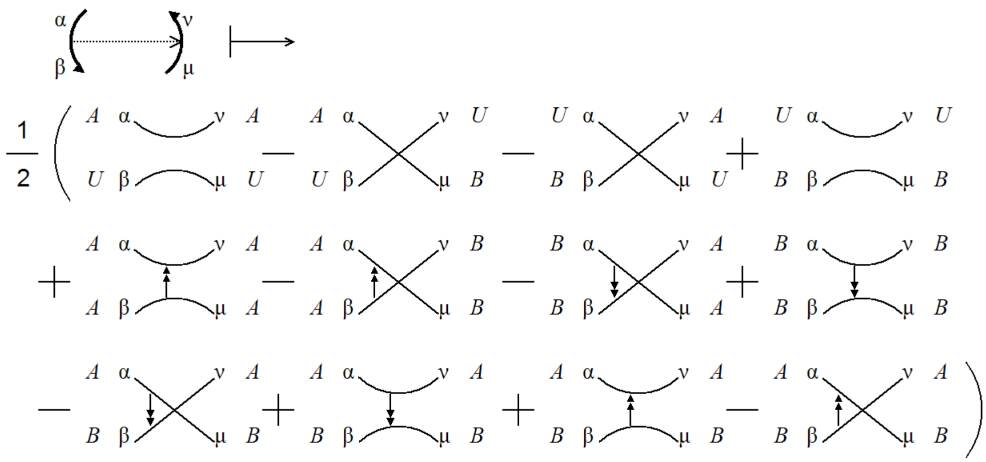

Figure 2.13. The $\mathfrak{so}(2N+1)$ tensor.

2.5 Composing the weight systems with the averaging map

Let an unoriented chord diagram with skeleton Γ be a diagram with unoriented chords joining distinct points of Γ, and $\mathcal{A}(\Gamma)$ be the space of (unoriented) chord diagrams modulo $4T$.

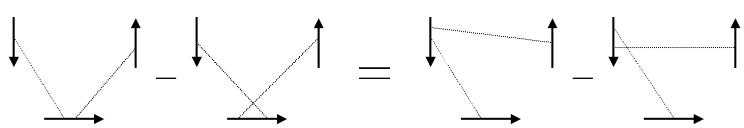

Figure 2.14. The 4T relation. Like in the figure for the 6T relation the part of the diagram which is not shown is the same for all terms.

We define the averaging map from $\mathcal{A}(\Gamma)$ to $\vec{\mathcal{A}}(\Gamma)$ as follows:

Definition 2.5.1. *The averaging map a takes a chord diagram to an arrow diagram by summing over all possible ways to direct each chord. (See figure 2.15 for an example.)*

It can be shown that a is well-defined, i.e., a maps the $4T$ relation to a linear combination of the $6T$ relation.

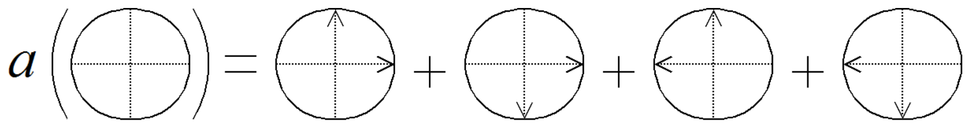

Figure 2.15. The averaging map.

Let $\mathcal{A}(\Gamma)$ be the vector space of unoriented chord diagrams on Γ modulo the 4T relations. We know that metrized Lie algebras give us weight systems on $\mathcal{A}(\Gamma)$. (See [BN1] and [BN2].) If we have a weight system on $\vec{\mathcal{A}}(\Gamma)$, then composing it with the averaging map a gives us a weight system on $\mathcal{A}(\Gamma)$. (4T is a consequence of 6T by repeatingly applying the averaging map.)

Since a complex simple Lie algebra $\mathfrak{g}$ has both a metric and a standard Manin triple structure, it gives us both a weight system on $\mathcal{A}(\Gamma)$ and a weight system on $\vec{\mathcal{A}}(\Gamma)$. Haviv ([Ha]) proved that composing the latter with the averaging map gives us the former at the universal enveloping algebra level. In the rest of this section we compose the formulae we obtained in the previous sections with the averaging map to show the following.

Formulae. *Composing our combinatorial formulae from the previous sections with the averaging map gives us the formulae for weight systems on chord diagrams mod 4T found in Bar-Natan's [BN1].*

We will look at each case separately. For $\mathfrak{gl}$ the unoriented tensor is calculated as in figure 2.16. Note the absence of restrictions on the values of unconnected Greek letters give us the weight system as given in [BN1].

For $\mathfrak{sp}(2N)$ we have figure 2.17, which again is the weight system given in [BN1].

For $\mathfrak{so}(2N)$, we have figure 2.18. We consider a new basis w_j so that $v_j = M(w_j)$ of $\mathbb{C}^{2N}$, where

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} iI & -iI \\ -I & -I \end{bmatrix} \tag{2.1}$$

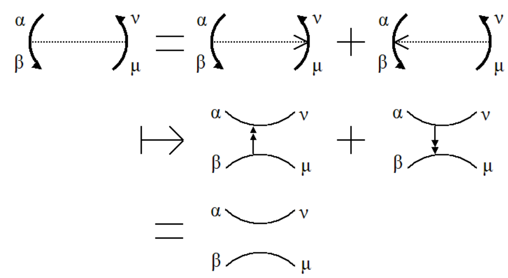

Figure 2.16. The unoriented $\mathfrak{gl}(N)$ tensor.

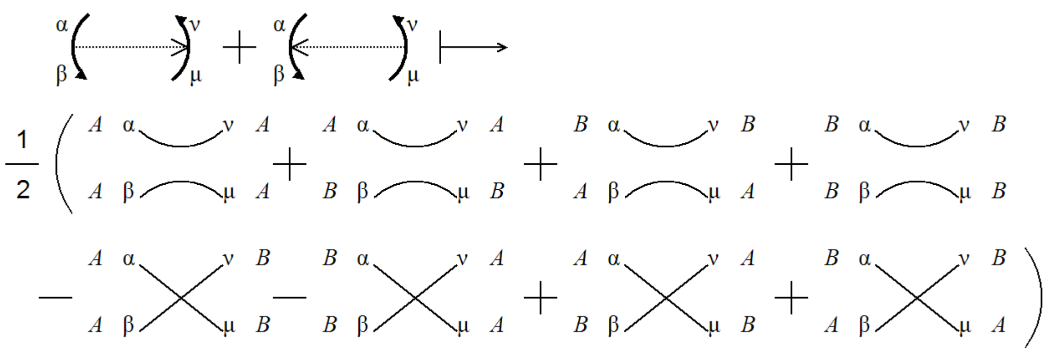

Figure 2.17. The unoriented $\mathfrak{sp}(2N)$ tensor.

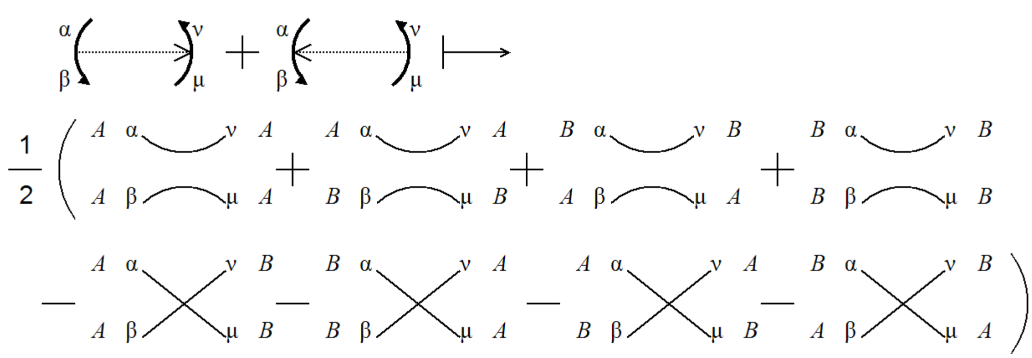

Figure 2.18. The unoriented $\mathfrak{so}(2N)$ tensor.

and I is the $N \times N$ identity matrix.

Note M is a unitary matrix, so $w^j := w_j^* = \langle w_j, . \rangle$ where $\langle ., . \rangle$ is the inner product

$\langle \sum c_i v_i, \sum c_i' v_i \rangle = \sum \bar{c}_i c_i'$. We therefore have the following relations for $1 \le j \le n$:

$$v_j = \tfrac{1}{\sqrt{2}}(iw_j - w_{j+N}) \tag{2.2}$$

$$v_{j+N} = \tfrac{1}{\sqrt{2}}(-iw_j - w_{j+N}) \tag{2.3}$$

$$v^j = \tfrac{1}{\sqrt{2}}(\langle iw_j - w_{j+N}, \cdot \rangle) = \tfrac{1}{\sqrt{2}}(-iw^j - w^{j+N}) \tag{2.4}$$

$$v^{j+N} = \tfrac{1}{\sqrt{2}}(\langle -iw_j - w_{j+N}, \cdot \rangle) = \tfrac{1}{\sqrt{2}}(iw^j - w^{j+N}). \tag{2.5}$$

We consider this change of basis diagramatically. Consider, for example, figure 2.19. It gives us the tensor

$$\sum_{i,j} v^i \otimes v^{i+N} \otimes v_{j+N} \otimes v_j.$$

Expressed in the new basis, we have

$$\frac{1}{4}\sum_{i,j}(-iw^i - w^{i+N}) \otimes (iw^i - w^{i+N}) \otimes (-iw_j - w_{j+N}) \otimes (iw_j - w_{j+N}),$$

which can be expanded into a sum of 16 terms by distribution. Expressed diagramatically the sum above becomes the sum of the 16 diagrams in figure 2.20, where constants may be factored out of the diagram (figure 2.21). If we allow for "distribution of tags", the sum in figure 2.20 can be contracted to one diagram, as shown in figure 2.22.

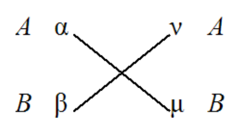

Figure 2.19. One of the term that appears in the $\mathfrak{so}(2N)$ weight system.

Following this procedure, whenever an A appears at the head of an arrow which is part of the skeleton we can replace it by $iA' - B'$. Similarly we can replace B by $-iA' - B'$. If they appear at the tail of an arrow that is part of the skeleton, however, we replace A and B by $-iA' - B'$ and $iA' - B'$, respectively, by equations (2.11) and (2.12) above. If we expand each of the diagrams linearly we obtain the same result as in [BN1]. See figure 2.23.

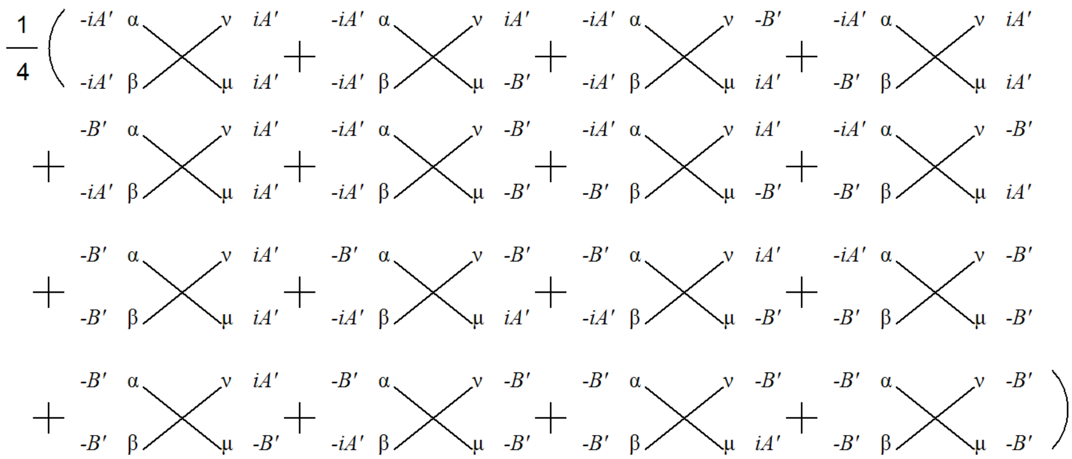

Figure 2.20. The diagram in figure 2.19 after a change of basis.

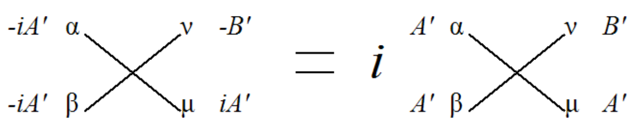

Figure 2.21. Constants may be factored out of a diagram.

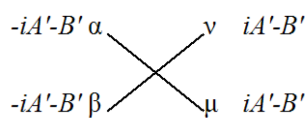

Figure 2.22. The sum in figure 2.20 expressed as one diagram if we allow for "distribution of tags".

For the case $\mathfrak{so}(2N+1)$ we use the change of basis matrix

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & iI & -iI \\ 0 & -I & -I \end{bmatrix} \tag{2.6}$$

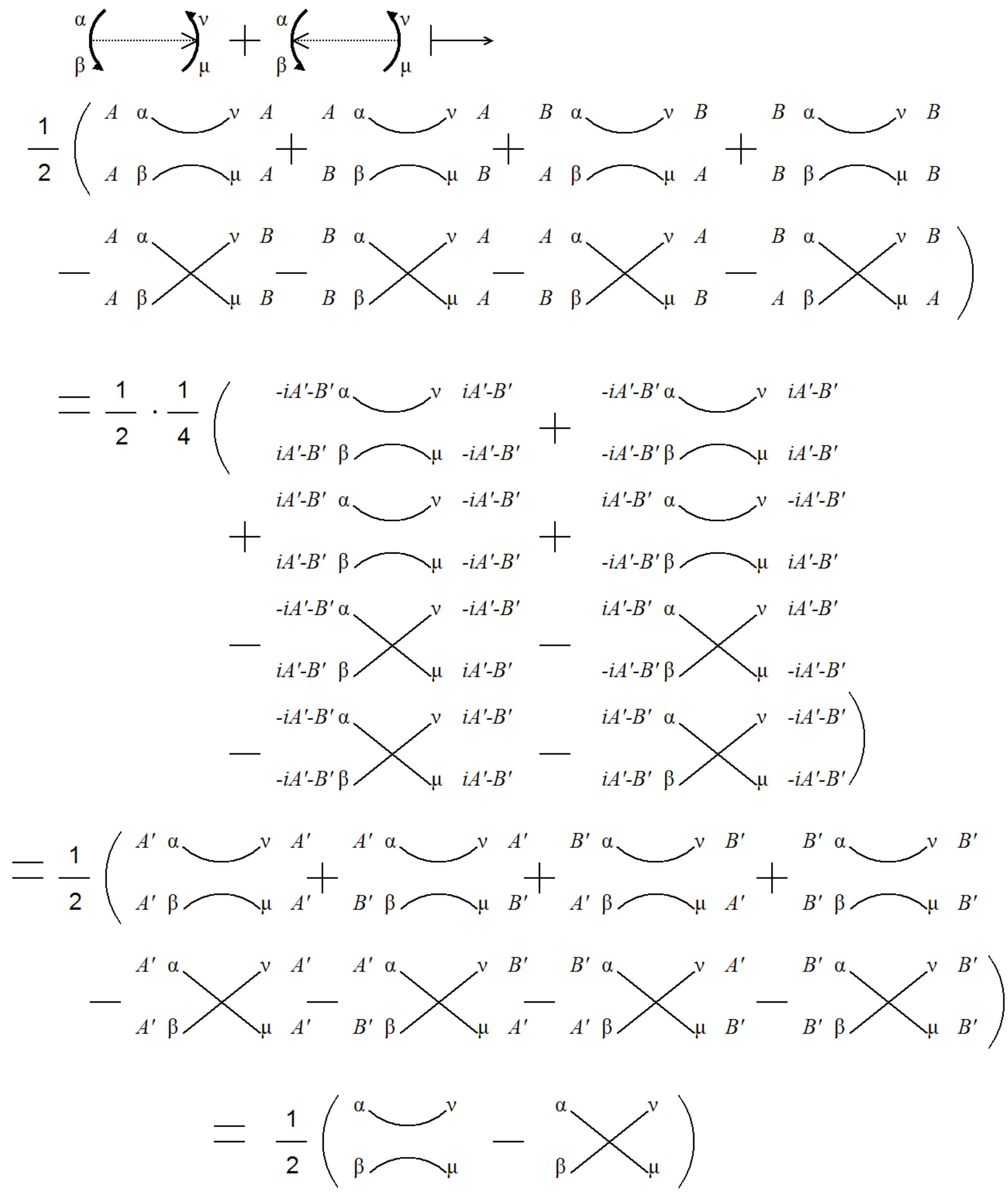

Figure 2.23. Calculating the unoriented $\mathfrak{so}(2N)$ tensor in the new basis. The third line is obtained from the seond line by expansion and cancelling. The fourth line is obtained from the third line by realizing that the latter is the former broken into cases.

The change of basis is therefore given by

$$v_0 = w_0 \tag{2.7}$$

$$v_j = \frac{1}{\sqrt{2}}(iw_j - w_{j+N}) \tag{2.8}$$

$$v_{j+N} = \frac{1}{\sqrt{2}}(-iw_j - w_{j+N}) \tag{2.9}$$

$$v^0 = w^0 \tag{2.10}$$

$$v^j = \frac{1}{\sqrt{2}}(\langle iw_j - w_{j+N}, \cdot\rangle) = \frac{1}{\sqrt{2}}(-iw^j - w^{j+N}) \tag{2.11}$$

$$v^{j+N} = \frac{1}{\sqrt{2}}(\langle -iw_j - w_{j+N}, \cdot\rangle) = \frac{1}{\sqrt{2}}(iw^j - w^{j+N}). \tag{2.12}$$

Like for $\mathfrak{so}(2N)$, we replace A and B at the heads of arrows which are parts of the skeleton by $iA' - B'$ and $-iA' - B'$, respectively. If they appear at the tail of an arrow that is part of the skeleton, we replace them by $-iA' - B'$ and $iA' - B'$, respectively. All U's are replaced by U''s. The change of basis can be expressed diagramatically as in figure 2.24. We get the same expression as in Bar-Natan's [BN1].

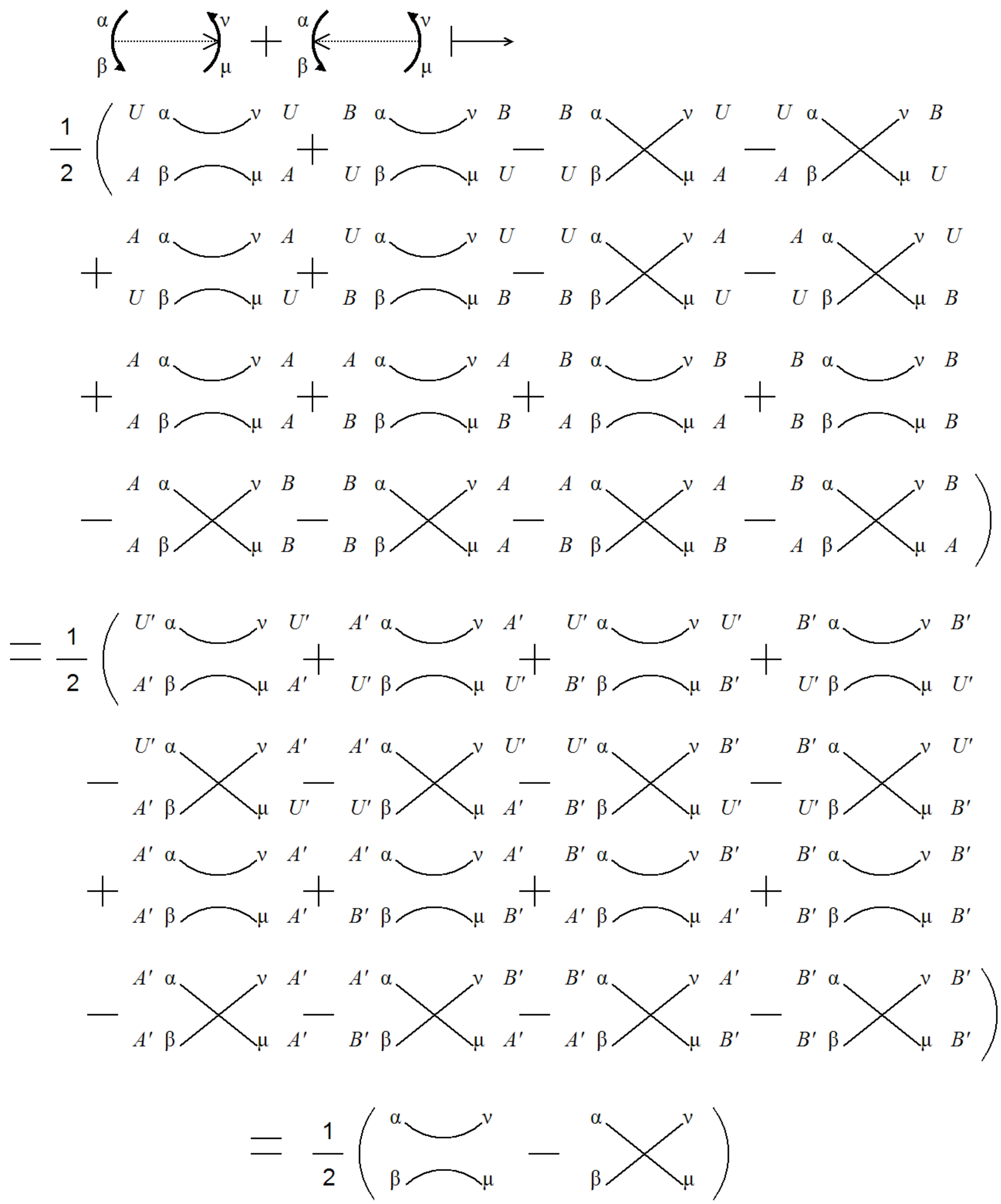

Figure 2.24. Calculating the unoriented $\mathfrak{so}(2N+1)$ tensor and expressing it in the new basis.

Chapter 3

Dimensions of $\vec{\mathcal{A}}_n(\uparrow)$ and of Their Images in Classical Lie Algebras

In this chapter we present results from computations, done jointly with Dror Bar-Natan, of the dimensions of $\vec{\mathcal{A}}_n(\uparrow)$, for $1 \leq n \leq 4$, and of the ranks of weight systems coming from all classical Lie algebras (using their standard Manin triple structures) and all representations. These ranks measure how well the standard Manin triple sturctures on classical Lie algebras capture $\vec{\mathcal{A}}_n(\uparrow)$. To consider all representations, instead of fixing a representation V and considering a weight system as a map from $\vec{\mathcal{A}}_n(\uparrow)$ to $End(V)$, we do not choose a particular representation and we take our weight systems to be maps from $\vec{\mathcal{A}}_n(\uparrow)$ to $U(\widetilde{\mathfrak{g}})$, where $\widetilde{\mathfrak{g}}$ is the Drinfeld double obtained through the standard construction presented in chapter 1. To do our computations for all classical Lie algebras $\mathfrak{gl}(N)$, $\mathfrak{so}(2N)$, $\mathfrak{so}(2N+1)$ and $\mathfrak{sp}(2N)$ for all N, we construct intermediate spaces where the parameter N does not appear but through which each weight system factors. We will explain how this is done in the following section.

In this chapter we use the Poincare-Birkhoff-Witt Theorem and the notion of a PBW-basis. The reader may refer to chapter 17 of [Hu] which covers these topics.

3.1 What to do with N?

3.1.1 $\mathfrak{gl}(N)$

We first define a map $T_{U(\widetilde{\mathfrak{gl}}(N))}$ from $\vec{\mathcal{A}}_n(\uparrow)$ to $U(\widetilde{\mathfrak{gl}}(N))$. Let D be an element of $\vec{\mathcal{A}}_n(\uparrow)$. We find $T_{U(\widetilde{\mathfrak{gl}}(N))}(D)$ as follows. At the tail and the head of the m^{th} arrow (how the arrows are ordered is not important) we put the letters $f^{i_m j_m}$ and $e_{i_m j_m}$, respectively, so in the end we have a word where each of the letters $f^{i_1 j_1}, \dots, f^{i_n j_n}, e_{i_1 j_1}, \dots, e_{i_n, j_n}$ appear exactly once, with the understanding that for each k, $i_k \leq j_k$. To get $T_{U(\widetilde{\mathfrak{gl}}(N))}(D)$ we sum over all possible values of the indices to find $T_{U(\widetilde{\mathfrak{gl}}(N))}(D)$. For example, for D in figure 3.1, the word is $f^{i_1 j_1} e_{i_1 j_1} f^{i_2 j_2} e_{i_2 j_2}$. Let $W_{i_1 j_1 i_2 j_2}$ denote the word. We have

$$T_{U(\widetilde{\mathfrak{gl}}(N))}(D) = \sum_{\substack{1 \leq i_m, j_m \leq N \\ i_m \leq j_m}} (\frac{1}{2})^q W_{i_1 j_1 i_2 j_2},$$

where q is the number of m such that $i_m = j_m$. (The factor $\frac{1}{2}$ is due to such an element being in the Cartan subalgebra.)

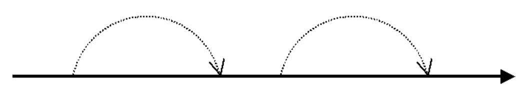

Figure 3.1. A diagram in $\vec{\mathcal{A}}_2(\uparrow)$.

Once a PBW basis is fixed (for example, $\mathfrak{gl}_+$ before $\mathfrak{gl}_-$, with pairs of indices ordered in lexicographic order), each summand in $T_{\widetilde{\mathfrak{gl}}}(D)$ can be PBW-reduced, and we can calculate the dimension of the image of $T_{\widetilde{\mathfrak{gl}}}$. We observe that in $\widetilde{\mathfrak{gl}}$ reduction to PBW basis is independent of the actual indices; rather what is of material is the order of the indices relative to each other.

Example 3.1.1. *The PBW reduced word for $e_{13} f^{24} e_{56}$ can be obtained from the PBW reduced word for $e_{15} f^{36} e_{78}$ by replacing 3 by 2, 5 by 3, 6 by 4, 7 by 5, and 8 by 6.*

This motivates our definition of the following vector space.

Definition 3.1.1. *Let V be the space spanned by generators $\langle k\rangle W$ where the following conditions are satisfied:*

1. *W is a string of letters of the form $f^{\langle ij\rangle}$ or $e_{\langle ij\rangle}$, where i, j are natural numbers greater than or equal to 1 and $i \leq j$.*

2. *k is greater than or equal to the largest index which appears in W.*

$V(\widetilde{\mathfrak{gl}})$ is the space V modulo relations between generators. The relations are

$$\langle k\rangle S_1(e_{\langle ij\rangle}e_{\langle pq\rangle})S_2 - \langle k\rangle S_1(e_{\langle pq\rangle}e_{\langle ij\rangle})S_2 = c^{rs}_{ij,pq}\langle k\rangle S_1(e_{\langle rs\rangle})S_2,$$

$$\langle k\rangle S_1(f^{\langle ij\rangle}f^{\langle pq\rangle})S_2 - \langle k\rangle S_1(f^{\langle pq\rangle}f^{\langle ij\rangle})S_2 = \gamma^{ij,pq}_{rs}\langle k\rangle S_1(f^{\langle rs\rangle})S_2, \text{ and}$$

$$\langle k\rangle S_1(f^{\langle ij\rangle}e_{\langle pq\rangle})S_2 - \langle k\rangle S_1(e_{\langle pq\rangle}f^{\langle ij\rangle})S_2 = c^{ij}_{pq,rs}\langle k\rangle S_1(f^{\langle rs\rangle})S_2 - \gamma^{ij,rs}_{pq}\langle k\rangle S_1(e_{\langle rs\rangle})S_2,$$

where S_1 and S_2 can be any pair of strings of letters (which may even be empty). The constants $c^{rs}_{ij,pq}$ and $\gamma^{ij,pq}_{rs}$ are structure constants from $U(\widetilde{\mathfrak{gl}}(N))$ so that

$$[e_{ij}, e_{pq}] = c^{rs}_{ij,pq}e_{rs} \text{ and } [f^{ij}, f^{pq}] = \gamma^{ij,pq}_{rs}f^{rs}.$$

Equivalently,

$$c^{rs}_{ij,pq} = \delta_{jp}\delta^r_i\delta^s_q - \delta_{qi}\delta^r_p\delta^s_j \text{ and } \gamma^{ij,pq}_{rs} = \delta^{iq}\delta^p_r\delta^j_s - \delta^{pj}\delta^i_r\delta^q_s.$$

Note. *By thinking of $\langle k\rangle e_{\langle ij\rangle}$ and $\langle k\rangle f^{\langle ij\rangle}$ as regular elements in $\widetilde{\mathfrak{gl}}(k)$ we can see that as a vector space $V(\widetilde{\mathfrak{gl}})$ is isomorphic to*

$$\bigoplus_{k=1}^{\infty} U(\widetilde{\mathfrak{gl}}(k)),$$

where the $\langle k\rangle$ in front of W indicates which $U(\widetilde{\mathfrak{gl}}(k))$ the word W belongs to. It is worth noting that we do not want to identify the sum of two words $\langle k\rangle W_1 + \langle k\rangle W_2$ with the same tag $\langle k\rangle$ with

$$\langle k\rangle(W_1 + W_2).$$

(Such a term is not even an element of V.) Nor do we want to define multiplication in V by juxtaposition

$$(\langle k\rangle W_1)\cdot(\langle k\rangle W_2) = \langle k\rangle W_1W_2,$$

where W_1W_2 is the concatenation of the two words W_1 and W_2. Such a multiplication would not commute with the interpretation map ι_N defined below.

The relations are symbolically the same as the bracket relations in $\widetilde{\mathfrak{gl}}$. Like in a universal enveloping algebra we can define a PBW basis for $V(\widetilde{\mathfrak{gl}})$. (Consider the vector space isomorphism between $V(\widetilde{\mathfrak{gl}})$ and $\bigoplus_{k=1}^{\infty} U(\widetilde{\mathfrak{gl}}(k))$.) The interpretation of an element $W \in V(\widetilde{\mathfrak{gl}})$ is that it represents the sum of all words in $U(\widetilde{\mathfrak{gl}}(N))$ such that the relative order of indices inside angled brackets are observed. To be more precise, given any N we can define an interpretation function ι_N from $V(\widetilde{\mathfrak{gl}})$ to $U(\widetilde{\mathfrak{gl}})$ as outlined in the following paragraph.

Let $W_{\langle n_1,\ldots,n_p\rangle}$ be a generator of $V(\widetilde{\mathfrak{gl}})$ whose set of indices is (in ascending order and with repeating indices listed only once) $\{n_1,\ldots,n_p\}$ and $W_{n_1,\ldots,n_p}$ be the corresponding word in $U(\widetilde{\mathfrak{gl}})$. (That is, $W_{n_1,\ldots,n_p}$ is obtained from $W_{\langle n_1,\ldots,n_p\rangle}$ with angled brackets around indices removed.) We define an "interpretation function" ι_N from $V(\widetilde{\mathfrak{gl}})$ to $U(\widetilde{\mathfrak{gl}}(N))$ which is given by

$$\iota_N : \langle k\rangle W_{\langle n_1,\ldots,n_p\rangle} \mapsto \sum_f W_{f(n_1),\ldots,f(n_p)}. \tag{3.1}$$

The sum ranges over all functions $f:\{1,\ldots,k\}\to\{1,\ldots,N\}$ such that $i\le j \Leftrightarrow f(i)\le f(j)$. If $k>N$ we sum over an empty set and get 0. Such a function can be thought of as giving actual values to the abstract indices while observing their relative order.

Proposition 3.1.1. *For each N, $\iota_N : V(\widetilde{\mathfrak{gl}}) \to U(\widetilde{\mathfrak{gl}}(N))$ is well-defined, i.e., a relation in $V(\widetilde{\mathfrak{gl}})$ is mapped to a relation in $U(\widetilde{\mathfrak{gl}}(N))$.*

Proof. Take, for example, the relation

$$\langle k\rangle S_1(e_{\langle ij\rangle}e_{\langle pq\rangle})S_2 - \langle k\rangle S_1(e_{\langle pq\rangle}e_{\langle ij\rangle})S_2 = c^{rs}_{ij,pq}\langle k\rangle S_1(e_{\langle rs\rangle})S_2.$$

The function ι_N takes the left hand side to

$$\sum_f (S_1^f(e_{f(i)f(j)}e_{f(p)f(q)})S_2^f - S_1^f(e_{f(p)f(q)}e_{f(i)f(j)})S_2^f)$$

(where the sum is over all functions $f : \{1, \ldots, k\} \to \{1, \ldots, N\}$ such that $i \leq j \Leftrightarrow f(i) \leq f(j)$, and S_1^f and S_2^f are obtained from S_1 and S_2 by removing the brackets around the indices and replacing each index i by $f(i)$). The right hand side is mapped to

$$\sum_f c_{ij,pq}^{rs} S_1^f(e_{f(r)f(s)})S_2^f. \tag{3.2}$$

Note that, since each f we sum over is an injection, we have

$$c_{ij,pq}^{rs} = \delta_{jp}\delta_i^r\delta_q^s - \delta_{qi}\delta_p^r\delta_j^s = \delta_{f(j)f(p)}\delta_{f(i)}^{f(r)}\delta_{f(q)}^{f(s)} - \delta_{f(q)f(i)}\delta_{f(p)}^{f(r)}\delta_{f(j)}^{f(s)} = c_{f(i)f(j),f(p)f(q)}^{f(r)f(s)}.$$

Therefore we can rewrite (3.2) as

$$\sum_f c_{f(i)f(j),f(p)f(q)}^{f(r)f(s)} S_1^f(e_{f(r)f(s)})S_2^f.$$

and ι_N maps the relation

$$\langle k\rangle S_1(e_{\langle ij\rangle}e_{\langle pq\rangle})S_2 - \langle k\rangle S_1(e_{\langle pq\rangle}e_{\langle ij\rangle})S_2 = c_{ij,pq}^{rs}\langle k\rangle S_1(e_{\langle rs\rangle})S_2$$

to

$$\sum_f (S_1^f(e_{f(i)f(j)}e_{f(p)f(q)})S_2^f - S_1^f(e_{f(p)f(q)}e_{f(i)f(j)})S_2^f) = \sum_f c_{f(i)f(j),f(p)f(q)}^{f(r)f(s)} S_1^f(e_{f(r)f(s)})S_2^f,$$

which is a relation in $U(\widetilde{\mathfrak{gl}}(N))$. We have similar proofs for the other relations. $\square$

Equivalently ι_N can be given as

$$\iota_N : \langle k\rangle W_{\langle n_1,\ldots,n_p\rangle} \mapsto \sum_{m_1<\ldots<m_p} \binom{m_1-1}{n_1-1}\binom{m_2-m_1-1}{n_2-n_1-1}\cdots\binom{m_p-m_{p-1}-1}{n_p-n_{p-1}-1}\binom{N-m_p}{k-n_p} W_{m_1,\ldots,m_p} \tag{3.3}$$

In the definition above it is understood that whenever $m < n$, $\binom{m}{n} = 0$. Note that in the process of PBW reduction some indices may disappear. For example, $f^{23}e_{13} = e_{13}f^{23} - e_{12}$, where 3 does not appear in the second term on the right. We therefore add an extra piece of data $\langle k \rangle$ to keep track of the number of distinct indices we started with. This information is not superfluous. For example, when mapped to $U(\widetilde{\mathfrak{gl}}(7))$, the index 2 in $\iota_7(\langle 2 \rangle e_{\langle 12 \rangle})$ can take on the value 7 in some of the summands, while the index 2 in $\iota_7(\langle 3 \rangle e_{\langle 12 \rangle})$ never takes on the value 7. The coefficients on the right in equation (3.3) above are there to represent the number of possible ways we have in choosing the value of the indices which get hidden in the process of PBW reduction. In what follows we summarize how we find the image of a diagram with n arrows in $V(\widetilde{\mathfrak{gl}})$. First we make a definition.

Definition 3.1.2. *An order type on a set S is a function $\tau : S \to \mathbb{N}\backslash\{0\}$ such that if $n_1 < n_2$ and $n_2 \in \tau(S)$, then $n_1 \in \tau(S)$.*

We can think of an order type on S as an ordering of elements in S, such that for any $s_1, s_2 \in S$, $s_1 \leq s_2$ iff $\tau(s_1) \leq \tau(s_2)$. (Different elements of S may be considered equal in a particular ordering.) This function is minimal in the sense that there are no gaps in the image of τ.

Definition 3.1.3. *The function $T_{V(\widetilde{\mathfrak{gl}})} : \vec{\mathcal{A}}_n(\uparrow) \to V(\widetilde{\mathfrak{gl}})$ is given as follows. (Steps A1 to A5.)*

A1. We label the arrows $a_1, \ldots, a_n$.

A2. For each arrow a_m we put the letter f^{i_m,j_m} at its tail and the letter e_{i_m,j_m} at its head.

A3. Given that the skeleton is an oriented line, we follow the orientation to string together the letters from the previous step to form a word W.

A4. We consider the sum

$$\sum_{\tau} c_{\tau} \langle p_{\tau} \rangle W_{\tau}$$

where the sum is over all order types on the set $\{i_1, j_1, \ldots, i_n, j_n\}$ such that for each m, $\tau(i_m) \leq \tau(j_m)$. W_τ is obtained from W by replacing each e_{ij} by $e_{\langle \tau(i), \tau(j) \rangle}$ and each f^{ij} by $f^{\langle \tau(i), \tau(j) \rangle}$. The term p_τ is $\max(\{\tau(i_1), \tau(j_1), \ldots, \tau(i_n), \tau(j_n)\})$ which is the number of distinct values that the indices take on. The term c_τ is given by $(\frac{1}{2})^q$ where q is the number of m's such that $\tau(i_m) = \tau(j_m)$. (The factor $\frac{1}{2}$ is due to such an element being in the Cartan subalgebra.)

A5. We reduce each W_τ to a linear combination of PBW basis elements.

We compare the above with the algorithm which finds $T_{U(\widetilde{\mathfrak{gl}}(N))}(D)$.

Definition 3.1.4. *The function* $T_{U(\widetilde{\mathfrak{gl}}(N))} : \vec{\mathcal{A}}_n(\uparrow) \to U(\widetilde{\mathfrak{gl}}(N))$ *(the "universal* $\widetilde{\mathfrak{gl}}(N)$ *weight system") is given as follows. (Steps B1 to B5.)*

B1. We label the arrows $a_1, \ldots, a_n$.

B2. For each arrow a_m we put the letter f^{i_m, j_m} at its tail and the letter e_{i_m, j_m} at its head.

B3. Given that the skeleton is an oriented line, we follow the orientation to string together the letters from the previous step to form a word W.

B4. We consider the sum

$$\sum_{\phi} c_{\phi} W_{\phi}$$

where the sum is over all functions $\phi : \{i_1, j_1, \ldots, i_n, j_n\} \to \{1, \ldots, N\}$ such that for each m, $\phi(i_m) \leq \phi(j_m)$. W_ϕ is obtained from W by replacing each $e_{i_m j_m}$ by $e_{\phi(i_m), \phi(j_m)}$ and each $f^{i_m j_m}$ by $f^{\phi(i_m), \phi(j_m)}$. The term c_ϕ is given by $(\frac{1}{2})^q$ where q is the number of m's such that $\phi(i_m) = \phi(j_m)$. (The factor $\frac{1}{2}$ is due to such an element being in the Cartan subalgebra.)

B5. We reduce each W_ϕ to a linear combination of PBW basis elements.

Each word W_ϕ obtained in step B4 corresponds to exactly one word $\langle p_\tau \rangle W_\tau$ obatined in A4. (τ is the unique order type on $\{i_1, j_1, \ldots, i_n, j_n\}$ for which $\tau(i_m)$ (or $\tau(j_m)$) is less than or equal to $\tau(i_{m'})$ (or $\tau(j_{m'})$) if and only if $\phi(i_m)$ (or $\phi(j_m)$) is less than or equal to $\phi(i_{m'})$ (or $\phi(j_{m'})$). The tag p_τ is then the cardinality of the image of ϕ.) Given $U(\widetilde{\mathfrak{gl}}(N))$, $c_\phi W_\phi$ is one of the summands of $\iota_N(c_\tau \langle p_\tau \rangle W_\tau)$ (See step A4). We therefore have the commutative diagram:

$$\begin{array}{ccc} & & \vec{\mathcal{A}}_n(\uparrow) \\ & \swarrow^{T_{V(\widetilde{\mathfrak{gl}})}} & \downarrow T_{U(\widetilde{\mathfrak{gl}}(N))} \\ V(\widetilde{\mathfrak{gl}}) & \xrightarrow[\iota_N]{} & U(\widetilde{\mathfrak{gl}}(N)) \end{array}$$

By considering the map

$$\bigoplus_{m=1}^{N} \iota_m : V(\widetilde{\mathfrak{gl}}) \to \bigoplus_{m=1}^{N} U(\widetilde{\mathfrak{gl}}(m)),$$

and the direct sum of weight systems

$$\bigoplus_{m=1}^{N} T_{U(\widetilde{\mathfrak{gl}}(m))} : \vec{\mathcal{A}}_n(\uparrow) \longrightarrow \bigoplus_{m=1}^{N} U(\widetilde{\mathfrak{gl}}(m)),$$

we have

$$\begin{array}{ccc} & & \vec{\mathcal{A}}_n(\uparrow) \\ & \swarrow^{T_{V(\widetilde{\mathfrak{gl}})}} & \downarrow \bigoplus_{m=1}^{N} T_{U(\widetilde{\mathfrak{gl}}(m))} \\ V(\widetilde{\mathfrak{gl}}) & \xrightarrow[\bigoplus_{m=1}^{N} \iota_m]{} & \bigoplus_{m=1}^{N} U(\widetilde{\mathfrak{gl}}(m)) \end{array}$$

From this diagram we have, for each N,

$$\mathrm{rank}(T_{V(\widetilde{\mathfrak{gl}})}) \geq \mathrm{rank}(\bigoplus_{m=1}^{N} T_{U(\widetilde{\mathfrak{gl}}(m))}). \tag{3.4}$$

In section 3.2 we will show that for $N \geq 2n$ the inequality above becomes an equality.

3.1.2 The Other Classical Lie Algebras

Let $\tilde{\mathfrak{g}}$ be a Manin triple of type $\widetilde{\mathfrak{so}}(even)$, $\widetilde{\mathfrak{so}}(odd)$ or $\widetilde{\mathfrak{sp}}$. We define $V(\tilde{\mathfrak{g}})$ in a similar manner. In the definition below each $e_{\langle r \rangle}$ or $e_{\langle s \rangle}$ stands for $e_{\langle i \rangle 0}$ (for $V(\widetilde{\mathfrak{so}}(odd))$ only),

$e_{\langle ij\rangle 1}$ or $e_{\langle ij\rangle 2}$. ($e_{\langle ij\rangle 1}$, for example, stands for e_{ij1} where the only information we keep is the relative order of i and j, between themselves and with respect to indices from other letters in the same word. The reader may refer to chapter 2 for an explanation of the notations e_{i0}, e_{ij1} and e_{ij2}.) Similarly each $f^{\langle r\rangle}$ or $f^{\langle s\rangle}$ stands for $f^{\langle i\rangle 0}$ (for $V(\widetilde{\mathfrak{so}}(odd))$ only), $f^{\langle ij\rangle 1}$ or $f^{\langle ij\rangle 2}$. The terms $c^t_{r,s}$ and $\gamma^{r,s}_t$ are corresponding structure constants.

Definition 3.1.5. *Let V be the space spanned by generators $\langle k\rangle W$ where the following conditions are satisfied:*

1. *W is a string of letters of the form $f^{\langle r\rangle}$ or $e_{\langle s\rangle}$, where i, j are natural numbers greater than or equal to 1.*

2. *k is greater than or equal to the largest index which appears in W.*

$V(\widetilde{\mathfrak{g}})$ is the space V modulo relations between generators. The relations are

$$\langle k\rangle S_1(e_{\langle r\rangle}e_{\langle s\rangle})S_2 - \langle k\rangle S_1(e_{\langle s\rangle}e_{\langle r\rangle})S_2 = \langle k\rangle S_1(c^t_{r,s}e_{\langle t\rangle})S_2,$$

$$\langle k\rangle S_1(f^{\langle r\rangle}f^{\langle s\rangle})S_2 - \langle k\rangle S_1(f^{\langle s\rangle}f^{\langle r\rangle})S_2 = \langle k\rangle S_1(\gamma^{r,s}_t f^{\langle t\rangle})S_2, \text{ and}$$

$$\langle k\rangle S_1(f^{\langle r\rangle}e_{\langle s\rangle})S_2 - \langle k\rangle S_1(e_{\langle s\rangle}f^{\langle r\rangle})S_2 = c^r_{s,t}\langle k\rangle S_1(f^{\langle t\rangle})S_2 - \gamma^{r,t}_s\langle k\rangle S_1(e_{\langle t\rangle})S_2,$$

where S_1 and S_2 can be any (even empty) string of letters.

Like in the $\widetilde{\mathfrak{gl}}$ case, the relations are symbolically the same as the bracket and cobracket relations. We define functions ι_N which map $V(\widetilde{\mathfrak{so}}(even))$, $V(\widetilde{\mathfrak{sp}})$ and $V(\widetilde{\mathfrak{so}}(odd))$ to $U(\widetilde{\mathfrak{so}}(2N))$, $U(\widetilde{\mathfrak{sp}}(2N))$ and $U(\widetilde{\mathfrak{so}}(2N+1))$, respectively, in a way similar to the $\widetilde{\mathfrak{gl}}$ case. (The maps ι_N take an element from V to the sum of all ways to assign actual values to indices within brackets while respecting their relative order. The proof that such maps are well defined is similar to the proof of Proposition 3.1.1.) The following is the algorithm for finding $T_{V(\widetilde{\mathfrak{g}})}(D)$ given $D \in \vec{\mathcal{A}}_n(\uparrow)$.

Definition 3.1.6. *The function $T_{V(\widetilde{\mathfrak{g}})} : \vec{\mathcal{A}}_n(\uparrow) \to V(\widetilde{\mathfrak{g}})$ is given as follows. (Steps A1 to A5.)*

A1. We label the arrows $a_1, \dots, a_n$.

A2. For each arrow a_m we put the letter f^{i_m,j_m,k_m} at its tail and the letter e_{i_m,j_m,k_m} at its head. (If $k_m = 0$, then j_m is empty.)

A3. Given that the skeleton is an oriented line, we follow the orientation to string together the letters from the previous step to form a word W.

A4. We consider the sum

$$\sum_{\tau,k_m} c_\tau \langle p_\tau \rangle W_\tau$$

where the sum is over all order types on the set $\{i_1, j_1, \dots, i_n, j_n\}$ such that for each m, $\tau(i_m) \leq \tau(j_m)$ and $k_m = 1, 2$ (for $\widetilde{\mathfrak{so}}(even)$ and $\widetilde{\mathfrak{sp}}$) or $k_m = 0, 1, 2$ (for $\widetilde{\mathfrak{so}}(odd)$, keeping in mind that j_m is empty when $k_m = 0$). W_τ is obtained from W by replacing each e_{ijk} by $e_{\langle \tau(i), \tau(j) \rangle k}$ and each f^{ijk} by $f^{\langle \tau(i), \tau(j) \rangle k}$. The term p_τ is $\max(\{\tau(i_1), \tau(j_1), \dots, \tau(i_n), \tau(j_n)\})$, which is the number of distinct values that the indices take on. The term c_τ is given by $(\frac{1}{2})^q$ where q is the number of m's such that $k_m = 1$ and $\tau(i_m) = \tau(j_m)$. (The factor $\frac{1}{2}$ is due to such an element being in the Cartan subalgebra.)

A5. We reduce each W_τ to a linear combination of PBW basis elements.

Like in the last section we compare with the following algorithm which maps a diagram D to actual universal enveloping algebras. If we let $\widetilde{\mathfrak{g}}$ be a Lie algebra of type $\widetilde{\mathfrak{so}}(even)$, $\widetilde{\mathfrak{sp}}$, or $\widetilde{\mathfrak{so}}(odd)$, we use $\widetilde{\mathfrak{g}}(N)$ to denote $\widetilde{\mathfrak{so}}(2N)$, $\widetilde{\mathfrak{sp}}(2N)$, or $\widetilde{\mathfrak{so}}(2N+1)$, respectively.

Definition 3.1.7. *The function* $T_{U(\widetilde{\mathfrak{g}}(N))} : \vec{\mathcal{A}}_n(\uparrow) \to U(\widetilde{\mathfrak{g}}(N))$ *is given as follows. (Steps B1 to B5.)*

B1. We label the arrows $a_1, \dots, a_n$.

B2. For each arrow a_m we put the letter f^{i_m,j_m,k_m} at its tail and the letter e_{i_m,j_m,k_m} at its head. (If $k_m = 0$, then j_m is empty.)

B3. Given that the skeleton is an oriented line, we follow the orientation to string together the letters from the previous step to form a word W.

B4. We consider the sum

$$\sum_{\phi,k_m} c_\phi W_\phi$$

where the sum is over all functions $\phi : \{i_1, j_1, \ldots, i_n, j_n\} \to \{1, \ldots, N\}$ such that for each m, $\phi(i_m) \leq \phi(j_m)$ and $k_m = 1, 2$ (for $\widetilde{\mathfrak{so}}(2N)$ and $\widetilde{\mathfrak{sp}}(2N)$) or $k_m = 0, 1, 2$ (for $\widetilde{\mathfrak{so}}(2N+1)$, keeping in mind that j_m is empty when $k_m = 0$). W_ϕ is obtained from W by replacing each e_{ijk} by $e_{\phi(i),\phi(j),k}$ and each f^{ijk} by $f^{\phi(i),\phi(j),k}$. The term c_ϕ is given by $(\frac{1}{2})^q$ where q is the number of m's such that $k_m = 1$ and $\phi(i_m) = \phi(j_m)$. (The factor $\frac{1}{2}$ is due to such an element being in the Cartan subalgebra.)

B5. We reduce each W_ϕ to a linear combination of PBW basis elements.

Like in the $\widetilde{\mathfrak{gl}}$ case any word W_ϕ obtained in B4 corresponds, through ι_N, to exactly one word $\langle p_\tau \rangle W_\tau$ obtained in A4. We therefore have the commutative diagram

$$\begin{array}{ccc} & & \vec{\mathcal{A}}_n(\uparrow) \\ & \swarrow^{T_{V(\widetilde{\mathfrak{g}})}} & \downarrow{\scriptstyle \bigoplus_{m=1}^N T_{U(\widetilde{\mathfrak{g}}(m))}} \\ V(\widetilde{\mathfrak{g}}) & \xrightarrow[\bigoplus_{m=1}^N \iota_m]{} & \bigoplus_{m=1}^N U(\widetilde{\mathfrak{g}}(m)) \end{array} .$$

The above implies

$$\operatorname{rank}(T_{V(\widetilde{\mathfrak{g}})}) \geq \operatorname{rank}(\bigoplus_{m=1}^{N} T_{\widetilde{\mathfrak{g}}(m)}). \tag{3.5}$$

When $N \geq 2n$, as in the $\widetilde{\mathfrak{gl}}$ case, the inequality above becomes an equality. This is the topic for the next section. After that, in section 3.3, we will present results of our computations on the rank of the map

$$\vec{\mathcal{A}}_n(\uparrow) \to V(\widetilde{\mathfrak{gl}}) \oplus V(\widetilde{\mathfrak{so}}) \oplus V(\widetilde{\mathfrak{sp}}).$$

3.2 The Rank of T_V and the Rank of the Weight Systems

In this section we relate the rank of T_V to the ranks of the actual weight systems. The first step is the following proposition.

Proposition 3.2.1. *Let V_K be the subspace of V spanned by elements $\langle k\rangle W$ such that $k \leq K$. The map $\bigoplus_{n=1}^{K} \iota_n$ restricted to V_K is injective.*

Proof. We fix a positive integer K and consider a linear combination of distinct words

$$\sum_i \alpha_i \langle k_i\rangle W_i \tag{3.6}$$

such that each $k_i \leq K$. Let $k = \min\{k_i\}$. We consider the map

$$\bigoplus_{n=1}^{K} \iota_n : V(\widetilde{\mathfrak{gl}}) \to \bigoplus_{n=1}^{K} U(\widetilde{\mathfrak{gl}}(n))$$

and suppose

$$\iota_1 \oplus \ldots \oplus \iota_K(\sum_i \alpha_i \langle k_i\rangle W_i) = 0.$$

This implies, in particular, that

$$\iota_k(\sum_i \alpha_i \langle k_i\rangle W_i) = 0.$$

By our choice of k, if $k_i \neq k$, $k_i > k$. By (3.1), the above is given by

$$\sum_{i \text{ for which } k_i = k} \alpha_i \iota_k(\langle k_i\rangle W_i) = \sum_{i \text{ for which } k_i = k} \alpha_i \iota_k(\langle k\rangle W_i).$$

If W_i is a word whose list of indices is $\{n_1, \ldots, n_p\}$ (we denote it by $W_{\langle n_1,\ldots,n_p\rangle}$), then we have

$$\iota_k(\langle k\rangle W_i) = W_{n_1,\ldots,n_p},$$

where $W_{n_1,\ldots,n_p} \in U(\widetilde{\mathfrak{gl}}(k))$ is the word obtained from $W_{\langle n_1,\ldots,n_p\rangle}$ by removing all brackets around indices. We denote the word obtained from W_i in such a manner by W_i' and we have

$$\sum_{i \text{ for which } k_i=k} \alpha_i W_i' = 0.$$

Since the W_i''s are distinct PBW-reduced words in $U(\widetilde{\mathfrak{gl}}(k))$, we must have all $\alpha_i = 0$, whenever $k_i = k$. By induction all α_i's in the word given in (3.6) must be 0. Therefore $\bigoplus_{n=1}^{K} \iota_n$ must be injective when restricted to V_K. $\square$

If D is a diagram in $\vec{\mathcal{A}}_n(\uparrow)$, since each arrow is responsible for at most two distinct indices, $T_V(D)$ contains only terms of the form $\langle k \rangle W$ where $k \leq 2n$. The function $\bigoplus_{m=1}^{2n} \iota_m$ therefore maps the image of T_V injectively to $\bigoplus_{m=1}^{2n} U(\widetilde{\mathfrak{gl}}(m))$. This together with (3.4) give us the following corollary.

Corollary 3.2.1. *The rank of $T_{V(\widetilde{\mathfrak{gl}})}$ on $\vec{\mathcal{A}}_n(\uparrow)$ equals to the rank of the direct sum of weight systems*

$$\bigoplus_{m=1}^{2n} T_{\widetilde{\mathfrak{gl}}(m)} : \vec{\mathcal{A}}_n(\uparrow) \longrightarrow \bigoplus_{m=1}^{2n} U(\widetilde{\mathfrak{gl}}(m)).$$

An important implication of the corollary above is that the weight systems coming from $\widetilde{\mathfrak{gl}}(1), \ldots, \widetilde{\mathfrak{gl}}(2n)$ capture the FULL strength of all $\widetilde{\mathfrak{gl}}(N)$ weight systems on $\vec{\mathcal{A}}_n$. The corollary above was proved for $\widetilde{\mathfrak{gl}}$, but by similar arguments and (3.5) we can also show the following.

Corollary 3.2.2. *1. The rank of $T_{V(\widetilde{\mathfrak{sp}})}$ on $\vec{\mathcal{A}}_n(\uparrow)$ equals to the rank of the direct sum of weight systems*

$$\bigoplus_{m=1}^{2n} T_{\widetilde{\mathfrak{sp}}(2m)} : \vec{\mathcal{A}}_n(\uparrow) \longrightarrow \bigoplus_{m=1}^{2n} U(\widetilde{\mathfrak{sp}}(2m)).$$

2. The rank of $T_{V(\widetilde{\mathfrak{so}}(even))}$ on $\vec{\mathcal{A}}_n(\uparrow)$ equals to the rank of the direct sum of weight systems

$$\bigoplus_{m=1}^{2n} T_{\widetilde{\mathfrak{so}}(2m)} : \vec{\mathcal{A}}_n(\uparrow) \longrightarrow \bigoplus_{m=1}^{2n} U(\widetilde{\mathfrak{so}}(2m)).$$

3. *The rank of* $T_{V(\widetilde{\mathfrak{so}}(odd))}$ *on* $\vec{\mathcal{A}}_n(\uparrow)$ *equals to the rank of the direct sum of weight systems*

$$\bigoplus_{m=1}^{2n} T_{\widetilde{\mathfrak{so}}(2m+1)} : \vec{\mathcal{A}}_n(\uparrow) \longrightarrow \bigoplus_{m=1}^{2n} U(\widetilde{\mathfrak{so}}(2m+1)).$$

3.3 Results

In this section we present our computational results. We consider the direct sum of all classical Lie algebra weight systems:

$$T_V = T_{V(\widetilde{\mathfrak{gl}})} \oplus T_{V(\widetilde{\mathfrak{so}}(even))} \oplus T_{V(\widetilde{\mathfrak{so}}(odd))} \oplus T_{V(\widetilde{\mathfrak{sp}})}.$$

Given a diagram $D \in \vec{\mathcal{A}}_n(\uparrow)$, T_V takes D to $V(\widetilde{\mathfrak{gl}}) \oplus V(\widetilde{\mathfrak{so}}(even)) \oplus V(\widetilde{\mathfrak{so}}(odd)) \oplus V(\widetilde{\mathfrak{sp}})$. The rank of T_V on $\vec{\mathcal{A}}_n(\uparrow)$ measures the dimensions of $\vec{\mathcal{A}}_n(\uparrow)$ seen by the standard Manin triple structures on classical Lie algebras. To get a better strength out of the weight systems, we compute the rank of $(T_V \otimes T_V) \circ \Delta$. For the reader's reference we also present the rank of the $\widetilde{\mathfrak{gl}}$ weight systems alone. Our results are contained in the following table.

n	1	2	3	4
Number of generating diagrams	2	12	120	1680
Number of $6T$-relations	0	6	120	2520
$dim(\vec{\mathcal{A}}_n(\uparrow))$	2	7	27	139
$dim(T_{V(\widetilde{\mathfrak{gl}})}(\vec{\mathcal{A}}_n(\uparrow)))$ (no coproduct)	2	7	27	118
$dim(T_{V(\widetilde{\mathfrak{gl}})} \otimes T_{V(\widetilde{\mathfrak{gl}})}) \circ \Delta(\vec{\mathcal{A}}_n(\uparrow)))$	2	7	27	122
$dim((T_V \otimes T_V) \circ \Delta(\vec{\mathcal{A}}_n(\uparrow)))$	2	7	27	125

Comments:

1. In computer programming bugs are always a possibility. We therefore welcome and appreciate independent verification of our numbers.

2. The third row of the table contains the number of all $6T$-relations generated using our algorithm. They are *not* the number of *independent* $6T$-relations.

3. The doubled Cartan is necessary to get the dimensions listed above, i.e., we do not want to take the further step $\pi : U(\tilde{\mathfrak{g}}) \to U(\mathfrak{g})$. In particular, the subspace of $\vec{\mathcal{A}}_2(\uparrow)$ seen without the doubled Cartan has dimension 6. The kernel is spanned by the diagram in figure 3.2.

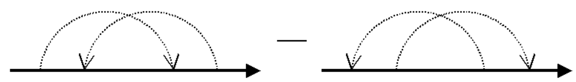

Figure 3.2. Without the doubled Cartan subalgebra this non-trivial element of $\vec{\mathcal{A}}_2(\uparrow)$ is mapped to 0.

4. Since $125 < 139$, our results suggest that the standard Manin triple structures on classical Lie algebras do not detect all diagrams of degree 4.

5. For degree $n = 4$ the rank 125 can be obtained just by using Manin triples of type $\widetilde{\mathfrak{gl}}$ and $\widetilde{\mathfrak{so}}(2N)$. In other words, including $\widetilde{\mathfrak{sp}}$ and $\widetilde{\mathfrak{so}}(2N+1)$ does not lead to a higher rank.

6. We also computed the weight system arising from the triangular Lie bialgebra structure on $\mathfrak{sl}(2)$, with r-matrix $H \wedge E$ (Example 2.1.8, [CP]). The image of this weight system on $\vec{\mathcal{A}}_4(\uparrow)$ has dimension 20. When restricted to the kernel of the classical weight systems, its image has dimension 3. We also computed the CYBE weight system on all degree-4 diagrams, and the resulting dimension is 1. This provides evidence with non-trivial r that, given a triangular Lie bialgebra, the CYBE weight system is different from the Drinfeld double weight system.

Appendix A

Sample Calculations in the Defining Representations

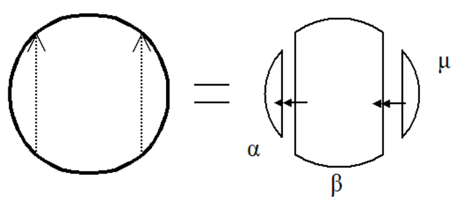

Figure A.1. Calculating the $\mathfrak{gl}(N)$ weight system.

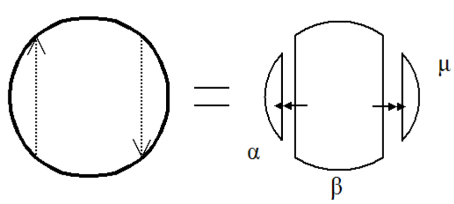

Figure A.2. Calculating the $\mathfrak{gl}(N)$ weight system.

We now do some sample calculations. In this section the skeleton is always a circle oriented counterclockwise. First we calculate the two diagrams shown in figures A.1 and A.2, using the $\mathfrak{gl}(N)$ weight system. For the first picture, each triple $(\alpha, \beta, \mu) \in$

$\{1, ..., N\}^3$ such that $\alpha \leq \beta \leq \mu$ gives us a value of either 1, $\frac{1}{2}$ or $\frac{1}{4}$, depending on whether $\alpha = \beta$ or $\beta = \mu$. Therefore the diagram should have weight $a + \frac{1}{2}b + \frac{1}{4}c$, where a is the number of triples (α, β, μ) such that $\alpha < \beta < \mu$, b is the number of triples (α, β, μ) such that $\alpha < \beta = \mu$ or $\alpha = \beta < \mu$, and c is the number of triples (α, β, μ) such that $\alpha = \beta = \mu$. The number is

$$\binom{N}{3} + \binom{N}{2} + \frac{N}{4}.$$

Using a similar argument we know the weight of the picture in figure A.2 is

$$2\binom{N}{3} + 2\binom{N}{2} + \frac{N}{4},$$

so the $\mathfrak{gl}(N)$ weight system is capable of telling the two diagrams apart.

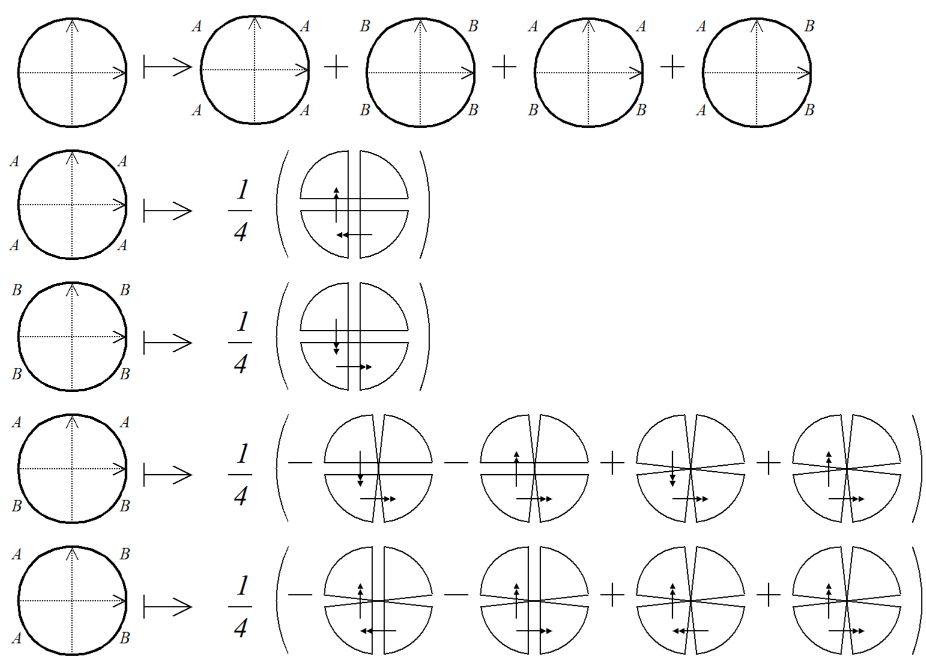

Figure A.3. Calculating the $\mathfrak{so}(2N)$ weight system. The first line gives us the only possible combinations of tags.

Now we calculate the $\mathfrak{so}(2N)$ weight of the picture in figure A.3. We assign tags A, B to each arc following the rules from section 2.2 (see figure A.3). (We want the combination of tags around each arrow to give us non-zero tensors.) For each assignment we have one or four ways to resolve the diagram, and for a resolution with k loops we count the number of k-tuples in $\{1, ..., N\}$ such that the inequalities as denoted by the double-headed arrows are satisfied, bearing in mind that each equality comes with a weight $\frac{1}{2}$. The weights of the first two diagrams and the last two diagrams on the right hand side of the first equation in figure A.3 are therefore $\frac{1}{4}(\frac{N}{4})$ and $\frac{1}{4}(-\frac{N}{4} - \frac{N}{4} + \frac{N}{4} + (\binom{N}{2} + \frac{N}{4}))$, respectively. The weight of the diagram is therefore

$$\frac{N}{8} + \frac{1}{2}\binom{N}{2}.$$

Appendix B

A Partial Sample Calculation of $T_{V(\widetilde{\mathfrak{gl}})}$ on an Element of $\vec{\mathcal{A}}_2(\uparrow)$

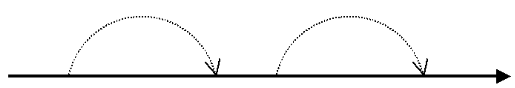

Figure B.1. A sample diagram in $\vec{\mathcal{A}}_2(\uparrow)$.

Let D be the degree-2 diagram shown in figure B.1. We have

$$T_{U(\widetilde{\mathfrak{gl}}(N))}(D) = \sum_{\substack{i \le j, k \le l \\ 1 \le i,j,k,l \le N}} (\frac{1}{2})^{\delta_{ij}+\delta_{kl}} f^{ij} e_{ij} f^{kl} e_{kl}.$$

Given the restriction $i \le j, k \le l$, all possible relative orders on i, j, k, l are listed in table B.1. We can therefore write the following.

$$
\begin{aligned}
T_{V(\widetilde{\mathfrak{gl}})}(D) &= (\frac{1}{2})^{\delta_{ij}+\delta_{kl}} f^{\langle ij\rangle} e_{\langle ij\rangle} f^{\langle kl\rangle} e_{\langle kl\rangle} \\
= \quad & \frac{1}{4}\langle 1\rangle f^{\langle 11\rangle} e_{\langle 11\rangle} f^{\langle 11\rangle} e_{\langle 11\rangle} + \frac{1}{4}\langle 2\rangle f^{\langle 11\rangle} e_{\langle 11\rangle} f^{\langle 22\rangle} e_{\langle 22\rangle} + \frac{1}{4}\langle 2\rangle f^{\langle 22\rangle} e_{\langle 22\rangle} f^{\langle 11\rangle} e_{\langle 11\rangle} \\
& + \frac{1}{2}\langle 3\rangle f^{\langle 11\rangle} e_{\langle 11\rangle} f^{\langle 23\rangle} e_{\langle 23\rangle} + \frac{1}{2}\langle 2\rangle f^{\langle 11\rangle} e_{\langle 11\rangle} f^{\langle 12\rangle} e_{\langle 12\rangle} + \frac{1}{2}\langle 3\rangle f^{\langle 22\rangle} e_{\langle 22\rangle} f^{\langle 13\rangle} e_{\langle 13\rangle} \\
& + \frac{1}{2}\langle 2\rangle f^{\langle 22\rangle} e_{\langle 22\rangle} f^{\langle 12\rangle} e_{\langle 12\rangle} + \frac{1}{2}\langle 3\rangle f^{\langle 33\rangle} e_{\langle 33\rangle} f^{\langle 12\rangle} e_{\langle 12\rangle} + \frac{1}{2}\langle 3\rangle f^{\langle 23\rangle} e_{\langle 23\rangle} f^{\langle 11\rangle} e_{\langle 11\rangle} \\
& + \frac{1}{2}\langle 2\rangle f^{\langle 12\rangle} e_{\langle 12\rangle} f^{\langle 11\rangle} e_{\langle 11\rangle} + \frac{1}{2}\langle 3\rangle f^{\langle 13\rangle} e_{\langle 13\rangle} f^{\langle 22\rangle} e_{\langle 22\rangle} + \frac{1}{2}\langle 2\rangle f^{\langle 12\rangle} e_{\langle 12\rangle} f^{\langle 22\rangle} e_{\langle 22\rangle} \\
& + \frac{1}{2}\langle 3\rangle f^{\langle 12\rangle} e_{\langle 12\rangle} f^{\langle 33\rangle} e_{\langle 33\rangle} + \langle 4\rangle f^{\langle 12\rangle} e_{\langle 12\rangle} f^{\langle 34\rangle} e_{\langle 34\rangle} + \langle 4\rangle f^{\langle 13\rangle} e_{\langle 13\rangle} f^{\langle 24\rangle} e_{\langle 24\rangle} \\
& + \langle 4\rangle f^{\langle 14\rangle} e_{\langle 14\rangle} f^{\langle 23\rangle} e_{\langle 23\rangle} + \langle 4\rangle f^{\langle 23\rangle} e_{\langle 23\rangle} f^{\langle 14\rangle} e_{\langle 14\rangle} + \langle 4\rangle f^{\langle 24\rangle} e_{\langle 24\rangle} f^{\langle 13\rangle} e_{\langle 13\rangle} \\
& + \langle 4\rangle f^{\langle 34\rangle} e_{\langle 34\rangle} f^{\langle 12\rangle} e_{\langle 12\rangle} + \langle 2\rangle f^{\langle 12\rangle} e_{\langle 12\rangle} f^{\langle 12\rangle} e_{\langle 12\rangle} + \langle 3\rangle f^{\langle 12\rangle} e_{\langle 12\rangle} f^{\langle 13\rangle} e_{\langle 13\rangle} \\
& + \langle 3\rangle f^{\langle 12\rangle} e_{\langle 12\rangle} f^{\langle 23\rangle} e_{\langle 23\rangle} + \langle 3\rangle f^{\langle 13\rangle} e_{\langle 13\rangle} f^{\langle 12\rangle} e_{\langle 12\rangle} + \langle 3\rangle f^{\langle 13\rangle} e_{\langle 13\rangle} f^{\langle 23\rangle} e_{\langle 23\rangle} \\
& + \langle 3\rangle f^{\langle 23\rangle} e_{\langle 23\rangle} f^{\langle 12\rangle} e_{\langle 12\rangle} + \langle 3\rangle f^{\langle 23\rangle} e_{\langle 23\rangle} f^{\langle 13\rangle} e_{\langle 13\rangle} \qquad \text{(B.1)}
\end{aligned}
$$

Each term can now be PBW reduced by the relations of $V(\widetilde{\mathfrak{gl}})$ given in Definition 3.1.1.

Relative order on indices	$\tau(i)$	$\tau(j)$	$\tau(k)$	$\tau(l)$
$i = j = k = l$	1	1	1	1
$i = j < k = l$	1	1	2	2
$i = j > k = l$	2	2	1	1
$i = j < k < l$	1	1	2	3
$i = j = k < l$	1	1	1	2
$k < i = j < l$	2	2	1	3
$k < l = i = j$	2	2	1	2
$k < l < i = j$	3	3	1	2
$k = l < i < j$	2	3	1	1
$k = l = i < j$	1	2	1	1
$i < k = l < j$	1	3	2	2
$i < j = k = l$	1	2	2	2
$i < j < k = l$	1	2	3	3
$i < j < k < l$	1	2	3	4
$i < k < j < l$	1	3	2	4
$i < k < l < j$	1	4	2	3
$k < i < j < l$	2	3	1	4
$k < i < l < j$	2	4	1	3
$k < l < i < j$	3	4	1	2
$i = k < j = l$	1	2	1	2
$i = k < j < l$	1	2	1	3
$i < j = k < l$	1	2	2	3
$i = k < l < j$	1	3	1	2
$i < k < j = l$	1	3	2	3
$k < i = l < j$	2	3	1	2
$k < i < j = l$	2	3	1	3

Table B.1. A table listing all possible orders of i, j, k, l given $i \leq j$ and $k \leq l$.

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