Day 1 − u, v, w: topology and philosophy

u, v, and w−Knots: Topology, Combinatorics and Low and High Algebra

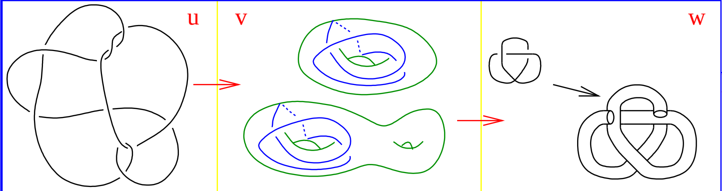
Dror Bar−Natan, Goettingen, April 2010

http://www.math.toronto.edu/~drorbn/Talks/Goettingen−1004/

Plans and Dreams

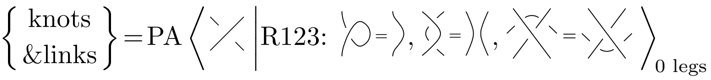
$$\begin{pmatrix}\text{arbitrary algebraic}\\ \text{structure}\end{pmatrix} \xrightarrow[\text{machine}]{\text{projectivization}} \begin{pmatrix}\text{a problem in}\\ \text{graded algebra}\end{pmatrix}$$

- Feed knot-things, get Lie algebra things.
- Feed u-knots, get Drinfel'd associators.
- Feed w-knots, get Kashiware-Vergne-Alekseev-Torossian.
- Dream: Feed v-knots, get Etingof-Kazhdan.
- Dream: Knowing the question whose answer is 42, or E-K, will be useful to algebra and topology.

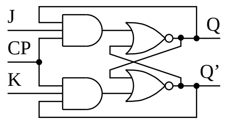


u-Knots

(PA :=Planar Algebra)



Circuit Algebras

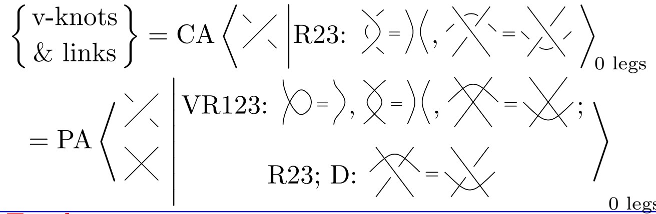


A J-K Flip Flop

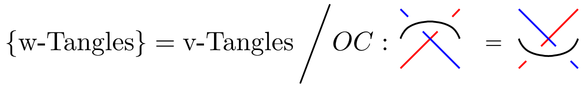
Infineon HYS64T64020HDL-3.7-A 512MB RAM

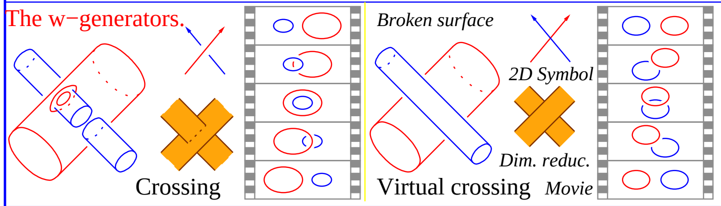
v-Knots

(CA :=Circuit Algebra)

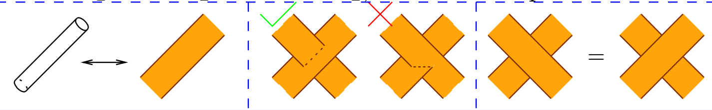


w-Tangles

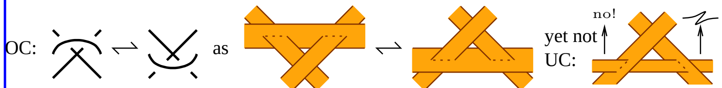




A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

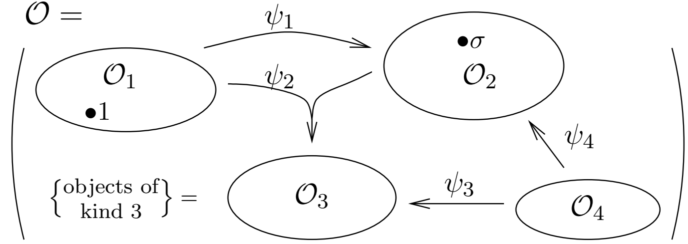


The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC:



Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

"An Algebraic Structure"



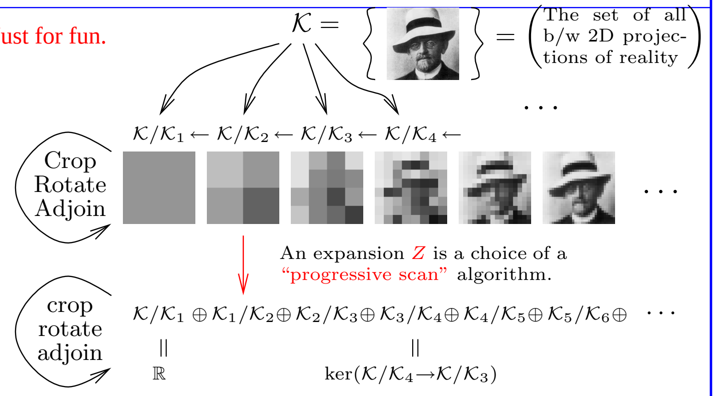
- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccccc} \text{ops} \circlearrowright \mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset & \dots \\ \Downarrow & & & & & & \downarrow Z & & & & \\ \text{ops} \circlearrowright \operatorname{gr}\mathcal{K} & := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 & \oplus & \dots \end{array}$$

An expansion is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Just for fun.



An expansion Z is a choice of a "progressive scan" algorithm.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").

Examples. 1. The projectivization of a group is a graded associative algebra. 2. Quandle: a set Q with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x, \qquad \text{(appetizers)}$$

$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

$\operatorname{proj} Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Our case(s).

$$\begin{tikzcd} \mathcal{K} \arrow[r, "Z\text{: high algebra}", "\substack{\text{solving finitely many}\\ \text{equations in finitely}\\ \text{many unknowns}}"'] & \mathcal{A} := \operatorname{proj}\mathcal{K} \arrow[r, "\substack{\text{given a ``Lie''}\\ \text{algebra } \mathfrak{g}}", "\substack{\text{low algebra: pic-}\\ \text{tures represent}\\ \text{formulas}}"'] & \text{``}\mathcal{U}(\mathfrak{g})\text{''} \end{tikzcd}$$

$\mathcal{K}$ is knot theory or topology; $\operatorname{proj}\mathcal{K} = \bigoplus \mathcal{I}^m/\mathcal{I}^{m+1}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.

"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

Day 2 − u, v, w: combinatorics, low and high algebra

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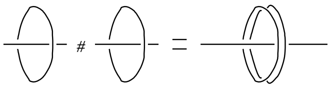
The Scheme. **Topology** → **Combinatorics** → **Lie Theory** via

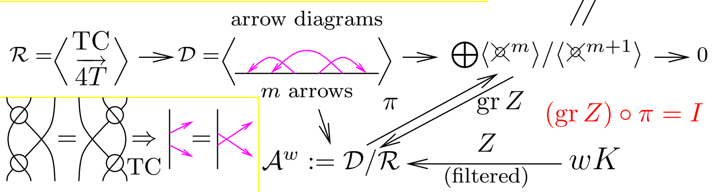
$$\mathcal{K} \xrightarrow[\text{equations, unknowns}]{Z:\ \text{high algebra}} \mathcal{A} = \operatorname{proj}\mathcal{K} = \bigoplus \mathcal{I}^m/\mathcal{I}^{m+1} \xrightarrow[\text{pictures}\to\text{formulas}]{\mathcal{T}_\mathfrak{g}:\ \text{low algebra}} \text{“}\mathcal{U}(\mathfrak{g})\text{”}$$

$1+1=2$, on an abacus, implies Duflo's $\mathcal{U}(\mathfrak{g})^\mathfrak{g} \cong \mathcal{S}(\mathfrak{g})^\mathfrak{g}$ (with T. Le and D. Thurston).

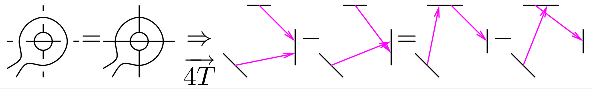


The Finite Type Story. With ⊗ := ⤫ − ⤬ set $\mathcal{V}_m := \{V : wK \to \mathbb{Q} : V(\otimes^{>m}) = 0\}$.

$\bigoplus(\mathcal{V}_m/\mathcal{V}_{m-1})^*$

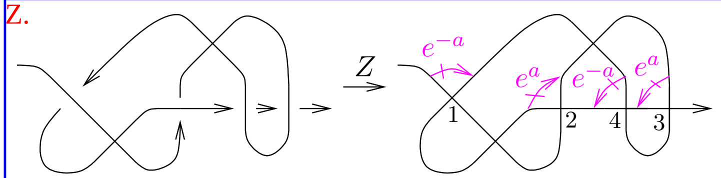


$(\text{gr } Z)\circ\pi = I$

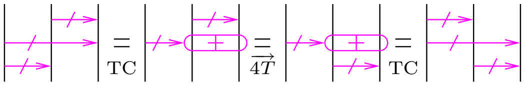


I take pride in this box

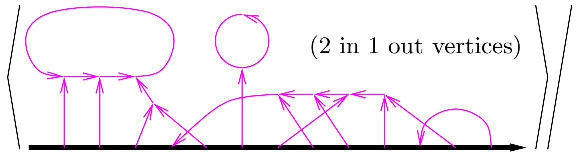
Z.



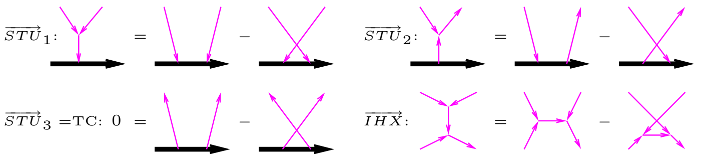
R3.



The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to

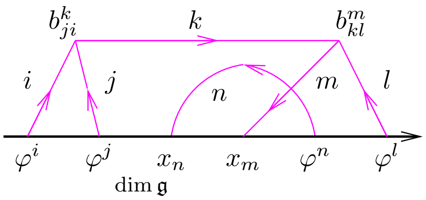


$\overrightarrow{STU}$, $\overrightarrow{AS}$, and $\overrightarrow{IHX}$ relations



Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

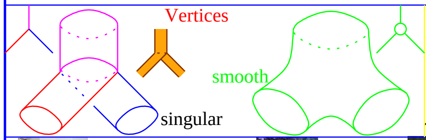
Low Algebra. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via

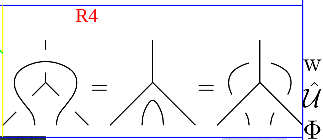


Penrose

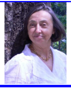
Cvitanovic

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$





Kashiwara

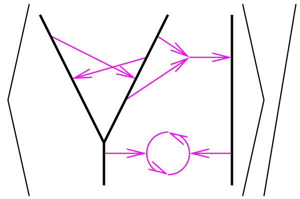


Vergne

Alekseev

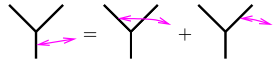
Torossian

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



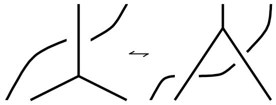
same relations, plus

VI:

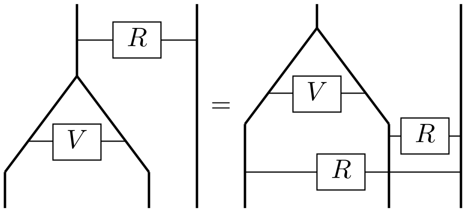


deg$=\frac{1}{2}\#\{\text{vertices}\}=6$

Knot-Theoretic statement (simplified). There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$.



Diagrammatic statement (simplified). Let $R = \exp {\mid\!\!\leftarrow\!\!\mid} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



Algebraic statement (simplified). With $r \in \mathfrak{g}^*\otimes\mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that $V(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}\otimes\hat{\mathcal{U}}(\mathfrak{g})$

Unitary statement (simplified). There exists a unitary tangential differential operator V defined on $\operatorname{Fun}(\mathfrak{g}_x\times\mathfrak{g}_y)$ so that $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

Unitary $\Longleftrightarrow$ Algebraic. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\operatorname{Fun}(\mathfrak{g})$: $\varphi\in\mathfrak{g}^*$ becomes a multiplication operator, and $x\in\mathfrak{g}$ becomes a tangential derivation, in the direction of the action of ad x: $(x\varphi)(y) := \varphi([x,y])$.

Group-Algebra statement (simplified). For every $\phi,\psi\in\operatorname{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$:

$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)e^x e^y.$$

(shhh, this is Duflo)

Unitary $\Longrightarrow$ Group-Algebra. $\iint e^{x+y}\phi(x)\psi(y) = \langle 1, e^{x+y}\phi(x)\psi(y)\rangle = \langle V1, Ve^{x+y}\phi(x)\psi(y)\rangle = \langle 1, e^x e^y V\phi(x)\psi(y)\rangle = \langle 1, e^x e^y\phi(x)\psi(y)\rangle = \iint e^x e^y\phi(x)\psi(y)$.

Convolutions statement (Kashiwara-Vergne, simplified). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, and let $\Phi : \operatorname{Fun}(G)\to\operatorname{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := f(\exp x)$. Then if $f,g\in\operatorname{Fun}(G)$ are Ad-invariant and supported near the identity, then $\Phi(f)\star\Phi(g) = \Phi(f\star g)$.

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G\to A$ is multiplicative then $(\operatorname{Fun}(G),\star)\to(A,\cdot)$ via $L : f\mapsto\sum f(a)\tau(a)$. For Lie $(G,\mathfrak{g})$,

$$\begin{tikzcd}
(\mathfrak{g},+)\ni x \arrow[r, "\tau_0=\exp_{\mathcal{S}}"] \arrow[d, "\exp_G"] \arrow[dr, "\exp_{\mathcal{U}}"] & e^x\in\hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] \\
(G,\cdot)\ni e^x \arrow[r, "\tau_1"] & e^x\in\hat{\mathcal{U}}(\mathfrak{g})
\end{tikzcd}$$

so

$$\begin{tikzcd}
\operatorname{Fun}(\mathfrak{g}) \arrow[r, "L_0"] \arrow[d, "\Phi^{-1}"] & \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] \\
\operatorname{Fun}(G) \arrow[r, "L_1"] & \hat{\mathcal{U}}(\mathfrak{g})
\end{tikzcd}$$

with $L_0\psi = \int\psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int\psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i\in\operatorname{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1)\star\Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1\star\psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$:

(shhh, $L_{0/1}$ are “Laplace transforms”)

$$\star \text{ in } G: \iint \psi_1(x)\psi_2(y)e^x e^y \qquad \star \text{ in } \mathfrak{g}: \iint \psi_1(x)\psi_2(y)e^{x+y}$$

Day 3 − A w−Map for General Orientation

u, v, and w−Knots: Topology, Combinatorics and Low and High Algebra

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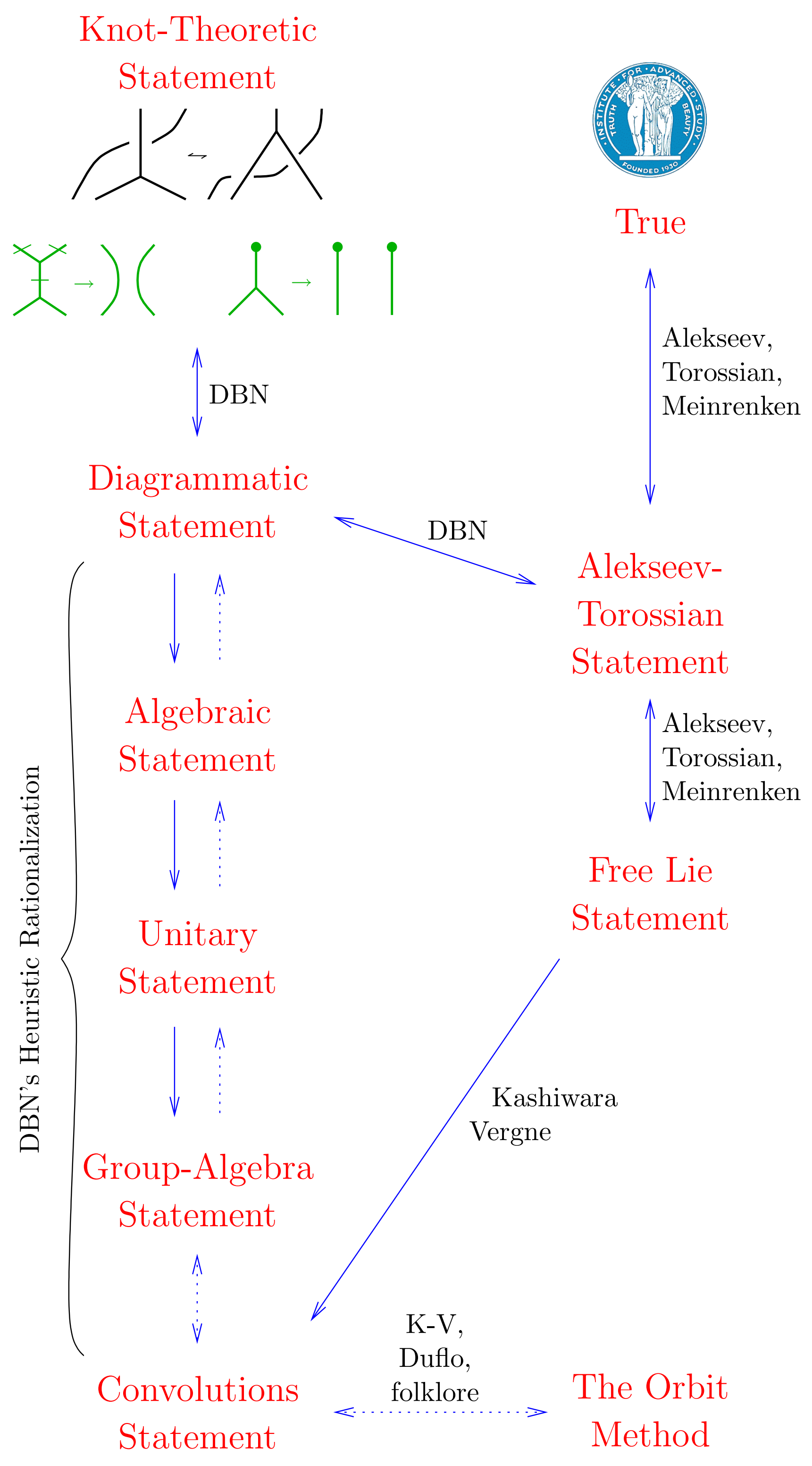


Knot-Theoretic Statement

True

DBN

Alekseev, Torossian, Meinrenken

Diagrammatic Statement

DBN

Alekseev-Torossian Statement

Algebraic Statement

Alekseev, Torossian, Meinrenken

Free Lie Statement

Unitary Statement

Kashiwara Vergne

Group-Algebra Statement

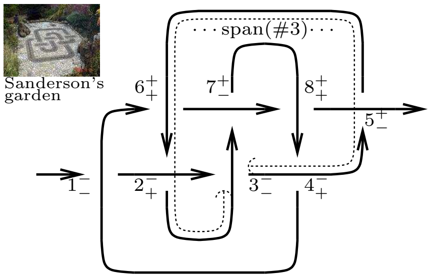
K-V, Duflo, folklore

Convolutions Statement

The Orbit Method

DBN's Heuristic Rationalization

The Alexander Theorem.



$T_{ij} = |\mathrm{low}(\#j) \in \mathrm{span}(\#i)|$,
$s_i = \mathrm{sign}(\#i)$, $d_i = \mathrm{dir}(\#i)$,
$S = \mathrm{diag}(s_i d_i)$,
$A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$$X^{-S} = \mathrm{diag}\left(\tfrac{1}{X}, X, \tfrac{1}{X}, X, X, \tfrac{1}{X}, X, \tfrac{1}{X}\right).$$

Conjecture. For u-knots, A is the Alexander polynomial.

Theorem. With $w : x^k \mapsto w_k =$ (the k-wheel),

$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right).$$

Mod $w_k w_k = w_{k+l}$,

$$Z = N \cdot A^{-1}(e^x).$$

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Talk Video

320 720 download ogg/Goettingen-100427_320.ogg

-60s -10s Comment on 0:31:37 +10s +60s

Videography by Karene Chu

0:01:49 [add] Handout view 2: Plans and Dreams
0:07:28 [add] Handout view 3: u,v,w knots
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0:16:38 [add] Handout view 5: PA to CA
0:23:21 [add] Handout view 6: v-knots
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0:30:59 [add] Handout view 8: The w-Generators
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0:31:09 [add] Handout view 10: The w-Relations
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0:33:59 [add] Handout view 9: Ribbon 2-Knots
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0:34:42 [add] Handout view 10: The w-Relations
0:34:45 [add] Handout view 11: Algebraic Structures
0:38:06 [add] Handout view 12: Homomorphic Expansions

Kashiwara

Vergne

The Orbit Method. By Fourier analysis, the characters of $(\mathrm{Fun}(\mathfrak{g})^G, \star)$ correspond to coadjoint orbits in $\mathfrak{g}^*$. By averaging representation matrices and using Schur's lemma to replace intertwiners by scalars, to every irreducible representation of G we can assign a character of $(\mathrm{Fun}(G)^G, \star)$.

Free Lie statement (Kashiwara-Vergne). There exist convergent Lie series F and G so that with $z = \log e^x e^y$

$$x + y - \log e^y e^x = (1 - e^{-\operatorname{ad} x})F + (e^{\operatorname{ad} y} - 1)G$$

$$\mathrm{tr}(\operatorname{ad} x)\partial_x F + \mathrm{tr}(\operatorname{ad} y)\partial_y G = \frac{1}{2}\,\mathrm{tr}\left(\frac{\operatorname{ad} x}{e^{\operatorname{ad} x} - 1} + \frac{\operatorname{ad} y}{e^{\operatorname{ad} y} - 1} - \frac{\operatorname{ad} z}{e^{\operatorname{ad} z} - 1} - 1\right)$$

Alekseev-Torossian statement. There is an element $F \in \mathrm{TAut}_2$ with

$$F(x + y) = \log e^x e^y$$

and $j(F) \in \operatorname{im}\tilde{\delta} \subset \mathrm{tr}_2$, where for $a \in \mathrm{tr}_1$, $\tilde{\delta}(a) := a(x) + a(y) - a(\log e^x e^y)$.

Alekseev

Torossian