Quantized $\mathrm{SL}(2)$ representations of knot groups

(joint with R. van der Veen)

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Outline:

1. Wirtinger presentation for closed braid
2. $\mathrm{SL}(2)$ representation space
3. Braided Hopf algebra
4. Quantized $\mathrm{SL}(2)$ representation space
5. Examples

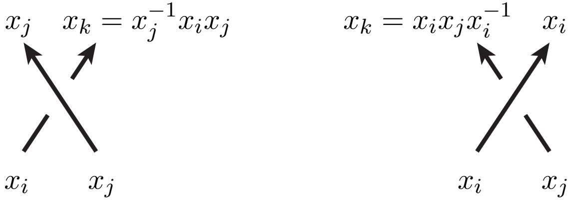
1. Wirtinger presentation for closed braid

Wirtinger presentation

Let K be a knot in S^3 and D be its diagram. Then the fundamental group of the complement of K $\pi_1(S^3 \setminus K)$ has the following presentation.

$$\pi_1(S^3 \setminus K) = \langle x_1, x_2, \cdots, x_n \mid r_1, r_2, \cdots, r_n \rangle$$

where n is the number of crossings or D, the generators $x_1, \cdots, x_n$ corresponds to the overpasses of D and r_i is the relation coming from the i-th crossing as follows.



Remark

The relation r_n comes from $r_1, \cdots, r_{n-1}$.

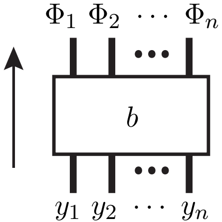
1. Wirtinger presentation for closed braid

Presentation coming from a braid

Every knot can be expressed as a closed braid. For a knot K, let $b \in B_n$ be a braid whose closure is isotopic to K.

Let y_1, y_2, $\cdots$, y_n be elements of $\pi_1(S^3 \setminus K)$ corresponding to the overpasses at the bottom (and the top) of b. By applying the relations of the Wirtinger presentation at every crossings from bottom to top, we get $\Phi_1(y_1, \cdots, y_n)$, $\cdots$, $\Phi_n(y_1, \cdots, y_n)$ at the top of b, and the Wirtinger presentation is equivalent to

$$\pi_1(S^3 \setminus K) = \langle y_1, \cdots, y_n \mid y_1 = \Phi_1(y_1, \cdots, y_n), \cdots, y_n = \Phi_n(y_1, \cdots, y_n) \rangle .$$



2. $\mathrm{SL}(2)$ representation space

$\mathrm{SL}(2)$ representation of $\pi_1(S^3 \setminus K)$

An $\mathrm{SL}(2)$ representation ρ of $\pi_1(S^3 \setminus K)$ is determined by $\rho(y_1), \cdots,$ $\rho(y_n) \in \mathrm{SL}(2)$ satisfying

$$\Phi_1(\rho(y_1), \cdots, \rho(y_n)) = \rho(y_1),$$
$$\cdots$$
$$\Phi_n(\rho(y_1), \cdots, \rho(y_n)) = \rho(y_n).$$

Let I_b be the ideal in the tensor $\mathbb{C}[\mathrm{SL}(2)]^{\otimes n}$ of the coordinate space of $\mathrm{SL}(2)$ generated by the above relations.

Theorem

$\mathbb{C}[\mathrm{SL}(2)]^{\otimes n}/I_b$ *does not depend on the presentation of* $\pi_1(S^3 \setminus K)$ *and is called* **the $\mathrm{SL}(2)$ representation space** *of* $\pi_1(S^3 \setminus K)$.

G. Brumfiel and H. Hilden: $\mathrm{SL}(2)$ *representations of finitely presented groups.* Contemporary Mathematics **187** Amer. Math. Soc. 1994, Proposition 8.2.

2. $\mathrm{SL}(2)$ representation space

Hopf algebra interpretation

$\mathbb{C}[\mathrm{SL}(2)]$ is generated by a, b, c, d representing a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

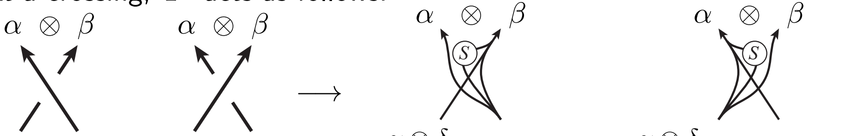
$\mathbb{C}[\mathrm{SL}(2)]$ has natural Hopf algebra structure coming from the group structure of $\mathrm{SL}(2)$.

Δ: $\mathbb{C}[\mathrm{SL}(2)] \to \mathbb{C}[\mathrm{SL}(2)] \otimes \mathbb{C}[\mathrm{SL}(2)]$ with $\Delta(f)(x \otimes y) = f(xy)$,

S: $\mathbb{C}[\mathrm{SL}(2)] \to \mathbb{C}[\mathrm{SL}(2)]$ with $S(f)(x) = f(x^{-1})$,

ε: $\mathbb{C}[\mathrm{SL}(2)] \to \mathbb{C}$ with $\varepsilon(f) = f(1)$.

Let $\Phi^* : \mathbb{C}[\mathrm{SL}(2)]^{\otimes n} \to \mathbb{C}[\mathrm{SL}(2)]^{\otimes n}$ be the dual map of $\Phi = (\Phi_1, \cdots, \Phi_n)$. At a crossing, Φ^* acts as follows.



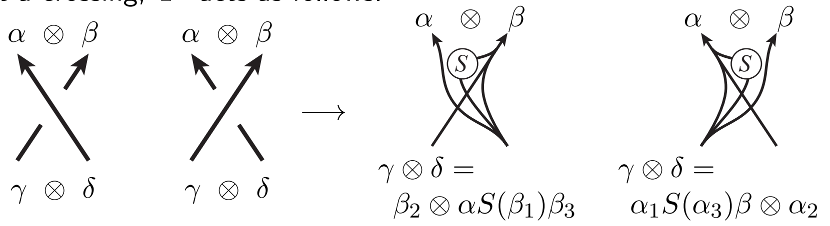
$\gamma \otimes \delta = \beta^{(2)} \otimes \alpha S(\beta^{(1)})\beta^{(3)}$ $\quad$ $\gamma \otimes \delta = \alpha^{(1)} S(\alpha^{(3)})\beta \otimes \alpha^{(2)}$

$\alpha^{(1)} \otimes \alpha^{(2)} \otimes \alpha^{(3)}$ means $\Delta(\Delta(\alpha)) = \sum_j \alpha_j^{(1)} \otimes \alpha_j^{(2)} \otimes \alpha_j^{(3)}$ (Sweedler notation).

2. $\mathrm{SL}(2)$ representation space

Hopf algebra interpretation

Let $\Phi^* : \mathbb{C}[\mathrm{SL}(2)]^{\otimes n} \to \mathbb{C}[\mathrm{SL}(2)]^{\otimes n}$ be the dual map of $\Phi = (\Phi_1, \cdots, \Phi_n)$. At a crossing, Φ^* acts as follows.



Theorem

Let J_b be the ideal generated by the image of $\Phi^* - id^{\otimes n}$, then J_b is equal to the previous ideal I_b and $\mathbb{C}[\mathrm{SL}(2)]^{\otimes n}/J_b$ is the $\mathrm{SL}(2)$ representation space of $\pi_1(S^3 \setminus K)$.

Remark

This construction can be generalized to any commutative Hopf algebra.

3. Braided Hopf algebra

Braided Hopf algebra

Definition

An algebra A is called **a braided Hopf algebra** if it is equipped with following linear maps satisfying the relations given in the next picture.

Operations

multiplication $\mu : A \otimes A \to A$,

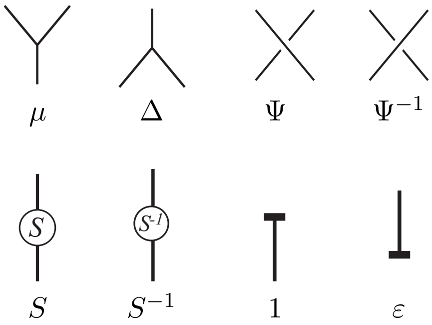
comultiplication $\Delta : A \to A \otimes A$,

unit $1 : \mathbb{C} \to A$,

counit $\varepsilon : A \to \mathbb{C}$,

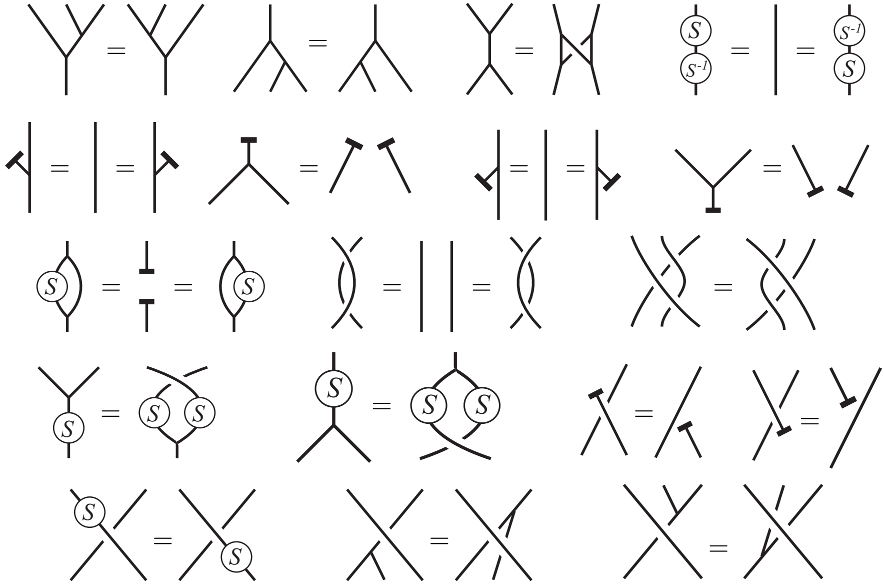
antipode $S : A \to A$,

braiding $\Psi : A \otimes A \to A \otimes A$.



3. Braided Hopf algebra

Relations of a braided Hopf algebra



3. Braided Hopf algebra

Adjoint coaction

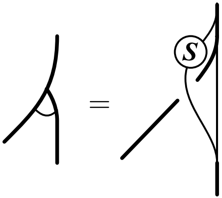
Dual of the adjoint action is given as follows.

Definition

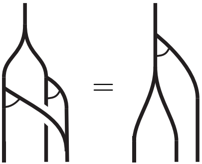
The adjoint coaction $\mathrm{ad} : A \to A \otimes A$ is defined by

$$\mathrm{ad}(x) = (id \otimes \mu)(\Psi \otimes id)(S \otimes \Delta)\Delta(x).$$

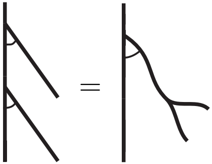
The adjoint coaction ad satisfies the following.



definition



$\mathrm{ad} \otimes \mathrm{ad}$ and Δ



two ad to $\mu \circ \mathrm{ad}$

3. Braided Hopf algebra

Braided commutativity

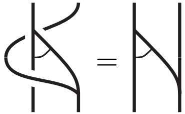
Braided commutativity is a weakened version of the commutativity.

Definition

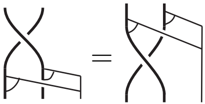
A braided Hopf algebra A is **braided commutative** if it satisfies

$$(id \otimes \mu)(\Psi \otimes id)(id \otimes \mathrm{ad})\Psi = (id \otimes \mu)(\mathrm{ad} \otimes id).$$

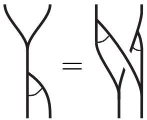
If A is braided commutative, the following commutativity holds.



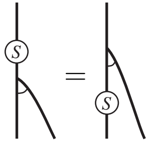
definition



$\mathrm{ad} \otimes \mathrm{ad}$ and Ψ



ad and μ



ad and S

3. Braided Hopf algebra

Braided $\mathrm{SL}(2)$

Definition (S. Majid)

A braided $\mathrm{SL}(2)$ is a one-parameter deformation of $\mathbb{C}[\mathrm{SL}(2)]$ defined by the following. It is denoted by $\mathrm{BSL}(2)$.

$$b\,a = t\,a\,b, \quad c\,a = t^{-1}\,a\,c, \quad d\,a = a\,d, \quad d\,b = b\,d + (1-t^{-1})\,a\,b,$$

$$c\,d = d\,c + (1-t^{-1})\,c\,a, \quad b\,c = c\,b + (1-t^{-1})\,a\,(d-a), \quad a\,d - t\,c\,b = 1,$$

$$\Delta(a) = a\otimes a + b\otimes c, \quad \Delta(b) = a\otimes b + b\otimes d, \quad \Delta(c) = c\otimes a + d\otimes c,$$

$$\Delta(d) = c\otimes b + d\otimes d, \quad S(a) = (1-t)\,a + t\,d \quad S(b) = -t\,b, \quad S(c) = -t\,c,$$

$$S(d) = a, \quad \varepsilon(a) = 1, \quad \varepsilon(b) = 0, \quad \varepsilon(c) = 0, \quad \varepsilon(d) = 1,$$

$$\Psi(x\otimes 1) = 1\otimes x, \quad \Psi(1\otimes x) = x\otimes 1, \quad \Psi(a\otimes a) = a\otimes a + (1-t)\,b\otimes c, \quad \Psi(a\otimes b) = b\otimes a,$$

$$\Psi(a\otimes c) = c\otimes a + (1-t)(d-a)\otimes c, \quad \Psi(a\otimes d) = d\otimes a + (1-t^{-1})\,b\otimes c,$$

$$\Psi(b\otimes a) = a\otimes b + (1-t)\,b\otimes(d-a), \quad \Psi(b\otimes b) = t\,b\otimes b, \quad \Psi(c\otimes a) = a\otimes c,$$

$$\Psi(b\otimes c) = t^{-1}\,c\otimes b + (1+t)(1-t^{-1})^2\,b\otimes c - (1-t^{-1})(d-a)\otimes(d-a),$$

$$\Psi(b\otimes d) = d\otimes b + (1-t^{-1})\,b\otimes(d-a), \quad \Psi(c\otimes b) = t^{-1}\,b\otimes c, \quad \Psi(c\otimes c) = t\,c\otimes c,$$

$$\Psi(c\otimes d) = d\otimes c, \quad \Psi(d\otimes a) = a\otimes d + (1-t^{-1})\,b\otimes c, \quad \Psi(d\otimes b) = b\otimes d,$$

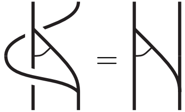
$$\Psi(d\otimes c) = c\otimes d + (1-t^{-1})(d-a)\otimes c, \quad \Psi(d\otimes d) = d\otimes d - t^{-1}\,(1-t^{-1})\,b\otimes c.$$

3. Braided Hopf algebra

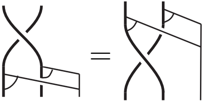
Braided $\mathrm{SL}(2)$ is braided commutative

Proposition

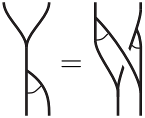
The braided Hopf algebra $\mathrm{BSL}(2)$ is braided commutative.



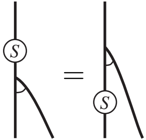
definition



$\mathrm{ad} \otimes \mathrm{ad}$ and Ψ



ad and μ



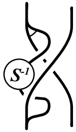
ad and S

4. Quantized $\mathrm{SL}(2)$ representation space

Braid group representation through a braided Hopf algebra

Let A be a braided Hopf algebra which may NOT be braided commutative.

We construct a representation of B_n on $A^{\otimes n}$ associated with the Wirtinger presentation. Let R and R^{-1} be elements of $\mathrm{End}(A^{\otimes 2})$ given by

$R =$  $R^{-1} =$ 

.

For $\sigma_i^{\pm 1} \in B_n$, let $\rho(\sigma_i) = id^{\otimes(i-1)} \otimes R \otimes id^{\otimes(n-i-1)}$ and $\rho(\sigma_i^{-1}) = id^{\otimes(i-1)} \otimes R^{-1} \otimes id^{\otimes(n-i-1)}$.

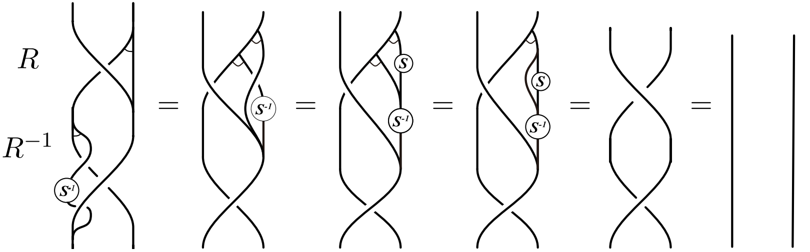
Theorem

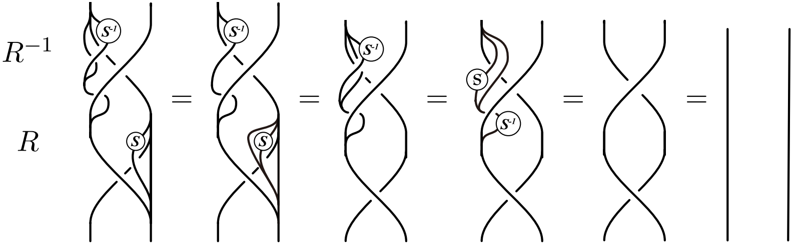
The above ρ defined for generators of B_n extends to a representation of B_n in $\mathrm{End}(A^{\otimes n})$.

S. Woronowicz, *Solutions of the braid equation related to a Hopf algebra.* Lett. Math. Phys. **23** (1991), 143–145. (for usual Hopf algebra)

4. Quantized $\mathrm{SL}(2)$ representation space

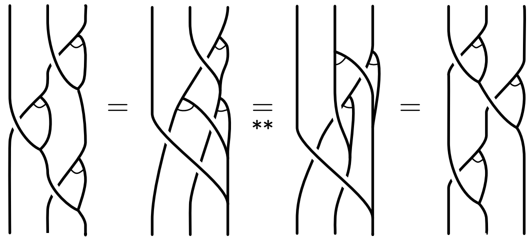
Proof for the inverse

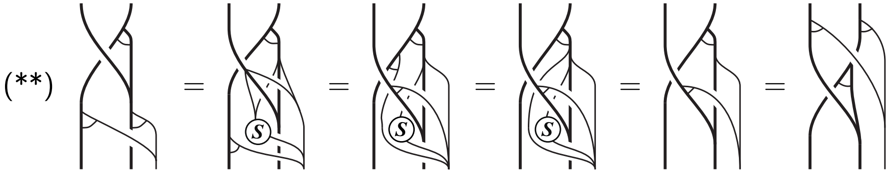




4. Quantized $\mathrm{SL}(2)$ representation space

Proof for the braid relation





4. Quantized $\mathrm{SL}(2)$ representation space

A representation space

From now on, we assume that the braided Hopf algebra A is **braided commutative**. For $b \in B_n$, let $\rho(b) \in \mathrm{End}(A^{\otimes n})$ be the representation of b defined as above. Let I_b be the left ideal of $A^{\otimes n}$ generated by the image of the map $\rho(b) - id^{\otimes n}$.

Proposition

The left ideal I_b is a two-sided ideal.

This proposition comes from the following lemma.

Lemma

For $\boldsymbol{x}, \boldsymbol{y} \in A^{\otimes n}$, we have

$$\rho(b)\,\mu(\boldsymbol{x} \otimes \boldsymbol{y}) = \mu\big((\rho(b)\,\boldsymbol{x}) \otimes (\rho(b)\,\boldsymbol{y})\big).$$

4. Quantized $\mathrm{SL}(2)$ representation space

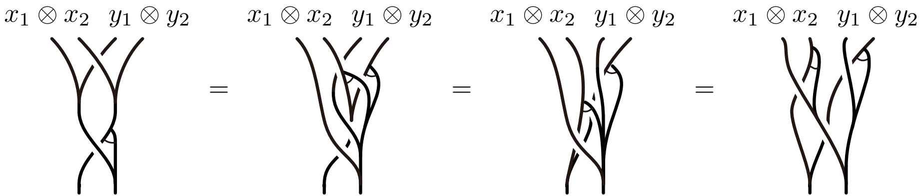
A representation space

Proof.

It is enough to show that

$$R\,\mu(\boldsymbol{x} \otimes \boldsymbol{y}) = \mu\,(R \otimes R)(\boldsymbol{x} \otimes \boldsymbol{y})$$

for the product $\mu : A^{\otimes 2} \otimes A^{\otimes 2} \to A^{\otimes 2}$ and $\boldsymbol{x} = x_1 \otimes x_2$, $\boldsymbol{y} = y_1 \otimes y_2 \in A^{\otimes 2}$, which is proved graphically as follows. □



4. Quantized $\mathrm{SL}(2)$ representation space

A **representation space**

Theorem

If the closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic, then A_{b_1} and A_{b_2} are isomorphic algebras. Moreover, A_{b_1} and A_{b_2} are isomorphic A-comodules with adjoint coaction. In other words, A_b is an invariant of the knot (or link) $\widehat{b}$.

Definition

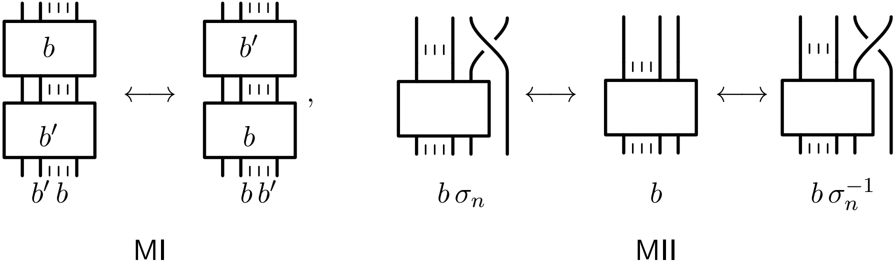
The quotient algebra $A_b = A^{\otimes n}/I_b$ is callde **the A representation space** of the closure $\widehat{b}$.

4. Quantized $\mathrm{SL}(2)$ representation space

Markov equivalence

Theorem

The closures of two braids $b_1 \in B_{n_1}$ and $b_2 \in B_{n_2}$ are isotopic in S^3 if and only if there is a sequence of the following two types of moves connecting b_1 to b_2. These moves are called the Markov moves and such b_1 and b_2 are called Markov equivalent.



We will see that A_b is invariant under MI and MII.

4. Quantized $\mathrm{SL}(2)$ representation space

Equivalent pair

Definition

For $b \in B_n$, we present $I_{\rho(b)}$ by $\rho(b) \sim \rho(1)$. Similarly, for two diagrams d_1, d_2 reresenting elements of $\mathrm{Hom}(A^{\otimes m}, A^{\otimes n})$, $d_1 \sim d_2$ present a two-sided ideal I_{d_1,d_2} in $A^{\otimes n}$ generated by

$$d_1(x_1 \otimes \cdots \otimes x_m) - d_2(x_1 \otimes \cdots \otimes x_m)$$

for $x_1, \cdots, x_m \in A$. Such d_1 and d_2 are called the **equivalent pair** of diagrams corresponding to the two-sided ideal I_{d_1,d_2} and the quotient algebra $A^{\otimes n}/I_{d_1,d_2}$.

Lemma

Let $d_1 \sim d_2$ be an equivalent pair and let $d_1' \sim d_2'$ be the equivalent pair where d_1' and d_2' are obtained from d_1 and d_2 respectively by one of the following operations (1), (2), (3), (3S), (4L), (4LS), (4R), (4RS) illustrated in the following. Then the corresponding ideals I_{d_1,d_2} and $I_{d_1',d_2'}$ are equal.

4. Quantized $\mathrm{SL}(2)$ representation space

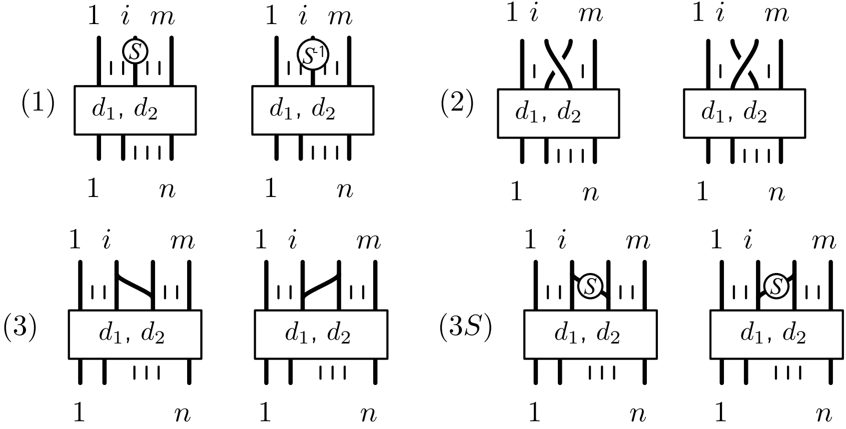
Operation (1), (2), (3), (3S)

(1) Add S or S^{-1} to the same position of d_1 and d_2 at the top.

(2) Apply a braiding to the same position of d_1 and d_2 at the top.

(3) Add an arc connecting the adjacent strings.

(3S) Add an arc with S connecting the adjacent arcs.



4. Quantized $\mathrm{SL}(2)$ representation space

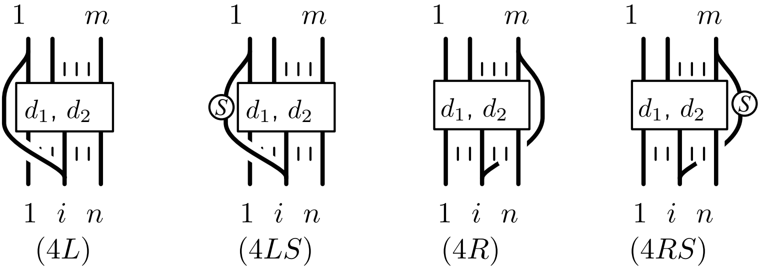
Operation (4L), (4R), (4R), (4RS)

(4L) Add an arc connecting the leftmost top arc and some bottom arc.

(4LS) Add an arc with S connecting the leftmost top arc and some bottom arc.

(4R) Add an arc connecting the rightmost top arc and some bottom arc.

(4RS) Add an arc with S connecting the rightmost top arc and some bottom arc.



4. Quantized $\mathrm{SL}(2)$ representation space

Invariance under MI, MII

Invariance under MI is rather easy.

Invariance under MII is proved by using the above lemma.

Quantized $\mathrm{SL}(2)$ representation space

Let A be $\mathrm{BSL}(2)$, then A_b is a one-parameter deformation of the $\mathrm{SL}(2)$ representation space, and we call it the **quantized** $\mathrm{SL}(2)$ **representation space** of $\pi_1(S^3 \setminus \widehat{b})$.

Generators of A_b

Since $\rho(b)(x_1 \otimes x_2) = \rho(b)(x_1 \otimes 1)\,\rho(b)(1 \otimes x_2)$ by the previous lemma, $\underline{\rho(b)(x_1 \otimes x_2) - x_1 \otimes x_2} = \rho(b)(x_1 \otimes 1)\,\rho(b)(1 \otimes x_2) - x_1 \otimes x_2 =$
$\left(\underline{\rho(b)(x_1 \otimes 1) - x_1 \otimes 1}\right)\,\rho(b)(1 \otimes x_2) + (x_1 \otimes 1)\,\left(\underline{\rho(b)(1 \otimes x_2) - 1 \otimes x_2}\right)$.
This implies that the ideal I_b is generated by

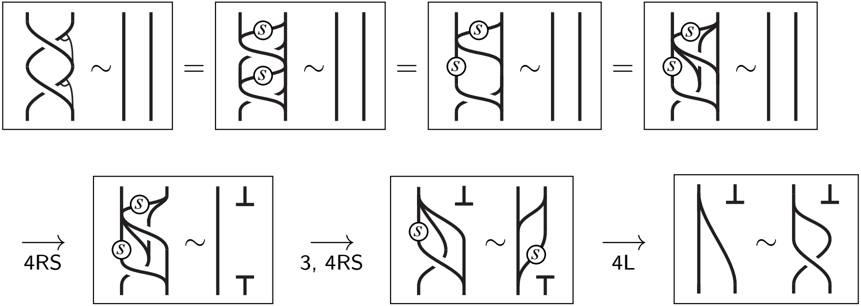
$$\rho(b)(1^{\otimes(i-1)} \otimes x_i \otimes 1^{\otimes(n-i)}) - 1^{\otimes(i-1)} \otimes x_i \otimes 1^{\otimes(n-i)}$$

for $x_i \in A$ and $i = 1, 2, \cdots, n$.

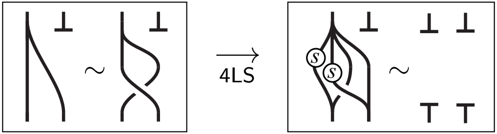
5. Examples

Hopf link

A representation space



Another presentation



Final remarks

- We have to prove that the quantized $\mathrm{SL}(2)$ character variety is generated by Tr_q.
- For surface group, such quantization is constructed by using the skein module by Bonahon-Wong and T. Le.
- By considering the quantum trace of the element representing the longitude, we may get some relation to the quantum version of the A-polynomial, which is introduced for the AJ-conjecture, where this polynomial gives the recurrence relation for the colored Jones invariant. In the A-polynomial, the variables come from the eigenvalues for the meridian and the longitude, and these two variables do not commute. On the other hand, the quantum characters for the meridian and the longitude commute each other. So it may be interesting to establish the notion of "eigenvalue" for $\mathrm{BSL}(2)$ and to find its relation to the quantum character.