Lie-algebras associated to multiple q-zeta values

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Expansions, Lie Algebras, and Invariants
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in parts joint work with:
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Leila Schneps, CNRS Jussieu, in progress

Multiple zeta values (MZV)

Definition

For natural numbers $s_1 \geqslant 2, s_2, ..., s_l \geqslant 1$ the sum

$$\zeta(s_1, ..., s_l) = \sum_{n_1 > ... > n_l > 0} \frac{1}{n_1^{s_1} \ldots n_l^{s_l}}$$

is called a multiple zeta value (MZV) of weight $s_1 + ... + s_l$ and depth l.

- The rules for the product of infinite sums imply that the product of MZV can be expressed as a linear combination of MZV with the same weight (stuffle product).
- MZV can be expressed as iterated integrals. This gives another way (shuffle product) to express the product of two MZV as a linear combination of MZV.
- These two products give a large number of $\mathbb{Q}$-linear relations (extended double shuffle relations) between MZV. Conjecturally these are all relations between MZV, e.g.

$$\zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2) \cdot \zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5)\,.$$

Dimension conjectures for $\mathcal{MZ}$

Consider the formal powerseries

$$\mathsf{E}_2(x) = \frac{x^2}{1-x^2} = x^2 + x^4 + x^6 + \ldots \qquad \text{"even zetas"},$$

$$\mathsf{O}_3(x) = \frac{x^3}{1-x^2} = x^3 + x^5 + x^7 + \ldots \qquad \text{"odd zetas"},$$

$$\mathsf{S}(x) = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12} + x^{16} + x^{18} + \ldots \qquad \text{"period polynomials"}.$$

Broadhurst-Kreimer Conjecture

The $\mathbb{Q}$-algebra $\mathcal{MZ}$ of multiple zeta values is a free polynomial algebra, which is graded for the weight and filtered for the depth ("depth drop for even zetas"). The numbers $g_{k,l}$ of generators in weight $k \geqslant 3$ and depth l are determined by

$$BK(x,y) = \sum_{k,l \geqslant 0} \dim_{\mathbb{Q}} \left(\mathrm{gr}^{W,D}_{k,l} \mathcal{MZ} \right) x^k y^l = \Big(1 + \mathsf{E}_2(x)\, y\Big) \prod_{k \geqslant 3, l \geqslant 1} \frac{1}{\left(1 - x^k y^l\right)^{g_{k,l}}}$$

where

$$BK(x,y) = \Big(1 + \mathsf{E}_2(x) y\Big) \frac{1}{1 - \mathsf{O}_3(x) y + \mathsf{S}(x) y^2 - \mathsf{S}(x) y^4}.$$

Dimension conjectures for $\mathcal{MZ}$

Zagier's Conjecture

The following identities hold:

$$Zag(x) = \sum_{k \geqslant 0} \dim_{\mathbb{Q}} \left(\mathrm{gr}_k^W \, \mathcal{MZ} \right) x^k = \frac{1}{1 - x^2 - x^3}.$$

Zagier's conjecture is implied by Broadhurst-Kreimer's conjecture. In order to neglect the depth we just have to set $y = 1$ and get

$$Zag(x) = BK(x,1) = \frac{1 + \mathsf{E}_2(x)}{1 - \mathsf{O}_3(x)} = \frac{1 + \frac{x^2}{1-x^2}}{1 - \frac{x^3}{1-x^2}} = \frac{1}{1 - x^2 - x^3}.$$

Brown's Theorem

The $\mathbb{Q}$-vector space of multiple zeta values is spanned by the "23"-MZV's, e.g. by those $\zeta(s_1, ..., s_l)$ with $s_i \in \{2,\ 3\}$.

By Brown's theorem the dimensions in Zagier's conjecture are the maximal possible ones.

MZV's and Lie-Algebras

$$\begin{tikzcd}
\text{Zagier's Conjecture for } \mathcal{MZ} \arrow[r, leftrightarrow, dashed] & \text{double shuffle Lie Algebra } \mathfrak{ds} \arrow[d, dashed] \\
\text{Broadhurst-Kreimer Conjecture for } \mathcal{MZ} \arrow[u, dashed] \arrow[r, leftrightarrow, dashed] & \text{linearized double-shuffle Lie-Algebra } \mathfrak{ls}
\end{tikzcd}$$

The connection to this workshop: We have $\mathfrak{grt} \hookrightarrow \mathfrak{ds} \hookrightarrow \mathfrak{kv}$.

Since we don't know all relations in the algebra multiple zeta values $\mathcal{MZ} \subset \mathbb{R}$, we consider formal multiple zeta values $\mathcal{MZ}^f$ instead. Roughly this algebra is

$$\mathcal{MZ}^f = \left\langle \zeta^f(s_1, ..., s_l) \middle| \, l \geqslant 0,\ s_i \geqslant 1 \right\rangle_{\mathbb{Q}} \Big/ \left\{ ш, \text{extended double shuffle}, \ \zeta^f(1) = 0 \right\}$$

Theorem (Racinet, Ecalle)

We have a non-canonical isomorphism

$$\mathcal{MZ}^f \cong \mathbb{Q}[\zeta^f(2)] \otimes \mathcal{U}(\mathfrak{ds})^{\vee},$$

in particular $\mathcal{MZ}^f$ is a free polynomial algebra.

Conjectures by Ihara and Zagier

Conjecture (Ihara,...)

The Lie algebra $\mathfrak{ds}$ is a free graded Lie algebra with one generator in each odd degree $k \geqslant 3$.

Corollary

If $\mathfrak{ds}$ satisfies Ihara's conjecture, then $\mathcal{MZ}^f$ satisfies Zagier's conjecture.

Idea of Proof: Since $\mathcal{MZ}^f \cong \mathbb{Q}[\zeta^f(2)] \otimes \mathcal{U}(\mathfrak{ds})^\vee$ we have

$$\begin{aligned}\sum_{k=0}^{\infty} \dim \mathcal{MZ}_k^f \; x^k &= H_{\mathcal{MZ}^f}(x) = H_{\mathbb{Q}[\zeta^f(2)]}(x) \cdot H_{\mathcal{U}(\mathfrak{ds})^\vee}(x) \\ &= \frac{1}{1-x^2} \frac{1}{1-x^3-x^5-x^7-\ldots} \\ &= \frac{1}{1-x^2-x^3}\,.\end{aligned}$$

$\square$

Main-MZV-Conjecture

The map $\mathcal{MZ}^f \to \mathcal{MZ}$ given by $\zeta^f(s_1,...,s_l) \mapsto \zeta^{\sqcup\!\sqcup}(s_1,...,s_l)$ is an isomorphism.

Multiple q-zeta values

Many of the most basic concepts in mathematics have so-called q-analogues, where q is a formal variable such that the specialisation $q = 1$ recovers the usual concept, e.g. Gauss q-integers

$$\{n\}_q = 1 + q + \ldots + q^{n-1} = \frac{1-q^n}{1-q}.$$

We will study the following q-analogues of multiple zeta values[1] .

Definition [(modified) multiple q-zeta value]

For natural numbers $s_1, \ldots, s_l \geqslant 1$ and $Q_1(t) \in t\mathbb{Q}[t]$ and $Q_2(t) \ldots, Q_l(t) \in \mathbb{Q}[t]$ we define

$$\zeta_q(s_1, \ldots, s_l; Q_1, \ldots, Q_l) = \sum_{n_1 > \cdots > n_l > 0} \frac{Q_1(q^{n_1}) \ldots Q_l(q^{n_l})}{(1-q^{n_1})^{s_1} \cdots (1-q^{n_l})^{s_l}} \in \mathbb{Q}[[q]].$$

This series can be seen as a q-analogue of multiple zeta values, since we have for $s_1 > 1$

$$\lim_{q \to 1} (1-q)^{s_1 + \cdots + s_l} \zeta_q(s_1, \ldots, s_l; Q_1, \ldots, Q_l) = Q_1(1) \ldots Q_l(1) \cdot \zeta(s_1, \ldots, s_l) \,.$$

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[1] Bachmann, Kühn: A dimension conjecture for q-analogues of multiple zeta values, arXiv:1708.07464 [math.NT]
Bachmann: Multiple Eisenstein series and q-analogues of multiple zeta values, arXiv:1704.06930 [math.NT]

Algebra of multiple q-zeta values

Definition

We set $\zeta_q(\varnothing;\varnothing) = 1$ and define the algebra of multiple q-zeta values to be the $\mathbb{Q}$-algebra

$$\mathcal{Z}_q := \left\langle \zeta_q(s_1,\dots,s_l;Q_1,\dots,Q_l) \;\middle|\; l \geqslant 0,\, s_1,\dots,s_l \geqslant 1,\, \deg(Q_j) \leqslant s_j \right\rangle_{\mathbb{Q}} .$$

Indeed $\mathcal{Z}_q$ is a $\mathbb{Q}$-algebra, for example, it is

$$\zeta_q(s_1;Q_1)\cdot\zeta_q(s_2;Q_2) = \zeta_q(s_1,s_2;Q_1,Q_2)+\zeta_q(s_2,s_1;Q_2,Q_1)+\zeta_q(s_1+s_2;Q_1\cdot Q_2)\,,$$

and clearly $\deg Q_1 \cdot Q_2 \leqslant s_1 + s_2$ if $\deg Q_j \leqslant s_j$ for $j = 1, 2$.

Caution: $s_1 + \cdots + s_l$ does not give a good notion of weight for the ζ_q. Also l will not be used to define the depth. Instead, we will consider a class of q-series which also span the space $\mathcal{Z}_q$ and use these series to define a weight and a depth filtration on $\mathcal{Z}_q$.

Subalgebras of $\mathcal{Z}_q$

For $d \geqslant 0$ we define the subspace

$$\mathcal{Z}_{q,d} = \Big\langle \zeta_q(s_1,\dots,s_l;Q_1,\dots,Q_l) \in \mathcal{Z}_q \;\Big|\; \deg(Q_j) \leqslant s_j - d \Big\rangle_{\mathbb{Q}}.$$

So in particular we have $\mathcal{Z}_q = \mathcal{Z}_{q,0}$ and $\mathcal{Z}_{q,d+1} \subset \mathcal{Z}_{q,d}$.

$$\mathcal{Z}_q^\circ = \Big\langle \zeta_q(s_1,\dots,s_l;Q_1,\dots,Q_l) \in \mathcal{Z}_q \;\Big|\; Q_1,\dots,Q_l \in t\mathbb{Q}[t] \Big\rangle_{\mathbb{Q}}.$$

For the spaces defined by

$$\mathcal{Z}_{q,d}^\circ = \mathcal{Z}_q^\circ \cap \mathcal{Z}_{q,d}$$

it holds $\mathcal{Z}_q^\circ = \mathcal{Z}_{q,0}^\circ$ and $\mathcal{Z}_{q,d+1}^\circ \subset \mathcal{Z}_{q,d}^\circ$.

Proposition

All of the above spaces are subalgebras of $\mathcal{Z}_q$.

These spaces recover previously known multiple q-zeta values, e.g. $\mathcal{Z}_{q,1}$ is the Bradley-Zhao model, $\mathcal{Z}_q^\circ$ is the Schlesinger-Zudilin model and $\mathcal{Z}_q$ its extension by Ebrahimi-Fard-Manchon-Singer, $\mathcal{Z}_{q,1}^\circ$ is the Okounkov-model, ...

Bi-brackets

For natural numbers $s_1, ..., s_l \geqslant 1$ and $r_1, ..., r_l \geqslant 0$ the bi-brackets are defined by

$$\begin{pmatrix} s_1, \ldots, s_l \\ r_1, \ldots, r_l \end{pmatrix}_q = \kappa \cdot \sum_{n_1 > \cdots > n_l > 0} \frac{n_1^{r_1} P_{s_1-1}(q^{n_1}) \ldots n_l^{r_l} P_{s_l-1}(q^{n_l})}{(1-q^{n_1})^{s_1} \ldots (1-q^{n_l})^{s_l}} \in \mathbb{Q}[[q]] \,,$$

where $\kappa = \big(r_1!(s_1-1)! \ldots r_l!(s_l-1)!\big)^{-1}$ and the $P_{k-1}(t)$ are the Eulerian polynomials defined by

$$\frac{P_{k-1}(t)}{(1-t)^k} = \mathrm{Li}_{1-k}(t) = \sum_{d>0} d^{k-1} t^d \,.$$

$$P_0(t) = P_1(t) = t \,, \quad P_2(t) = t^2 + t \,, \quad P_3(t) = t^3 + 4t^2 + t \,,$$

$$\begin{pmatrix} 1, 1 \\ 0, 1 \end{pmatrix}_q = \sum_{n_1 > n_2 > 0} \frac{q^{n_1} n_2 q^{n_2}}{(1-q^{n_1})(1-q^{n_2})} \,,$$

$$\begin{pmatrix} 4, 2, 1 \\ 2, 0, 5 \end{pmatrix}_q = \frac{1}{3! \cdot 2! \cdot 5!} \sum_{n_1 > n_2 > n_3 > 0} \frac{n_1^2 (q^{3n_1} + 4q^{2n_1} + q^{n_1}) \cdot q^{n_2} \cdot n_3^5 q^{n_3}}{(1-q^{n_1})^4 \cdot (1-q^{n_1})^2 \cdot (1-q^{n_1})^1} \,.$$

Multiple divisor sums and modular forms

If $r_1 = \dots = r_l = 0$, we set $[s_1, \dots s_l]_q = \begin{bmatrix} s_1, \dots, s_l \\ 0, \dots, 0 \end{bmatrix}_q$ and we find

$$[s_1, \dots, s_l]_q = \frac{1}{(s_1-1)! \dots (s_l-1)!} \sum_{n>0} \Bigg(\sum_{\substack{u_1 v_1 + \dots + u_l v_l = n \\ u_1 > \dots > u_l > 0 \\ v_1, \dots, v_l > 0}} v_1^{s_1} \dots v_l^{s_l} \Bigg) q^n \,.$$

We call the coefficients $\sigma_{s_1-1,\dots,s_l-1}(n)$ multiple divisor sums. In the case $l = 1$ we get the classical divisor sums $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and

$$[k]_q = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n \,.$$

These function appear in the Fourier expansion of classical Eisenstein series which are (quasi)-modular forms for $SL_2(\mathbb{Z})$, for example

$$G_2 = -\frac{1}{24} + [2]_q \,, \quad G_4 = \frac{1}{1440} + [4]_q \,, \quad G_6 = -\frac{1}{60480} + [6]_q \in \mathcal{Z}_q .$$

Bi-brackets as q-multiple zeta values

Proposition (Bachmann-K.)

We have

$$\mathcal{Z}_q^\circ = \Big\langle (s_1, \dots, s_l)_q \,\Big|\, l \geqslant 0,\ s_1, \dots, s_l \geqslant 1 \Big\rangle_{\mathbb{Q}},$$
$$\mathcal{Z}_{q,1}^\circ = \Big\langle (s_1, \dots, s_l)_q \,\Big|\, l \geqslant 0\,, s_1, \dots, s_l \geqslant 2 \Big\rangle_{\mathbb{Q}}.$$

In addition, $\mathcal{Z}_q^\circ$ is closed under the q-derivation $q\frac{\mathrm{d}}{\mathrm{d}\,q}$.

The (bi)-brackets have also direct connection to multiple zeta values, since they behave like multiple q-zeta values:

Theorem (Bachmann-K. , Zudilin)

Assume that $s_1 > r_1 + 1$ and $s_j \geqslant r_j + 1$ for $j = 2, .., l$. Then

$$\lim_{q \to 1} (1-q)^{s_1 + \ldots + s_l} \begin{pmatrix} s_1, \ldots, s_l \\ r_1, \ldots, r_j \end{pmatrix}_q = \frac{1}{r_1! \cdot \ldots \cdot r_l!} \zeta(s_1 - r_1, \ldots, s_l - r_l).$$

Remark: Another very interesting connection to MZV is given by the Fourier expansion of multiple Eisenstein series.

Bi-brackets and $\mathcal{Z}_q$

Theorem (Bachmann-K.)

The following equality holds

$$\mathcal{Z}_q = \left\langle \begin{pmatrix} s_1,\dots,s_l \\ r_1,\dots,r_l \end{pmatrix}_q \;\middle|\; l \geqslant 0, s_1,\dots,s_l \geqslant 1,\, r_1,\dots,r_l \geqslant 0 \right\rangle_{\mathbb{Q}} .$$

Idea of proof:

$$\begin{aligned}\begin{pmatrix} 1,1 \\ 0,1 \end{pmatrix}_q &= \sum_{n_1>n_2>0} \frac{q^{n_1}}{(1-q^{n_1})} \frac{n_2 q^{n_2}}{(1-q^{n_2})} \\ &= \sum_{n_1>n_2>0} \frac{q^{n_1}}{(1-q^{n_1})} \frac{q^{n_2}}{(1-q^{n_2})} + \sum_{n_1>n_2>n_3>0} \frac{q^{n_1}}{(1-q^{n_1})} \frac{q^{n_2}}{(1-q^{n_2})} \frac{1-q^{n_3}}{(1-q^{n_3})} \\ &= \zeta_q(1,1;t,t) + \zeta_q(1,1,1;t,t,1-t)\,.\end{aligned}$$

Depth- and weight-filtration

We endow the space of multiple q-zeta values $\mathcal{Z}_q$ with the depth- resp. weight-filtration induced by the notion of weight and depth defined on the bi-brackets

Bi-brackets - conjectures

Conjecture w.r.t. the algebra structures

(S1) The algebra $\mathcal{Z}_q$ is isomorphic to a free polynomial algebra.

Conjectures w.r.t. the vector space basis

(B1) Every bi-bracket equals a linear combination of brackets, i.e. $\mathcal{Z}_q^\circ = \mathcal{Z}_q$.

(B2) Every bi-bracket equals a linear combination of "123"-brackets.

Conjectures w.r.t. the graded dimensions

(D1) We have

$$\sum_{k\geqslant 0} \dim_{\mathbb{Q}} \mathrm{gr}_k^{\mathrm{W}}(\mathcal{Z}_q)x^k = \frac{1}{1-x-x^2-x^3+x^6+x^7+x^8+x^9}.$$

Conjectures w.r.t. the graded dimensions

(D2) We have

$$\sum_{w,l\geqslant 0} \dim \mathrm{gr}^{\mathrm{W},\mathrm{D}}_{k,l}(\mathcal{Z}_q)\, x^k y^l = \chi_{\widetilde{M}}(x,y) \cdot \chi_{\mathcal{A}}(x,y),$$

with

$$\chi_{\mathcal{A}}(x,y) = 1/\Big(1 - a_1(x)\,y + a_2(x)\,y^2 - a_3(x)\,y^3 - a_4(x)\,y^4 + a_5(x)\,y^5\Big),$$

$$\begin{aligned} a_1(x) &= \mathsf{D}(x)\,\mathsf{O}_1(x) \\ a_2(x) &= \mathsf{D}(x) \sum_{k\geqslant 4} \dim(M_k(\mathrm{SL}_2(\mathbb{Z}))^2\, x^k \\ a_3(x) &= \mathsf{D}(x)\,x\mathsf{S}(x) = a_5(x) \\ a_4(x) &= \mathsf{D}(x) \sum_{k\geqslant 12} \dim(S_k(\mathrm{SL}_2(\mathbb{Z}))^2\, x^k \end{aligned}$$

With $\mathsf{D}(x) = 1/(1-x^2)$, $\mathsf{O}_1(x) = x/(1-x^2)$, $\mathsf{S}(x) = x^{12}/((1-x^4)(1-x^6))$ and

$$\chi_{\widetilde{M}}(x,y) = 1 + \frac{x^2}{(1-x^2)^2}\,y + \frac{x^{12}}{(1-x^2)(1-x^4)(1-x^6)}\,y^2.$$

Bi-brackets - evidences for the conjectures

There are obvious implications, e.g. (B2) $\Longrightarrow$ (B1) and (D2) $\Longrightarrow$ (D1). If we assume that the lower bounds obtained by the numerical calculations equal the actual dimensions, then the conjectures hold within the range of our experiments. In particular for Conjecture (D2) we have

$$\begin{aligned}
a_2(x) &\equiv \sum_k a_{k,2}^{\mathrm{exp}} x^k &&\mod x^{32}\\
a_3(x) &\equiv \sum_k a_{k,3}^{\mathrm{exp}} x^k &&\mod x^{22}\\
a_4(x) &\equiv \sum_k a_{k,4}^{\mathrm{exp}} x^k &&\mod x^{19}\\
a_5(x) &\equiv \sum_k a_{k,5}^{\mathrm{exp}} x^k &&\mod x^{16}
\end{aligned}$$

Theorem (Bachmann-K.)

The conjectures (B1), (B2) and (D1) hold for all weights $k \leqslant 7$, i.e. every bi-bracket is a linear combination of "123"-brackets and there are exactly as many linear independent as expected.

q-MZV's and Lie-Algebras

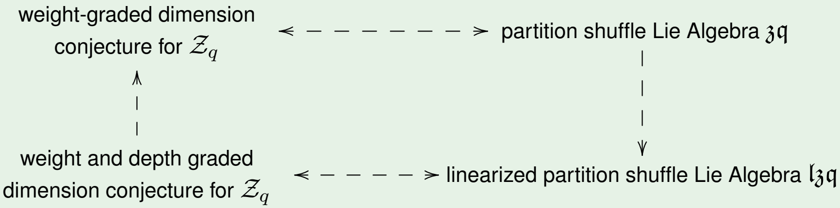
Refined conjecture w.r.t. the algebra structures

(S2) We have a decomposition of $\mathbb{Q}$-algebras

$$\mathcal{Z}_q \cong \widetilde{M}_{\mathbb{Q}}(SL_2(\mathbb{Z})) \otimes \mathcal{A},$$

moreover $\mathcal{A}$ equals the graded dual of the universal enveloping algebra of a bi-filtered Lie-algebra.

There are Lie algebras $\mathfrak{zq}$ and $\mathfrak{lzq}$, defined within Ecalle's theory of bimoulds, that very likely correspond to the associated graded of $\mathcal{A}$.



weight-graded dimension conjecture for $\mathcal{Z}_q$

partition shuffle Lie Algebra $\mathfrak{zq}$

weight and depth graded dimension conjecture for $\mathcal{Z}_q$

linearized partition shuffle Lie Algebra $\mathfrak{lzq}$

Moulds

Let A be an alphabet and denote by A^* its words. A Mould $M^\bullet$ is a map from A^* to a Ring R. Observe there is a bijection between moulds and non-commutative power series

$$\{M^\bullet : A^* \to R\} \cong R\langle\langle A\rangle\rangle,$$

since the coefficients of $\sum_{a\in A*}(M|a)\,a \in R\langle\langle A\rangle\rangle$ determine a mould M uniquely.

Example 1

Let $A = \{a_1, a_2, ...\}$ be an alphabet. Let $R = k[[u_1, u_2, u_3, ...]]$ and

$$M = (f_0, f_1, f_2, ...)$$

with $f_0 \in k$ and $f_l(u_1, u_2, ..., u_l) \in k[[u_1, u_2, ..., u_l]]$. Then we view M as mould via

$$\begin{aligned} M^\bullet :\ & A^* \to R \\ a_{i_1} a_{i_2} ... a_{i_l} & \mapsto f_l(u_{i_1}, u_{i_2}, ..., u_{i_l}). \end{aligned}$$

By abuse of notation we just write $M(u_{i_1}, u_{i_2}, ..., u_{i_l})$ instead of $(M|a_{i_1} a_{i_2} ... a_{i_l})$. Thus, the example explains the meaning of Ecalle's definition:

A mould is a collection of functions depending on a variable number of variables

Alternal moulds

Key remark

Most properties assigned to moulds correspond to functional equations.

Given a sequence $(f_1(u_1), f_2(u_1, u_2), f_3(u_1, u_2, u_3), \ldots)$ we use the notation

$$f_1(u_{j_1}) \mathbin{\text{ꝏ}} f_r(u_{j_2}, \ldots, u_{j_r}) = f_{r+1}(u_{j_1}, u_{j_2}, \ldots, u_{j_r}).$$

to define recursively a set of equations with the initial condition $1 ш f = f ш 1 = f$ and

$$\begin{aligned} & f_r(u_1, \ldots, u_r) ш f_s(u_{r+1}, \ldots, u_{r+s}) = \\ & \quad f_1(u_1) \mathbin{\text{ꝏ}} \big(f_{r-1}(u_2, \ldots, u_r) ш f_s(u_{r+1}, \ldots, u_{r+s}) \big) \\ & \quad + f_1(u_{r+1}) \mathbin{\text{ꝏ}} \big(f_r(u_1, \ldots, u_r) ш f_{s-1}(u_{r+2}, \ldots, u_{r+s}) \big) . \end{aligned}$$

We say a mould $(f_1, f_2, f_3, \ldots)$ is alternal, if for all $r, s \geqslant 1$ we have

$$f_r(u_1, \ldots, u_r) ш f_s(u_{r+1}, \ldots, u_{r+s}) = 0.$$

Note for example:

$$f_1(u_1) ш f_2(u_2, u_3) = 0 \quad \Longleftrightarrow \quad f_3(u_1, u_2, u_3) + f_3(u_2, u_1, u_3) + f_3(u_2, u_3, u_1) = 0.$$

Bimoulds

A bimould $M = (f_0, f_1, f_2, ...)$ in the pairs of variables $w_i = \binom{u_i}{v_i}$ is a mould

$$M^{\bullet} : A^* \to R = k[[u_1, v_1, u_2, v_2, u_3, v_3, ...]],$$

such that $f_0 \in k$ and $f_l \in k[[u_1, v_1, u_2, v_2, ..., u_l, v_l]]$ for all $l \geqslant 1$.

We will use the notation

$$(M|a_{i_1} a_{i_2} ... a_{i_l}) = M(w_{i_1}, w_{i_2}, ..., w_{i_l}) = M\binom{u_{i_1}, \ldots, u_{i_l}}{v_{i_1}, \ldots, v_{i_l}} = f_l(u_{i_1}, v_{i_1}, ..., u_{i_l}, v_{i_l}).$$

There are symmetries of moulds $M^{\bullet} : A^* \to R$ induced by endomorphisms of R. The involution *swap* is of particular interest for us

$$\mathrm{swap}\left(M\binom{u_1, u_2, \ldots, u_l}{v_1, v_2, \ldots, v_l}\right) = M\binom{v_l, v_{l-1} - v_l, \ldots, v_1 - v_2}{u_1 + \ldots + u_l, u_1 + \ldots + u_{l-1}, \ldots, u_1}$$

An alternal bimould $M\binom{\underline{u}}{\underline{v}}$ is alternal w.r.t. to both set of variables simultaneously.

Any mould $M(\underline{u})$ as in Example 1 becomes a bimould by setting $M(\underline{w}) = M\binom{\underline{u}}{\underline{v}} = M(\underline{u})$

A mould M is bi-alternal if $M(\underline{u})$ and $M^{\sharp}(\underline{v}) = \mathrm{swap}(M(\underline{u}))$ are alternal.

Alternil bimoulds

Given a sequence $(f_0, f_1(w_1), f_2(w_1, w_2), f_3(w_1, w_2, w_3), \ldots)$ we use the notation

$$f_1(w_{j_1}) \mathbin{\circ\!\!\circ} f_r(w_{j_2}, \ldots, w_{j_r}) = f_{r+1}(w_{j_1}, w_{j_2}, \ldots, w_{j_r}).$$

to define recursively a set of equations with the initial condition $1 * f = f * 1 = f$

$$\begin{aligned}
&f_r(w_1, \ldots, w_r) * f_s(w_{r+1}, \ldots, w_{r+s}) = \\
&\quad f_1(w_1) \mathbin{\circ\!\!\circ} \left(f_{r-1}(w_2, \ldots, w_r) * f_s(w_{r+1}, \ldots, w_{r+s})\right) \\
&\quad + f_1(w_{r+1}) \mathbin{\circ\!\!\circ} \left(f_r(w_1, \ldots, w_r) * f_{s-1}(w_{r+2}, \ldots, w_{r+s})\right) \\
&\quad + \frac{f_1\binom{u_1+u_{r+1}}{v_1} - f_1\binom{u_1+u_{r+1}}{v_r}}{v_1 - v_r} \mathbin{\circ\!\!\circ} \left(f_{r-1}(w_2, \ldots, w_r) * f_{s-1}(w_{r+2}, \ldots, w_{r+s})\right).
\end{aligned}$$

Example:

$$f_1(w_1) * f_1(w_2) = f_2(w_1, w_2) + f_2(w_2, w_1) + \frac{f_1\binom{u_1+u_2}{v_1} - f_1\binom{u_1+u_2}{v_2}}{v_1 - v_2}$$

We say a bimould $(f_1, f_2, f_3, \ldots)$ is alternil, if for all $r, s \geqslant 1$ we have

$$f_r(w_1, \ldots, w_r) * f_s(w_{r+1}, \ldots, w_{r+s}) = 0.$$

ARI Lie-bracket

Decompose a bi-word $w = \binom{u_1,\dots,u_l}{v_1,\dots,v_l}$ into $w = abc$ with

$$a = \binom{u_1,\dots,u_r}{v_1,\dots,v_r}\ , \quad b = \binom{u_{r+1},\dots,u_{r+s}}{v_{r+1},\dots,v_{r+s}}\ , \quad c = \binom{u_{r+s+1},\dots,u_l}{v_{r+s+1},\dots,v_l}\ .$$

then their flexions are defined by $\lceil c = c$ and $a\rceil = a$ if $b = \varnothing$, $b\rfloor = b$ if $c = \varnothing$, $\lfloor b = b$ if $a = \varnothing$ and else by

$$b\rfloor = \binom{u_{r+1},\dots,u_{r+s}}{v_{r+1}-v_{r+s+1},\dots,v_{r+s}-v_{r+s+1}}\ , \quad \lceil c = \binom{u_{r+1}+\dots+u_{r+s+1},u_{r+s+2},\dots,u_l}{v_{r+s+1},v_{r+s+2}\dots,v_l}$$

$$a\rceil = \binom{u_1,\dots,u_{r-1},u_r+u_{r+1}+\dots+u_{r+s}}{v_1,\dots,v_{r-1},v_r}\ , \quad \lfloor b = \binom{u_{r+1},\dots,u_{r+s}}{v_{r+1}-v_r,\dots,v_{r+s}-v_r}$$

Definition

The Ari Lie-bracket of two bimoulds is defined with above notation as

$$[A,B](w) = \sum_{\substack{w=abc\\ b\neq\varnothing}} A(a\lceil c)B(b\rfloor) - B(a\lceil c)A(b\rfloor) - \sum_{\substack{w=abc\\ a,b\neq\varnothing}} A(a\rceil c)B(\lfloor b) - B(a\rceil c)A(\lfloor b).$$

ARI Lie-bracket

Example

Let F and G be bi-moulds concentrated in depth 1, i.e., $F^{a_{i_1}\cdots a_{i_l}} = 0$, if $l \neq 1$. Then $H = \{F, G\}_{ARI}$ is concentrated in depth 2, with

$$H\binom{u_1, u_2}{v_1, v_2} = h\binom{u_1, u_2}{v_1, v_2} + h\binom{u_1+u_2, u_1}{v_2, v_1-v_2} + h\binom{u_2, u_1+u_2}{v_2-v_1, v_1},$$

where

$$h\binom{u_1, u_2}{v_1, v_2} = f_1\binom{u_1}{v_1} g_1\binom{u_2}{v_2} - g_1\binom{u_1}{v_1} f_1\binom{u_2}{v_2}.$$

ARI Lie-bracket

Theorem (Ecalle)

(i) The set of bimoulds $Bari$ equipped with the Ari Lie-bracket is a Lie Algebra.
(ii) $Bari^{pol}_{\underline{al},swap} = \{$polynomial, alternal, swap invariant bimoulds with m_1 even$\}$ is a sub Lie algebra.

The map $\delta : Bari \to Bari$ given in depth l by multiplication with $u_1 v_1 + ... + u_l v_l$ is a derivation, i.e., it satisfies the Leibniz rule

$$\delta[A, B] = [\delta A, B] + [A, \delta B].$$

Theorem

i) $\delta\left(Bari^{pol}_{\underline{al},swap}\right) \subset Bari^{pol}_{\underline{al},swap}$
(ii) $Ari^{pol}_{\underline{al},\underline{al}} = \{$polynomial, bi-alternal moulds with m_1 even$\}$ via the natural map

$$\iota : Ari^{pol}_{\underline{al},\underline{al}} \to Bari^{pol}_{\underline{al},swap}$$

given by $\iota(A)(w) = A(u) + swap(A)(v)$ is a sub Lie algebra.

Schneps: ARI, GARI, Zig and Zag: An introduction to Ecalle's theory of multiple zeta values, arXiv:1507.01534 [math.NT]

Another Lie-bracket

"Theorem" (in progress with L. Schneps)

(i) $Bari^{pol}_{\underline{il},swap} = \{$polynomial, alternil, swap invariant bimoulds and m_1 even$\}$ equipped with the pairing

$$\{A,B\} = ganit_{pic}\left([ganit_{poc}(A), ganit_{poc}(B)]\right)$$

is Lie algebra.
(ii) $Ari^{pol}_{\underline{al}*\underline{il}} = \{$polynomial, alternal moulds with alternil swap and m_1 even$\}$ via the natural map

$$\iota:\ Ari^{pol}_{\underline{al}*\underline{il}} \to Bari^{pol}_{\underline{il},swap}$$

given by $\iota(A)(w) = A(u) + swap(A)(v) + C_A$ is a sub Lie algebra of $Bari^{pol}_{\underline{il},swap}$.

Idea: By Ecalle the map $ganit_{poc}$ sends alternil to alternal bimoulds and $ganit_{pic}$ is the inverse map. Thus the pairing is defined on alternil bimoulds. The refinement for polynomial and swap-invariant bimould holds for depth $\leqslant 3$ and is work in progress for depth $\geqslant 4$.
And (ii), we proved yesterday. "□"

q-MZV and Moulds

For the multiple q-zeta values we expect the following

multiple q-Zeta values:

$$\begin{tikzcd}
 & \mathcal{MZ} \arrow[d, dashed] & \mathcal{Z}_q \arrow[l, "q\to 1"'] \arrow[d, dashed] & \\
Ari^{pol}_{\underline{al}*\underline{il}} \arrow[r, "\cong"] \arrow[d, dashed] & \mathfrak{ds} \arrow[r, hook, "\iota"'] \arrow[d, dashed] & \mathfrak{zq} \arrow[r, hook] \arrow[d, dashed] & Bari^{pol}_{\underline{il},swap} \arrow[d, dashed] \\
Ari^{pol}_{\underline{al},\underline{al}} \arrow[r, "\cong"] & \mathfrak{ls} \arrow[r, hook, "\iota"'] & \mathfrak{lzq} \arrow[r, hook] & Bari^{pol}_{\underline{al},swap}
\end{tikzcd}$$

It may be the case that $\mathfrak{lzq}$ is the extension of $\iota(\mathfrak{ls})$ and some extra generators corresponding to period polynomials by the action of the derivation δ on $Bari_{\underline{al},swap}$,i.e.

$$\mathfrak{lzq} \overset{?}{\cong} \mathrm{Lie}\left(\bigoplus_{i=0}^{\infty}\delta^{i}\big(\iota(\mathbb{1}\oplus\mathfrak{ls})\oplus\mathfrak{bc}\big)\right).$$

Bi-brackets - generating series

Key remark (Dimorphy)

The most inspiring feature of the bi-brackets is that there are two independent descriptions of the product of two of them. One of them lifts the stuffle-product and the other the shuffle-product

Today we focus on the functional equations satisfied by their generating function.

Definition

For the generating function of the bi-brackets in depth l we write

$$\left| \begin{matrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{matrix} \right| := \sum_{\substack{s_1,\dots,s_l \geqslant 1 \\ r_1,\dots,r_l \geqslant 0}} \begin{pmatrix} s_1, \dots, s_l \\ r_1, \dots, r_l \end{pmatrix}_q X_1^{s_1-1} \dots X_l^{s_l-1} \cdot Y_1^{r_1} \dots Y_l^{r_l}$$

The following results are based on the explicit description

$$\left| \begin{matrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{matrix} \right| = \sum_{u_1 > \dots > u_l > 0} \prod_{j=1}^{l} e^{u_j Y_j} \frac{e^{X_j} q^{u_j}}{1 - e^{X_j} q^{u_j}}.$$

Bi-brackets - functional equations for generating series

Proposition (stuffle product - special case of the algebra structure)

$$\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix} \cdot \begin{vmatrix} X_2 \\ Y_2 \end{vmatrix} \overset{st}{=} \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix} + \begin{vmatrix} X_2, X_1 \\ Y_2, Y_1 \end{vmatrix} + \frac{1}{X_1 - X_2} \left(\begin{vmatrix} X_1 \\ Y_1 + Y_2 \end{vmatrix} - \begin{vmatrix} X_2 \\ Y_1 + Y_2 \end{vmatrix} \right)$$
$$+ \sum_{k=1}^{\infty} \frac{B_k}{k!} (X_1 - X_2)^{k-1} \left(\begin{vmatrix} X_1 \\ Y_1 + Y_2 \end{vmatrix} + (-1)^{k-1} \begin{vmatrix} X_2 \\ Y_1 + Y_2 \end{vmatrix} \right) .$$

Theorem (Bachmann)

For all $l \geqslant 1$ we have the partition relation

$$\begin{vmatrix} X_1, \ldots, X_l \\ Y_1, \ldots, Y_l \end{vmatrix} = \begin{vmatrix} Y_1 + \cdots + Y_l, \ldots, Y_1 + Y_2, Y_1 \\ X_l, X_{l-1} - X_l, \ldots, X_1 - X_2 \end{vmatrix}$$

The partition relation (Ecalle notation: swap invariance) gives linear relations between bi-brackets in a fixed depth, for example

$$\begin{bmatrix} s \\ r \end{bmatrix}_q = \begin{bmatrix} r+1 \\ s-1 \end{bmatrix}_q \quad \text{or} \quad \begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix}_q = -2 \begin{bmatrix} 2,2 \\ 0,2 \end{bmatrix}_q + \begin{bmatrix} 2,2 \\ 1,1 \end{bmatrix}_q - 4 \begin{bmatrix} 3,1 \\ 0,2 \end{bmatrix}_q + 2 \begin{bmatrix} 3,1 \\ 1,1 \end{bmatrix}_q .$$

Bi-brackets - stuffle & shuffle product

The partition relation induces an involution P on the bi-brackets and this implies the double shuffle relations (on the level of representatives)

$$a \cdot b = P\big(P(a) \cdot P(b)\big) \qquad \forall\, a, b \in \mathcal{Z}_q$$

These double shuffle relations are indeed lifts of the double shuffle relations for MZV's.

Example:

$$\begin{pmatrix} 1,2 \\ 3,4 \end{pmatrix}_q + \begin{pmatrix} 2,1 \\ 4,3 \end{pmatrix}_q - \frac{35}{2} \begin{pmatrix} 2 \\ 7 \end{pmatrix}_q + 35 \begin{pmatrix} 3 \\ 7 \end{pmatrix}_q \overset{st}{=} \begin{pmatrix} 1 \\ 3 \end{pmatrix}_q \cdot \begin{pmatrix} 2 \\ 4 \end{pmatrix}_q$$

$$\overset{sh}{=} -35 \begin{pmatrix} 1,2 \\ 0,7 \end{pmatrix}_q + 15 \begin{pmatrix} 1,2 \\ 1,6 \end{pmatrix}_q - 5 \begin{pmatrix} 1,2 \\ 2,5 \end{pmatrix}_q + \begin{pmatrix} 1,2 \\ 3,4 \end{pmatrix}_q - 5 \begin{pmatrix} 2,1 \\ 1,6 \end{pmatrix}_q$$

$$+ 5 \begin{pmatrix} 2,1 \\ 2,5 \end{pmatrix}_q - 3 \begin{pmatrix} 2,1 \\ 3,4 \end{pmatrix}_q + \begin{pmatrix} 2,1 \\ 4,3 \end{pmatrix}_q - \frac{1}{6048} \begin{pmatrix} 2 \\ 2 \end{pmatrix}_q + \frac{1}{720} \begin{pmatrix} 2 \\ 4 \end{pmatrix}_q + \begin{pmatrix} 2 \\ 8 \end{pmatrix}_q$$

Conjecture (Bachmann)

All linear relations between bi-brackets come from the partition and the double shuffle relations.

From bi-brackets to bimoulds

Recall the generating series for bi-brackets satisfies the partition relation

$$\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix} = \begin{vmatrix} Y_1 + \cdots + Y_l, \dots, Y_1 + Y_2, Y_1 \\ X_l, X_{l-1} - X_l, \dots, X_1 - X_2 \end{vmatrix}$$

and the formula for the product

$$\begin{vmatrix} X_1, \dots, X_j \\ Y_1, \dots, Y_j \end{vmatrix} \cdot \begin{vmatrix} X_{j+1}, \dots, X_l \\ Y_{j+1}, \dots, Y_l \end{vmatrix} = \left.\begin{vmatrix} X_1, \dots, X_l \\ Y_1, \dots, Y_l \end{vmatrix}\right|_{\mathrm{Sh}_j} + \text{ lower weight and depth terms.}$$

where $|_{\mathrm{Sh}_j}$ denotes the sum over all (j, l)-shuffles. Now, if we decompose the generating series into a sum of polynomials

$$\left(\begin{vmatrix} X_1 \\ Y_1 \end{vmatrix}, \begin{vmatrix} X_1, X_2 \\ Y_1, Y_2 \end{vmatrix}, \dots \right) \equiv \sum_{\alpha} \alpha \left(f_1^{\alpha}(X_1, Y_1), f_2^{\alpha}(X_1, X_2, Y_1, Y_2), \dots \right) ,$$

where α runs through a vector space basis of $\mathcal{Z}_q$ modulo products and lower weight resp. lower depth, then $f^{\alpha} = (f_1^{\alpha}, f_2^{\alpha}, \dots)$ is a polynomial bimould[2] which is swap-invariant and alternil resp. swap-invariant and alternal.

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[2] after the coordinate change $u_i = Y_i$ and $v_i = X_i$

Partition shuffle Lie-Algebra

Definition

We define partition shuffle Lie-Algebra $\mathfrak{zq}$ as the sub Lie algebra of $Bari_{\underline{il},swap}^{pol}$ generated by the sequences of polynomials coming from the generating series of bi-brackets modulo quasi-modular forms, products and lower weight terms

For example we have in degrees 1 to 5 the elements

$$
\begin{aligned}
\widehat{\xi}_{1,0} &= (1, 0, \ ...), \\
\widehat{\xi}_{3,0} &= (u_1^2 + v_1^2, \ v_1 - 2v_2 - u_1 + u_2, \tfrac{1}{3}, 0, \ ...), \\
\widehat{\xi}_{2,1} &= (u_1 v_1, -2v_1 + 2v_2 - 2u_2, 0, \ ....), \\
\widehat{\xi}_{5,0} &= (u_1^4 + v_1^4, \ ...), \\
\widehat{\xi}_{4,1} &= (u_1 v_1 (u_1^2 + v_1^2), \ ...), \\
\widehat{\xi}_{3,2} &= ((u_1 v_1)^2, \ ...).
\end{aligned}
$$

these correspond to the classes of the bi-brackets $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q$, $\begin{bmatrix} 3 \\ 0 \end{bmatrix}_q$, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q$, $\begin{bmatrix} 5 \\ 0 \end{bmatrix}_q$, $\begin{bmatrix} 4 \\ 1 \end{bmatrix}_q$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q$

Conjecture

The partition shuffle Lie-algebra $\mathfrak{zq}$ is generated by the $\widehat{\xi}_{r,s}$ and quadratic relations generate the ideal of relations. The Hilbert-Poincare series equals

$$H_{\mathcal{U}(\mathfrak{zq})^\vee}(x) = \sum_{k=0}^{\infty} \dim \operatorname{gr}_k^{\mathrm{W}} S(\mathfrak{zq})\, x^k = \frac{1}{1 - \mathsf{D}(x)\mathsf{O}_1(x) + \mathsf{D}(x)\mathsf{R}(x)},$$

where $\mathsf{D}(x) = 1/(1-x^2)$, $\mathsf{O}_1(x) = x/(1-x^2)$ and $\mathsf{R}(x) = \sum_{k \geqslant 4} \dim(S_k \oplus M_k)x^k$.

For example the relation $\{\widehat{\xi}_{1,0}, \widehat{\xi}_{2i+1,0}\} = 0$ for all $i \geqslant 1$ gives the "Eisenstein-part".

Interpretation

If the missing link[a] from $\mathcal{Z}_q$ to $\mathfrak{zq}$ is filled, then the weight graded dimension conjecture would be implied, i.e. $\mathcal{Z}_q \cong \widetilde{M}(SL_2(\mathbb{Z})) \otimes \mathcal{U}(\mathfrak{zq})^\vee$. At the level of Hilbert-Poincare series, the above isomorphism is reflected by the identity

$$\frac{1}{1 - x - x^2 - x^3 + x^6 + x^7 + x^8 + x^9} = \frac{1}{(1-x^2)(1-x^4)(1-x^6)} \frac{1}{1 - \mathsf{D}(x)\mathsf{O}_1(x) + \mathsf{D}(x)\mathsf{R}(x)}.$$

---

[a] e.g. similarily like Racinet's construction for MZV by using appropriate "formal multiple q-zeta values"

Linearised partition shuffle Lie-Algebra

Definition

We define the linearised partition shuffle Lie-Algebra $\mathfrak{lzq}$ as the sub Lie algebra of $Bari^{pol}_{\underline{al},swap}$ generated by the polynomials coming from the generating series for bi-brackets modulo quasi-modular forms, products and lower depth terms.

The elements

$$\xi_{r,s} = (u_1 v_1)^s (u_1^{r-s-1} + v_1^{r-s-1})$$

correspond to the class of the bi-brackets $\begin{pmatrix} r \\ s \end{pmatrix}_q$. More generally, there is a map $\mathfrak{zq}_k \to \mathfrak{lzq}_{k,l}$ given by

$$(\underbrace{0, ..., 0}_{\#=l-1}, f_l(u_1, v_1, ..., u_l, v_l), *, ..., *, 0, ...) \mapsto f_l.$$

Using quadratic relations in $\mathfrak{lzq}$ we obtain this way "new" generators in depth 4, e.g.,

$$\left\{\widehat{\xi}_{3,0}, \widehat{\xi}_{9,0}\right\} - 3\left\{\widehat{\xi}_{5,0}, \widehat{\xi}_{7,0}\right\} = \left(0, 0, 0, \chi_{\triangle,0}, *, ...\right).$$

Conjecture

The algebra $\mathcal{U}(\mathfrak{lzq})$ has the Hilbert-Poincare series

$$H_{\mathcal{U}(\mathfrak{lzq})}(x,y) = \frac{1}{1-a_1(x)\,y+a_2(x)\,y^2-a_3(x)\,y^3-a_4(x)\,y^4+a_5(x)\,y^5},$$

with

$$\begin{aligned}
a_1(x) &= \mathsf{D}(x)\,\mathsf{O}_1(x) && \text{"Generators } \xi_{r,s}\text{ "} \\
a_2(x) &= \mathsf{D}(x)\sum_{k\geqslant 4}\dim(M_k(\mathrm{SL}_2(\mathbb{Z}))^2\,x^k && \text{"Periodpoly Relations"} \\
a_3(x) &= \mathsf{D}(x)\,x\mathsf{S}(x) && \text{"Homology"} \\
a_4(x) &= \mathsf{D}(x)\sum_{k\geqslant 12}\dim(S_k(\mathrm{SL}_2(\mathbb{Z}))^2\,x^k && \text{"new Generators } \chi_{cusp}\text{"} \\
a_5(x) &= \mathsf{D}(x)\,x\mathsf{S}(x) && \text{"}\xi_{1,0}\text{-orthogonality"}
\end{aligned}$$

With $\mathsf{D}(x) = 1/(1-x^2)$, $\mathsf{O}_1(x) = x/(1-x^2)$ and $\mathsf{S}(x) = x^{12}/((1-x^4)(1-x^6))$.

Some evidence

- Experiments using Pari/GP with parallel algorithms support the conjectures, e.g.

$$\dim \mathrm{gr}^{\mathrm{W,D}}_{k,l}(\text{space spanned by depth } 1 \text{ elements } \xi_{r,s})$$

is as conjectured for

depth l	1	2	3	4	5	6
weight $k \leqslant$	∞	52	35	26	23	18

- Also we calculated in depth 4 the dimension of the spaces spanned by the new generators for weight $k \leqslant 26$. For this we used a Computer at DESY Hamburg with 128 cores and 1 terabyte RAM. For $k = 26$ the calculation took about a week.
- Some of the statements in that conjecture are actually proven, e.g.

Theorem

For the vector spaces $\mathfrak{R}_k$ spanned by the relations in weight k and depth 2 in the Lie algebra spanned by depth 1 elements $\xi_{r,s}$ we have the generating series

$$a_2(x) = \sum_{k\geqslant 4} \dim \mathfrak{R}_k x^k = \mathsf{D}(x) \sum_{n\geqslant 4} \dim(M_n(\mathrm{SL}_2(\mathbb{Z}))^2\, x^n, \qquad \left(\mathsf{D}(x) = \frac{1}{1-x^2}\right).$$

Idea of Proof: Explicitly these spaces are given by

$$\mathfrak{R}_k = \Big\{ P \in \mathbb{Q}[u_1,u_2,v_1,v_2] \,\Big|\, \text{homogenous, } \deg P = k-2, \text{ such that} \\ P\binom{u_1,u_2}{v_1,v_2} + P\binom{u_1+u_2,u_1}{v_2,v_1-v_2} + P\binom{u_2,u_1+u_2}{v_2-v_1,v_1} = 0, \\ P\binom{u_1,u_2}{v_1,v_2} + P\binom{u_2,u_1}{v_2,v_1} = 0, \quad P\binom{u_1,u_2}{v_1,v_2} = P\binom{\epsilon u_1,\mu u_2}{\epsilon v_1,\mu v_2}, (\epsilon,\mu \in \{\pm 1\}), \\ P\binom{u_1,u_2}{v_1,v_2} = P\binom{v_1,u_2}{u_1,v_2}, \quad P\binom{u_1,u_2}{v_1,v_2} = P\binom{u_1,v_2}{v_1,u_2} \quad \Big\}$$

We will use harmonic analysis for the symplectic Laplacian

$$\Delta = \partial_{u_1}\partial_{v_1} + \partial_{u_2}\partial_{v_2} \,,$$

because following an idea of Zagier we can recover tensor products of period polynomials in the subspace spanned by Δ-harmonic solutions of the above functional equation:
Consider $P \in \mathbb{Q}[u_1,u_2,v_1,v_2]$ as a function on (2×2)-matrices via

$$P : X = \begin{pmatrix} u_1 & u_2 \\ -v_2 & v_1 \end{pmatrix} \mapsto P(X) = P\binom{u_1,u_2}{v_1,v_2}.$$

Relations in depth 2

Let SL_2 act by the multiplication of matrices and then $\mathfrak{R}_k$ is given by the set of P such that

$$\begin{aligned} P(X) + P(XU) + P(XU^2) &= 0 \\ P(X) + P(XS) &= 0 \\ P(X^t) &= P(X) \\ P(\epsilon X \epsilon) &= P(X), \end{aligned}$$

where $U = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\epsilon = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We have with $\Delta = \partial_{u_1}\partial_{v_1} + \partial_{u_2}\partial_{v_2}$

$$\mathfrak{R}_k = \ker \Delta \oplus (u_1v_1 + u_2v_2)\mathfrak{R}_{k-2}.$$

Via the map

$$P(X) \mapsto P\left(\left(\begin{smallmatrix} ac & ad \\ bc & bd \end{smallmatrix}\right)\right) = P\left(\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\left(\begin{smallmatrix} c & d \end{smallmatrix}\right)\right),$$

we identify

$$\ker \Delta = \left\langle f(a,b) \otimes g(c,d) + g(a,b) \otimes f(c,d) \;\middle|\; f, g \in W_k^+ \text{ or } f, g \in W_k^- \right\rangle_{\mathbb{Q}}.$$

From $\dim W_k^- = s_k + 1$ and $\dim W_k^+ = s_k$ it follows the claimed identity

$$\dim \ker \Delta = \frac{(s_k+2)(s_k+1)}{2} + \frac{(s_k+1)s_k}{2} = \dim(M_n(\mathrm{SL}_2(\mathbb{Z}))^2.$$

$\square$

The relations in depth 2 contain two families.

$$\mathfrak{R}_{Eis} = \Big\{\delta^r[\xi_{1,0}, \iota(\xi_s)]\Big\}, \qquad \mathfrak{R}_{Cusp} = \Big\{\delta^r\, \iota\big(\text{periodpoly relations for } \mathfrak{ls}\ \big)\Big\}\,.$$

Because of the Jacobi identity the ideals of the relations $\mathfrak{R}_{Cusp}$ and $\mathfrak{R}_{Eis}$ have a non-trivial intersection in depth 3. More precisely we have

$$\big[\xi_{1,0}, \mathfrak{R}_{Cusp}\big] \subset \big[\mathfrak{lzq}, \mathfrak{R}_{Eis}\big].$$

Lemma.

The generating series of the numbers of these "relations in the relations" in depth 3 equals $a_3(x)$.

In depth 5 we find the relations

$$\mathfrak{R}_{new} = \big\{\delta^r[\xi_{1,0}, \iota(\text{ new generators in depth } 4 \text{ for } \mathfrak{ls}\ )]\big\},$$

Lemma.

The generating series of the numbers of these orthogonalities in depth 5 equals $a_5(x)$.

Summary

- The $\mathbb{Q}$-algebra of multiple q-zeta values $\mathcal{Z}_q$ is spanned by bi-brackets, i.e., q-series whose coefficients are rational numbers given by sums over partitions. It contains all quasi-modular forms.
- The elements in $\mathcal{Z}_q$ have a direct connection to multiple zeta values.
- There are conjectural formulas for the dimensions $\dim \mathrm{gr}_{k,l}^{\mathrm{W},\mathrm{D}} \mathcal{Z}_q$, and other subspaces.
- Conjecturally every element in $\mathcal{Z}_q$ can be written as a linear combination of 123-brackets. In particular $\mathcal{Z}_q^{\circ} = \mathcal{Z}_q$.
- The algebra $\mathcal{Z}_q$ is dimorphic, i.e. there are two different ways to express a product of bi-brackets in terms of the generators. This gives rise to a lot of, conjecturally all, linear relations between bi-brackets.
- The functional equations satisfied by the generating series of bi-brackets modulo products and lower depth give rise to a subspace in the Lie-algebra of bimoulds.
- Conjecturally the generators of $\mathcal{Z}_q$ give a basis of a Lie-algebra contained in the Lie-algebra of swap-invariant, alternal, polynomial bimoulds.
- Massive computer calculations give striking evidence for those conjectures.