Rational Homotopy and Intrinsic Formality of E_n-operads Part I

Benoit Fresse

Université de Lille 1

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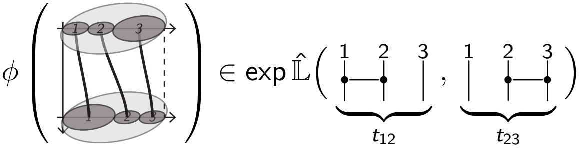


Introduction

- Idea: an E_n-operad is an object E_n, defined by a reference model, the operad of little n-discs (or n-cubes) D_n, used to encode the nth layer of a hierarchy of homotopy commutative structures, from fully homotopy associative but non-commutative ($n = 1$), up to fully homotopy associative and commutative ($n = \infty$).
- Problems:
  - give an intrinsic characterization of the class of E_n-operads (today's talk);
  - understand the spaces of homotopy automorphisms $\mathtt{Aut}^h_{\mathcal{O}p}(\mathsf{E}_n)$, which represent internal symmetries of these homotopy commutative structures (Thomas's talk).

- The operads of little n-discs (and the class of E_n-operads) have been introduced in topology in order to model structures attached to n-fold loop spaces (Boardman-Vogt, May).
- New motivating applications of E_n-operads have arisen in mathematical physics:
  - second generation proofs of Kontsevich's formality theorem (giving the existence of deformation-quantizations) relies on an interpretation of the Drinfeld associator in terms of a formality quasi-isomorphism for chain E_2-operads;
  - the new arguments imply the existence of an action of the Grothendieck-Teichmüller group on moduli spaces of deformation-quantizations.
- New applications of E_n-operads also occur in the study of the spaces of compactly-supported embeddings (modulo immersions):
  - we have

$$\overline{\mathtt{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n) \sim \mathtt{Map}^h_{\mathcal{A}b\mathcal{B}i\mathcal{M}od(E_m)}(E_m, E_n) \sim \Omega^{m+1}\,\mathtt{Map}^h_{\mathcal{O}p}(E_m, E_n)$$

    when $n \geq m+3$ (by theorems of Arone-Turchin and Dwyer-Hess).

- Theorem (rational homotopy theory interpretation of Drinfeld's associators): Any Drinfeld rational associator can be used to construct a formality map:

$$\mathrm{D}_2 \xleftarrow{\sim} \mathrm{R}_2 \underset{\exists}{\overset{\sim^{\mathbb{Q}}}{\dashrightarrow}} \langle \mathtt{H}^*(\mathrm{D}_2, \mathbb{Q}) \rangle$$

  where:
  - R_2 denotes a (cofibrant) resolution of the little 2-discs operad D_2,
  - $\langle \mathtt{H}^*(\mathrm{D}_2, \mathbb{Q}) \rangle$ is an operad in spaces deduced from the rational cohomology of the little 2-discs operad,
  - $\sim^{\mathbb{Q}}$ denotes a rational homotopy equivalence of operads.
- Theorem (rational homotopy theory interpretation of Kontsevich's formality theorem): We have an analogous formality map

$$\mathrm{D}_n \xleftarrow{\sim} \mathrm{R}_n \underset{\exists}{\overset{\sim^{\mathbb{R}}}{\dashrightarrow}} \langle \mathtt{H}^*(\mathrm{D}_n, \mathbb{R}) \rangle \ ,$$

  defined over $\mathbb{R}$, for every dimension $n \geq 2$.

- Intrinsic Formality Theorem (BF, Willwacher): Let P be an operad in topological space, with $P(0) = P(1) = pt$. If we have:
  - a rational cohomology isomorphism $\mathtt{H}^*(P, \mathbb{Q}) \simeq \mathtt{H}^*(D_n, \mathbb{Q})$, for some $n \geq 3$,
  - an involutive isomorphism $J : P \xrightarrow{\simeq} P$ which mirrors the action of a hyperplane reflection on D_n in the case $4 \mid n$,

  then we can produce a map:

$$P \xleftarrow{\sim} R \underset{\exists}{\overset{\sim^{\mathbb{Q}}}{\dashrightarrow}} \langle \mathtt{H}^*(D_n, \mathbb{Q}) \rangle \ ,$$

  where:
  - R denotes a cofibrant resolution of P,
  - and $\sim^{\mathbb{Q}}$ denotes a rational homotopy equivalence of operads (as in the $n = 2$ case).
- Corollary: We have a rational counterpart of Kontsevich's formality map.

Plan

- Part I:
  - §0. The notion of an operad
  - §1. The little n-discs operads
  - §2. The (co)homology of the little discs operads
  - §3. The rational homotopy theory of operads
  - §4. The statement of the intrinsic formality theorem
- Part II:
  - §1. The Drinfeld-Kohno Lie algebra operad
  - §2. The realization of the $(n-1)$-Poisson cooperad
  - §3. The obstruction theory proof of the intrinsic formality theorem
  - Appendix: The fundamental groupoid of the little 2-discs operad
  - Appendix: The rational homotopy theory interpretation of Drinfeld's associators

§0. The notion of an operad

Intuitively, the notion of an operad formalizes the abstract structure defined by collections of operations $\mathsf{P}(r) = \{p(x_1, \dots, x_r)\}$.

Definition: An operad P in a symmetric monoidal category $\mathcal{M}$ (e.g. spaces, modules, ...) is a collection of objects $\mathsf{P}(r) \in \mathcal{M}$, $r \in \mathbb{N}$, equipped with:

- an action of the symmetric group Σ_r on $\mathsf{P}(r)$, for each $r \in \mathbb{N}$;
- composition operations

$$\circ_i : \mathsf{P}(k) \otimes \mathsf{P}(l) \to \mathsf{P}(k + l - 1),$$

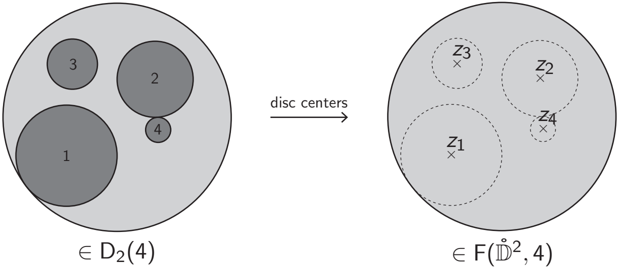
defined for each $k, l \in \mathbb{N}$, $i = 1, \dots, k$, and which satisfy natural equivariance, unit and associativity relations.

Example: Let $M \in \mathcal{M}$. The collection of hom-objects $\mathrm{End}_M(r) = \mathrm{Hom}(M^{\otimes r}, M)$ forms an operad associated to M, the endomorphism operad of M. In the point-set (module) context:

- The action of a permutation $\sigma \in \Sigma_r$ on an element $f \in \mathrm{End}_M(r)$ is defined by:
$$(\sigma f)(x_1, \dots, x_r) = f(x_{\sigma(1)}, \dots, x_{\sigma(r)}).$$
- The operadic composite of elements $f \in \mathrm{End}_M(m)$ and $g \in \mathrm{End}_M(n)$ is defined by:
$$(f \circ_i g)(x_1, \dots, x_{m+n-1}) \\ = f(x_1, \dots, g(x_i, \dots, x_{i+n-1}), \dots, x_{m+n-1}).$$

The structure of an operad is modeled on this fundamental example. Many usual algebra categories can be defined in terms of operad actions.

§1. The little discs operads

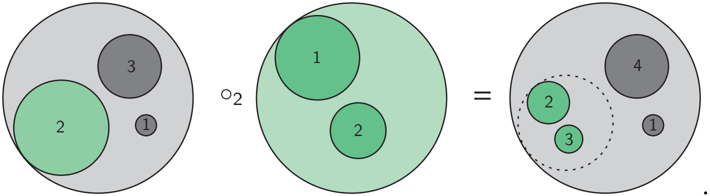


- The little n-discs spaces $D_n(r)$ consist of collections of r little n-discs with disjoint interiors inside a fixed unit n-disc $\mathbb{D}^n$ (see Figure).
- The configuration spaces $F(\mathring{\mathbb{D}}^n, r)$ consist of collections of r distinct points in the open disc $\mathring{\mathbb{D}}^n$ (see Figure).
- There is an obvious homotopy equivalence $D_n(r) \xrightarrow{\sim} F(\mathring{\mathbb{D}}^n, r)$.

- The symmetric group Σ_r acts on $\mathrm{D}_n(r)$ by permutation of the little disc indices (and on the configuration space similarly).
- The little n-discs spaces (unlike the configuration spaces) inherit operadic composition operations

$$\circ_i : \mathrm{D}_n(k) \times \mathrm{D}_n(l) \to \mathrm{D}_n(k+l-1)$$

given by the following substitution process



This gives the structure of the little n-discs operad D_n*. The elements of this operad* $\underline{c} \in \mathrm{D}_n(r)$ *define operations acting on n-loop spaces* $\Omega^n X = \mathtt{Map}_*(\mathbb{S}^n, X)$*.*

§2. The (co)homology of the little discs operads

- Observations: The homology modules of any operad in $\mathcal{T}op$

$$\mathrm{H}_*(\mathsf{P}(r)) = \mathrm{H}_*(\mathsf{P}(r), \mathbb{Q}), \quad r \in \mathbb{N},$$

form an operad in $gr\,\mathcal{M}od$ with the (action of the symmetric groups and the) composition operations

$$\mathrm{H}_*(\mathsf{P}(k)) \otimes \mathrm{H}_*(\mathsf{P}(l)) \to \mathrm{H}_*(\mathsf{P}(k) \times \mathsf{P}(l)) \xrightarrow{\circ_i} \mathrm{H}_*(\mathsf{P}(k+l-1))$$

yielded by the structure operations of our operad P.

- If we have $\dim \mathrm{H}_*(\mathsf{P}(r)) < \infty$ for all r, then the cohomology algebras

$$\mathrm{H}^*(\mathsf{P}(r)) = \mathrm{H}^*(\mathsf{P}(r), \mathbb{Q}), \quad r \in \mathbb{N},$$

dually form a cooperad in graded commutative algebras (a graded Hopf cooperad), with composition coproducts:

$$\mathrm{H}^*(\mathsf{P}(k+l-1)) \xrightarrow{\circ_i^*} \mathrm{H}^*(\mathsf{P}(k) \times \mathsf{P}(l)) \xleftarrow{\simeq} \mathrm{H}^*(\mathsf{P}(k)) \otimes \mathrm{H}^*(\mathsf{P}(l))$$

dual to the homological $\circ_i$.

- Theorem (F. Cohen): For any $n \geq 2$, we have an identity:

$$\mathrm{H}_*(\mathsf{D}_n) = \mathsf{Pois}_{n-1},$$

where Pois_{n-1} is the operad of $(n-1)$-Poisson algebras, for a product operation and a Poisson bracket operation given by:

$$x_1 x_2 = [pt] \in \mathrm{H}_0(\mathsf{D}_n(2)), \quad [x_1, x_2] = [\mathbb{S}^{n-1}] \in \mathrm{H}_{n-1}(\mathsf{D}_n(2)),$$

and that satisfy the graded symmetry relations:

$$x_1 x_2 = x_2 x_1, \quad [x_1, x_2] = (-1)^n [x_2, x_1],$$

together with the usual associativity, Jacobi, and Poisson relations

$$(x_1 x_2) x_3 = x_1 (x_2 x_3),$$
$$[[x_1, x_2], x_3] = [[x_1, x_3], x_2] + [x_1, [x_2, x_3]],$$
$$[x_1 x_2, x_3] = [x_1, x_3] x_3 + x_1 [x_2, x_3]$$

within the graded modules $\mathrm{H}_*(\mathsf{D}_n(r))$.

- Theorem (Arnold): Let $n \geq 2$. For each $r \in \mathbb{N}$, the graded commutative algebra $\mathtt{H}^*(\mathsf{D}_n(r)) \simeq \mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r))$ has a presentation of the form:

$$\mathtt{H}^*(\mathsf{F}(\mathring{\mathbb{D}}^n, r)) = \frac{\mathbb{S}(\omega_{ij}, 1 \leq i \neq j \leq r)}{(\omega_{ij}^2, \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij})}$$

where a generator ω_{ij} of degree $\deg(\omega_{ij}) = n - 1$ and such that $\omega_{ij} = (-1)^n \omega_{ji}$ is assigned to each pair $i \neq j$.

- Observation: We have

$$\langle \omega_{ij}, \underbrace{\pi(x_1, \dots, x_r)}_{\in \mathsf{Pois}_{n-1}(r)} \rangle = \begin{cases} 1, & \text{if } \pi(x_1, \dots, x_r) = x_1 \dots [x_i, x_j] \cdots \widehat{x_j} \cdots x_r, \\ 0, & \text{otherwise.} \end{cases}$$

§3. The rational homotopy of operads

Quick recollections on Sullivan's models:

- The model is given by Sullivan's functor of PL differential forms $\Omega^* : s\mathcal{S}et^{op} \to dg\,\mathcal{C}om$. For a simplex $\Delta^n = \{0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$, we have:

$$\Omega^*(\Delta^n) = \mathbb{Q}[x_1, \ldots, x_n, dx_1, \ldots, dx_n].$$

- This functor has a left adjoint $\mathtt{G}_\bullet : dg\,\mathcal{C}om \to s\mathcal{S}et^{op}$ such that $\mathtt{G}_n(A) = \mathtt{Mor}_{dg\,\mathcal{C}om}(A, \Omega^*(\Delta^n))$, for each $n \in \mathbb{N}$. Let:

$$\langle A \rangle := \text{derived functor of } \mathtt{G}(A) = \mathtt{Mor}_{dg\,\mathcal{C}om}(R_A, \Omega^*(\Delta^\bullet)),$$

where $R_A \xrightarrow{\sim} A$ is any cofibrant resolution of A in $dg\,\mathcal{C}om$.

- If X satisfies reasonable finiteness and nilpotence assumptions, then

$$X^\wedge_{\mathbb{Q}} := \langle \Omega^*(X) \rangle$$

defines a rationalization of the space X.

- Idea: Take the category of cooperads in commutative dg-algebras (the category of Hopf dg-cooperads) as a model for the category of operads in simplicial sets (and in topological spaces).
- Problem: If P is an operad in $s\mathcal{S}et$, then we only have a zig-zag of maps:

$$\Omega^*(\mathsf{P}(k+l-1)) \xrightarrow{\circ_i^*} \Omega^*(\mathsf{P}(k)\times\mathsf{P}(l)) \xleftarrow{\sim} \Omega^*(\mathsf{P}(k))\otimes\Omega^*(\mathsf{P}(l))$$

  and no strictly defined cooperad coproducts on P.
- Proposition: The functor $\mathtt{G}_\bullet(A) = \mathtt{Mor}_{dg\,\mathcal{C}om}(A, \Omega^*(\Delta^\bullet))$ is symmetric monoidal (in a strong sense).
- Corollary: The simplicial sets $\mathtt{G}_\bullet(\mathsf{K}(r))$ associated to a Hopf dg-cooperad K do inherit composition products

$$\mathtt{G}_\bullet(\mathsf{K}(k))\times\mathtt{G}_\bullet(\mathsf{K}(l)) \xleftarrow{\simeq} \mathtt{G}_\bullet(\mathsf{K}(k)\otimes\mathsf{K}(l)) \xrightarrow{\circ_i^{**}} \mathtt{G}_\bullet(\mathsf{K}(k+l-1))$$

  and form an operad in simplicial sets.

- Hence, the functor $\mathtt{G}_\bullet(A) = \mathtt{Mor}_{dg\,\mathcal{C}om}(A, \Omega^*(\Delta^\bullet))$ induces a functor on Hopf dg-cooperads $\mathtt{G}_\bullet : dg\,\mathcal{H}opf\,\mathcal{O}p^c \to s\mathcal{S}et\mathcal{O}p^{op}$.
- Idea: Take a right adjoint of this functor:

$$\mathtt{G}_\bullet : dg\,\mathcal{H}opf\,\mathcal{O}p^c \rightleftarrows s\mathcal{S}et\mathcal{O}p^{op} : \exists\,\Omega^*_\sharp$$

  to get an operadic upgrade of the Sullivan functor.
- Theorem (BF):
  - The functors $\mathtt{G}_\bullet : dg\,\mathcal{H}opf\,\mathcal{O}p^c \rightleftarrows s\mathcal{S}et\mathcal{O}p^{op} : \Omega^*_\sharp$ form a Quillen pair.
  - Let R be a cofibrant operad such that $\dim \mathtt{H}^*(\mathsf{R}(r)) < \infty$ for each r. Then we have a quasi-isomorphism of dg-algebras

$$\Omega^*_\sharp(\mathsf{R})(r) \xrightarrow{\sim} \Omega^*(\mathsf{R}(r))$$

    in each arity r.
  - The canonical map $\eta : \mathsf{R} \to \mathsf{R}^\wedge_{\mathbb{Q}}$, where:

$$\mathsf{R}^\wedge_{\mathbb{Q}} = \langle \Omega^*_\sharp(\mathsf{R}) \rangle,$$

    is equivalent to the Sullivan rationalization map of the space $\mathsf{R}(r)$ in each arity r.

§4. The statement of the intrinsic formality

- Reminder: $\mathrm{H}_*(\mathrm{D}_n) = \mathsf{Pois}_{n-1} \Rightarrow \mathrm{H}^*(\mathrm{D}_n) = \mathsf{Pois}^c_{n-1}$ (the dual cooperad of Pois_{n-1} in graded modules).
- Topological Intrinsic Formality Theorem: Let R be a cofibrant operad in simplicial sets, with $\mathsf{R}(0) = \mathsf{R}(1) = pt$. If we have:
  - a rational cohomology isomorphism $\mathrm{H}^*(\mathsf{R}, \mathbb{Q}) \simeq \mathsf{Pois}^c_{n-1}$, for some $n \geq 3$,
  - an involutive isomorphism $J : \mathsf{R} \xrightarrow{\simeq} \mathsf{R}$ which mirrors the action of a hyperplane reflection on D_n in the case $4 \mid n$,

  then we can produce a map:

$$\mathsf{R}^{\wedge}_{\mathbb{Q}} \overset{\sim}{\underset{\exists}{\dashrightarrow}} \langle \mathsf{Pois}^c_{n-1} \rangle \ .$$

- First purpose: Give an explicit description of $\langle \mathsf{Pois}^c_{n-1} \rangle$.

To be continued . . .