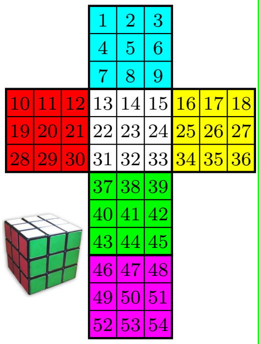
Dror Bar-Natan: Talks:
CUMC-0807:

Non-Commutative Gaussian Elimination and Rubik's Cube

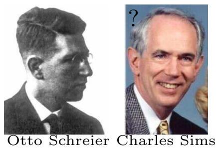
Joint study with
Itai Bar-Natan

The Problem. Let $G=\langle g_1,\dots,g_\alpha\rangle$ be a subgroup of S_n, with $n=O(100)$. Before you die, understand G:

1. Compute $|G|$.
2. Given $\sigma\in S_n$, decide if $\sigma\in G$.
3. Write a $\sigma\in G$ in terms of $g_1,\dots,g_\alpha$.
4. Produce *random* elements of G.



Based on an algorithm by



Otto Schreier Charles Sims

See also *Permutation Group Algorithms* by Ákos Seress.

The Commutative Analog. Let $V=\mathrm{span}(v_1,\dots,v_\alpha)$ be a subspace of $\mathbb{R}^n$. Before you die, understand V.

Solution: Gaussian Elimination. Prepare an empty table,

1	2	3	4	$\cdots$	$n-1$	n

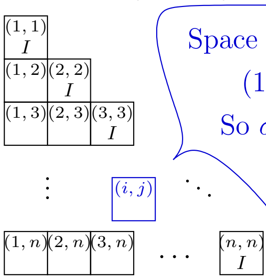
Space for a vector $u_4\in V$, of the form $u_4=(0,0,0,1,*,\dots,*)$; 1 :="the pivot".

Feed $v_1,\dots,v_\alpha$ in order. To feed a non-zero v, find its pivotal position i.

1. If box i is empty, put v there.
2. If box i is occupied, find a combination v' of v and u_i that eliminates the pivot, and feed v'.

Non-Commutative Gaussian Elimination

Prepare a mostly-empty table,



Space for a $\sigma_{i,j}\in S_n$ of the form $(1,2,\dots,i-2,i-1,j,*,*,\dots,*)$
So $\sigma_{i,j}$ fixes $1,\dots,i-1$,
sends "the pivot" i to j and
goes wild afterwards, and
$\sigma_{i,j}^{-1}$ "does sticker j".

Feed $g_1,\dots,g_\alpha$ in order. To feed a non-identity σ, find its pivotal position i and let $j:=\sigma(i)$.

1. If box (i,j) is empty, put σ there.
2. If box (i,j) contains $\sigma_{i,j}$, feed $\sigma':=\sigma_{i,j}^{-1}\sigma$.

The Twist. When done, for every occupied (i,j) and (k,l), feed $\sigma_{i,j}\sigma_{k,l}$. Repeat until the table stops changing.

Claim. The process stops in our lifetimes, after at most $O(n^6)$ operations. Call the resulting table T.

Claim. Anything fed in T is a monotone product in T:

$$f\text{ was fed }\Rightarrow f\in M_1:=\{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n}:\forall i,j_i\geq i\ \&\ \sigma_{i,j_i}\in T\}$$

The Generators

```
In[1]:= gs = {
  purple = P[18,27,36,4,5,6,7,8,9,3,11,12,13,14,15,16,17,
   45,2,20,21,22,23,24,25,26,44,1,29,30,31,32,33,34,35,43,
   37,38,39,40,41,42,10,19,28,52,49,46,53,50,47,54,51,48],
  white = P[1,2,3,4,5,6,16,25,34,10,11,9,15,24,33,39,17,
   18,19,20,8,14,23,32,38,26,27,28,29,7,13,22,31,37,35,36,
   12,21,30,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54],
  green = P[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,
   19,20,21,22,23,24,25,26,27,31,32,33,34,35,36,48,47,46,
   39,42,45,38,41,44,37,40,43,30,29,28,49,50,51,52,53,54],
  blue = P[3,6,9,2,5,8,1,4,7,54,53,52,10,11,12,13,14,15,
   19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,
   37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,18,17,16],
  red = P[13,2,3,22,5,6,31,8,9,12,21,30,37,14,15,16,17,
   18,11,20,29,40,23,24,25,26,27,10,19,28,43,32,33,34,35,
   36,46,38,39,49,41,42,52,44,45,1,47,48,4,50,51,7,53,54],
  yellow = P[1,2,48,4,5,51,7,8,54,10,11,12,13,14,3,18,27,
   36,19,20,21,22,23,6,17,26,35,28,29,30,31,32,9,16,25,34,
   37,38,15,40,41,24,43,44,33,46,47,39,49,50,42,52,53,45]
};        Enter
```

Theorem. $G=M_1$. G^{-1} is more fun!

$$G=M_1:=\{\sigma_{1,j_1}\sigma_{2,j_2}\cdots\sigma_{n,j_n}:\forall i,j_i\geq i\text{ and }\sigma_{i,j_i}\in T\}.$$

Proof. The inclusions $M_1\subset G$ and $\{g_1,\dots,g_\alpha\}\subset M_1$ are obvious. The rest follows from the following

Lemma. M_1 is closed under multiplication.

Proof. By backwards induction. Let

$$M_k:=\{\sigma_{k,j_k}\cdots\sigma_{n,j_n}:\forall i\geq k,j_i\geq i\text{ and }\sigma_{i,j_i}\in T\}.$$

Clearly $M_nM_n\subset M_n$. Now assume that $M_5M_5\subset M_5$ and show that $M_4M_4\subset M_4$. Start with $\sigma_{8,j}M_4\subset M_4$:

$$\sigma_{8,j}(\sigma_{4,j_4}M_5)\overset{1}{=}(\sigma_{8,j}\sigma_{4,j_4})M_5\overset{2}{\subset}M_4M_5$$

$$\overset{3}{=}\sigma_{4,j_4}(M_5M_5)\overset{4}{\subset}\sigma_{4,j_4}M_5\subset M_4$$

(1: associativity, 2: thank the twist, 3: associativity and tracing i_4, 4: induction). Now the general case

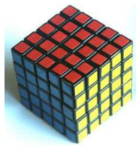
$$(\sigma_{4,j_4'}\sigma_{5,j_5'}\cdots)(\sigma_{4,j_4}\sigma_{5,j_5}\cdots)$$

falls like a chain of dominos.

Problem Solved!

A Demo Program

```
In[2]:= ($RecursionLimit = 2^16;
  n = 54;
  P /: p_P ** P[a___] := p[[{a}]];
  Inv[p_P] := P @@ Ordering[p];
  Feed[P @@ Range[n]] := Null;
  Feed[p_P] := Module[{i, j},
    For[i = 1, p[[i]] == i, ++i];
    j = p[[i]];
    If[Head[s[i, j]] === P,
      Feed[Inv[s[i, j]] ** p],
    (* Else *) s[i, j] = p;
      Do[If[Head[s[k, l]] == P,
        Feed[s[i, j] ** s[k, l]];
        Feed[s[k, l] ** s[i, j]]
      ], {k, n}, {l, n}]
    ]]);      Enter
```



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Homework Problem 1. Can you do cosets?

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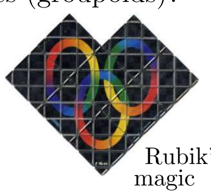
Homework Problem 2. Can you do categories (groupoids)?

7	9	2	5
1	4	8	3
6	10	11	12
13	14	15	



Rubik's magic

The Results

```
In[3]:= (Feed[#]; Product[1 + Length[Select[Range[n], Head[s[i, #]] === P &]], {i, n}]) & /@ gs    Enter
Out[3]= {4, 16, 159993501696000, 21119142223872000, 43252003274489856000, 43252003274489856000}
```

http://www.math.toronto.edu/~drorbn/Talks/CUMC-0807/ and http://www.math.toronto.edu/~drorbn/Misc/SchreierSimsRubik/