Handouts for the Montpellier Meeting

I understand Drinfel'd and Alekseev-Torossian, I don't understand Etingof-Kazhdan yet, and I'm clueless about Kontsevich

Dror Bar-Natan, June 2010

Abstract. The title, minus the last 5 words, completely describes what I want to share with you while we are in Montpellier. I'll tell you that Drinfel'd associators are the solutions of the homomorphic expansion problem for u-knots (really, knotted trivalent graphs), that Kashiwara-Vergne-Alekseev-Torossian series are the same for w-knots, that the two are related because u- and w- knots are related, and that there are strong indications that "v-knots" are likewise related to the Etingof-Kazhdan theory of quantization of Lie bialgebras, though some gaps remain and significant ideas are probably still missing. Kontsevich's quantization of Poisson structures seems like it could be similar, but I am completely clueless as for how to put it under the same roof.
I want as much as your air-time and attention as I can get! So I'll talk for as long as you schedule me or until you stop me, following parts of the following 8 handouts in an order that will be negotiated in real time.

Contents

See also `http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/`

1. proj $\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n)$

— All Signs Are Wrong! —

I understand Drinfel’d and Alekseev−Torossian, I don’t understand Etingof−Kazhdan yet, and I’m clueless about Kontsevich

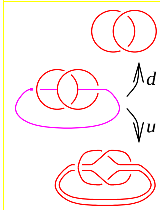
Dror Bar−Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier−1006/

Cans and Can’t Yets.

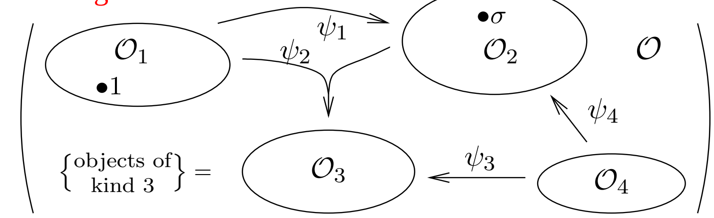
(arbitrary algebraic structure) $\xrightarrow{\text{projectivization machine}}$ (a problem in graded algebra)

www.katlas.org

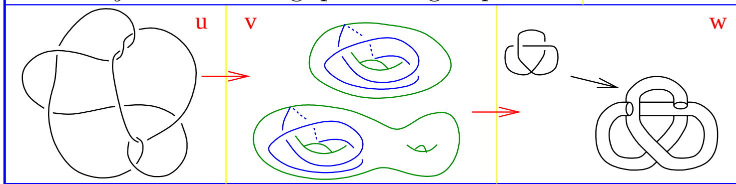
- Feed knot-things, get Lie algebra things.
- (u-knots)→(Drinfel’d associators).
- (w-knots)→(K-V-A-E-T).
- Dream: (v-knots)→(Etingof-Kazhdan).
- Clueless: (???)→(Kontsevich)?
- Goals: add to the Knot Atlas, produce a working AKT and touch ribbon 1-knots, rip benefits from *truly* understanding quantum groups.



"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

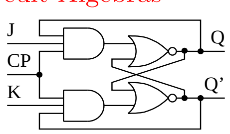


u-Knots

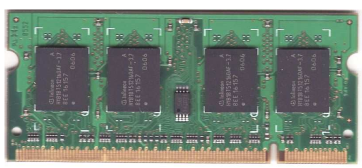
(PA :=Planar Algebra)

$\left\{\begin{matrix}\text{knots}\\ \text{\&links}\end{matrix}\right\} = \mathrm{PA}\langle \text{crossing} \mid \text{R123} \rangle_{0\text{ legs}}$

Circuit Algebras



A J-K Flip Flop



Infineon HYS64T64020HDL-3.7-A 512MB RAM

v-Tangles and w-Tangles

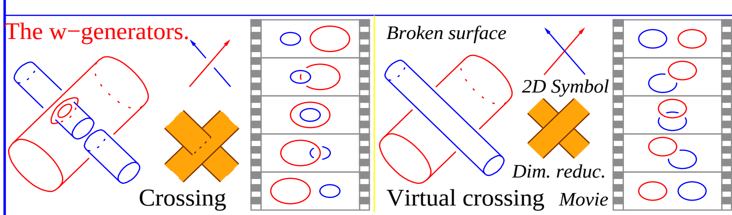
(CA :=Circuit Algebra)

$\left\{\begin{matrix}\text{v-knots}\\ \text{\& links}\end{matrix}\right\} = \mathrm{CA}\langle \text{crossing} \mid \text{R23} \rangle$

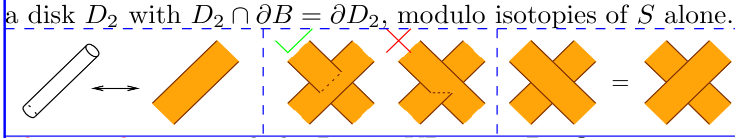
$= \mathrm{PA}\langle \text{crossings, virtual crossing} \mid \text{VR123: , R23; D} \rangle$

{w-Tangles} = v-Tangles / OC

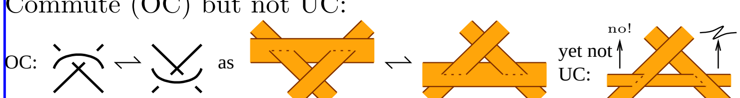
The w−generators.

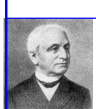


A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only “ribbon singularities”; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC:





"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

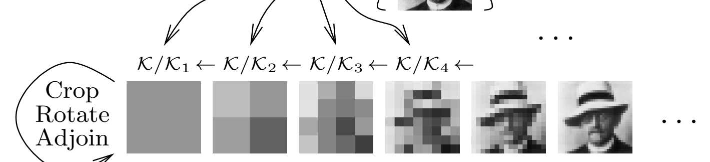
$$\begin{array}{ccccccccccc} \mathrm{ops} \circlearrowleft \mathcal{K} & = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset & \dots \\ \Downarrow & & & & & & \downarrow Z & & & & \\ \mathrm{ops} \circlearrowleft \mathrm{gr}\,\mathcal{K} & := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 & \oplus & \dots \end{array}$$

An **expansion** is a filtered $Z : \mathcal{K} \to \mathrm{gr}\,\mathcal{K}$ that “covers” the identity on $\mathrm{gr}\,\mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant “extra” operations.

Reality. $\mathrm{gr}\,\mathcal{K}$ is often too hard. An $\mathcal{A}$-expansion is a graded “guess” $\mathcal{A}$ with a surjection $\tau : \mathcal{A} \to \mathrm{gr}\,\mathcal{K}$ and a filtered $Z : \mathcal{K} \to \mathcal{A}$ for which $(\mathrm{gr}\,Z) \circ \tau = I_{\mathcal{A}}$. An $\mathcal{A}$-expansion confirms $\mathcal{A}$ and yields an ordinary expansion. Same for “homomorphic”.

Just for fun.

$\mathcal{K} = \{$  $\} = \left(\begin{matrix}\text{The set of all b/w 2D projections of reality}\end{matrix}\right)$



An expansion Z is a choice of a “progressive scan” algorithm.

crop rotate adjoin $\quad \mathcal{K}/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \mathcal{K}_4/\mathcal{K}_5 \oplus \mathcal{K}_5/\mathcal{K}_6 \oplus \cdots$

$\mathcal{K}/\mathcal{K}_1 = \mathbb{R}$, $\quad \mathcal{K}_3/\mathcal{K}_4 = \ker(\mathcal{K}/\mathcal{K}_4 \to \mathcal{K}/\mathcal{K}_3)$

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1 = \mathcal{I}$ be the ideal generated by differences (the “augmentation ideal”), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available “products”). In this case, set $\mathrm{proj}\,\mathcal{K} := \mathrm{gr}\,\mathcal{K}$.

Examples. 1. The projectivization of a group is a graded associative algebra.

2. Pure braids — PB_n is generated by x_{ij}, “strand i goes around strand j once”, modulo “Reidemeister moves”. $A_n := \mathrm{gr}\,PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel’d theory of associators.

3. Quandle: a set Q with an op $\wedge$ s.t.

$$1 \wedge x = 1, \quad x \wedge 1 = x, \qquad \text{(appetizers)}$$
$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \qquad \text{(main)}$$

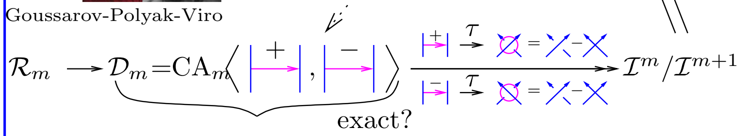
$\mathrm{proj}\,Q$ is a graded Leibniz algebra: Roughly, set $\bar{v} := (v - 1)$ (these generate I!), feed $1 + \bar{x}$, $1 + \bar{y}$, $1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

Kashiwara, Vergne, Alekseev, Enriquez, Torossian.

1. $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong_j \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n)$, continued.

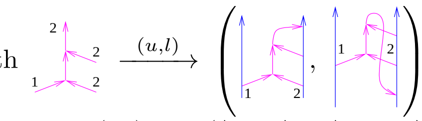
Goussarov-Polyak-Viro



Wheels and Trees. With $\mathcal{P}$ for $\mathcal{P}$rimitives,

$$0 \longrightarrow \langle\text{wheels}\rangle \stackrel{\iota}{\longrightarrow} \mathcal{PA}^w(\uparrow_n) \underset{l}{\overset{u}{\underset{\pi}{\rightleftarrows}}} \langle\text{trees}\rangle \longrightarrow 0,$$

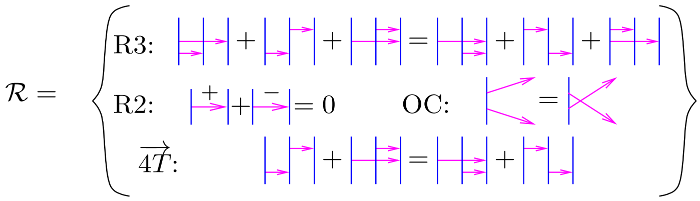
with



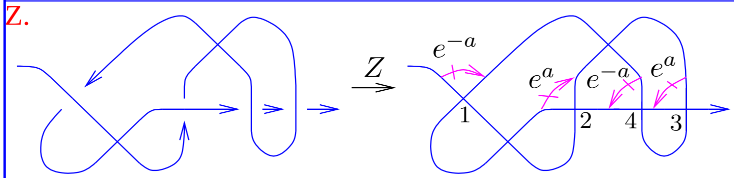
So $\operatorname{proj}\mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle\text{trees}\rangle \ltimes \langle\text{wheels}\rangle)$.

trees atop a wheel, and a little prince.

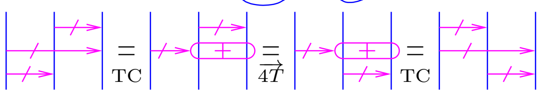
Imperfect Thumb-Rule. Take R3 (say), substitute $\times \to \times +$ (diagram), keep the lowest degree terms that don't immediately die:



Z.

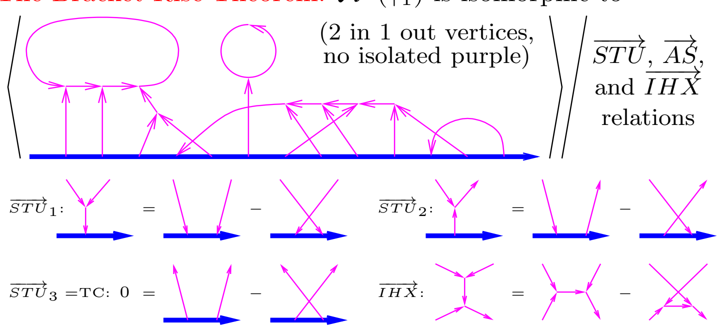


R3.

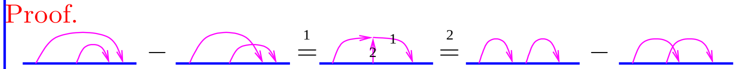


Thm. $Z \succ A$, maybe $\succ\!\succ$.

The Bracket-Rise Theorem. $\mathcal{A}^w(\uparrow_1)$ is isomorphic to

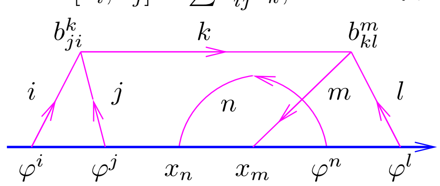


Proof.



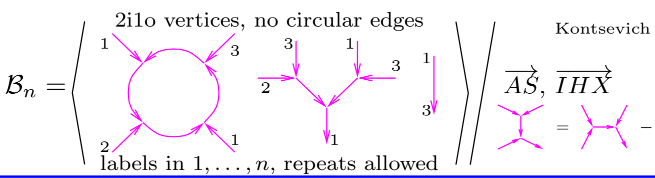
Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

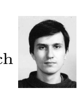
To Lie Algebras. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g})$$

Theorem (PBW, "$\mathcal{U}(I\mathfrak{g})^{\otimes n} \cong \mathcal{S}(I\mathfrak{g})^{\otimes n}$"). As vector spaces, $\mathcal{A}^w(\uparrow_n) \cong \mathcal{B}_n$, where





Kontsevich

Some A-T Notions. $\mathfrak{a}_n$ is the vector space with basis $x_1, \ldots, x_n$, $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\mathrm{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algera "of words", $\mathrm{tr} : \mathrm{Ass}_n^+ \to \mathfrak{tr}_n = \mathrm{Ass}_n^+ / (x_{i_1}x_{i_2}\cdots x_{i_m} = x_{i_2}\cdots x_{i_m}x_{i_1})$ is the "trace" into "cyclic words", $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i\ \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

are "tangential derivations", so $D \leftrightarrow (a_1, \ldots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\operatorname{div} : \mathfrak{tder}_n \to \mathfrak{tr}_n$ is $(a_1, \ldots, a_n) \mapsto \sum_k \mathrm{tr}(x_k(\partial_k a_k))$, where for $a \in \mathrm{Ass}_n^+$, $\partial_k a \in \mathrm{Ass}_n$ is determined by $a = \sum_k (\partial_k a) x_k$, and $j : \mathrm{TAut}_n = \exp(\mathfrak{tder}_n) \to \mathfrak{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \operatorname{div} D$.

Theorem. Everything matches. $\langle\text{trees}\rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle\text{wheels}\rangle$ is $\mathfrak{tr}_n$ as $\langle\text{trees}\rangle$ / $\mathfrak{tder}_n$-modules, $\operatorname{div} D = \iota^{-1}(u - l)(D)$, and $e^{uD}e^{-lD} = e^{jD}$.

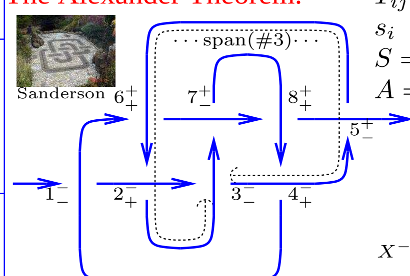
Differential Operators. Interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\operatorname{ad} x$: $(x\varphi)(y) := \varphi([x, y])$.

Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\operatorname{div} D$ is $D - D^*$ and $jD = \log(e^D(e^D)^*) = \int_0^1 dt e^{tD} \operatorname{div} D$.

Special Derivations. Let $\mathfrak{sder}_n = \{D \in \mathfrak{tder}_n : D(\sum x_i) = 0\}$.
Theorem. $\mathfrak{sder}_n = \pi\alpha(\operatorname{proj}\text{u-tangles})$, where α is the obvious map $\operatorname{proj}\text{u-tangles} \to \operatorname{proj}\text{w-tangles}$.
Proof. After decoding, this becomes Lemma 6.1 of Drinfel'd's amazing $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ paper.

The Alexander Theorem.



$T_{ij} = |\mathrm{low}(\#j) \in \mathrm{span}(\#i)|$,
$s_i = \mathrm{sign}(\#i)$, $d_i = \mathrm{dir}(\#i)$,
$S = \mathrm{diag}(s_i d_i)$,
$A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$X^{-S} = \mathrm{diag}(\frac{1}{X}, X, \frac{1}{X}, X, X, \frac{1}{X}, X, \frac{1}{X})$.

Conjecture. For u-knots, A is the Alexander polynomial.
Theorem. With $w : x^k \mapsto w_k = $ (the k-wheel),

$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right)$$

mod $w_k w_l = w_{k+l}$, $Z = N \cdot A^{-1}(e^x)$

This is the ultimate Alexander invariant! computable in polynomial time, local, composes well, behaves under cabling. Seems to significantly generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers. But it's ugly, and much work remains.

2. w−Knots, Alekseev−Torossian, and baby Etingof−Kazhdan

I understand Drinfel'd and Alekseev−Torossian, I don't understand Etingof−Kazhdan yet, and I'm clueless about Kontsevich

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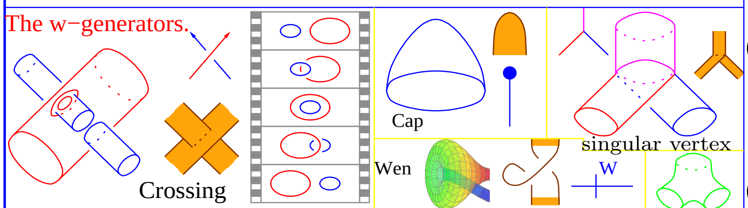
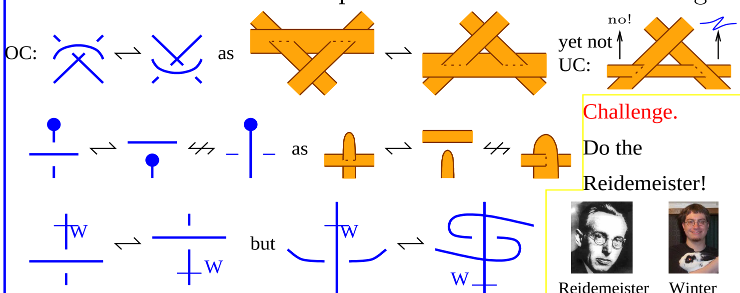
Trivalent w-Tangles.

$$\text{wTT} = \text{CA}\left\langle \text{w-generators} \;\middle|\; \text{w-relations} \;\middle|\; \text{unary w-operations} \right\rangle$$

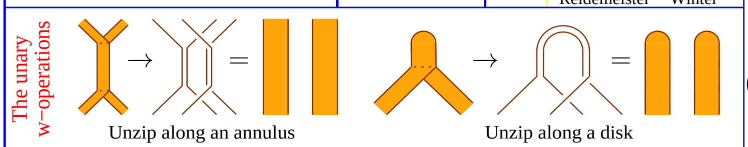
The w−generators.



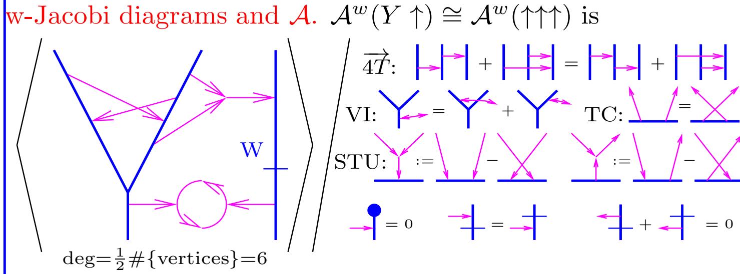
The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:



The unary w−operations



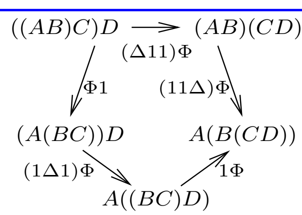
w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



An Associator:

$$(AB)C \xrightarrow{\Phi\in\mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

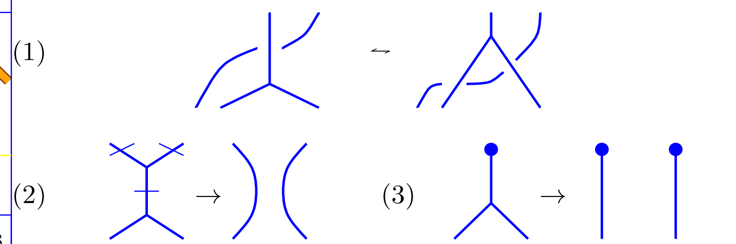
satisfying the “pentagon”,



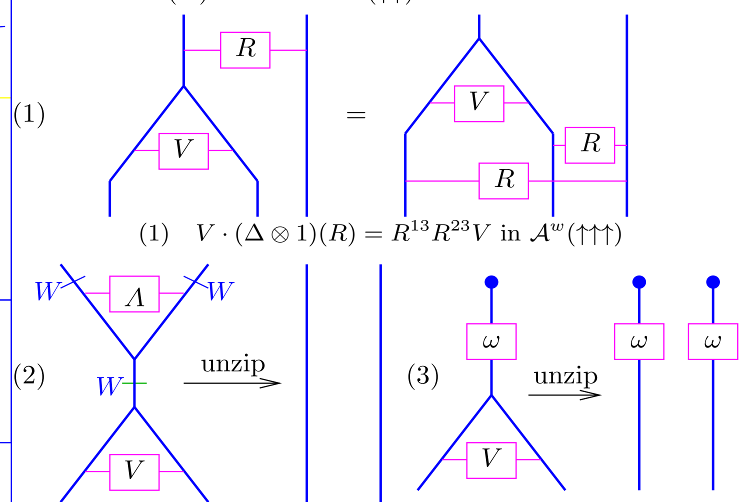
$$\Phi 1\cdot(1\Delta 1)\Phi\cdot 1\Phi = (\Delta 11)\Phi\cdot(11\Delta)\Phi$$

The hexagon? Never heard of it.

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



(1) $V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$

(2) $VV^* = I$ in $\mathcal{A}^w(\uparrow\uparrow)$

(3) $V\cdot\Delta(\omega) = \omega\otimes\omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

Alekseev-Torossian statement. There are elements $F \in \text{TAut}_2$ and $a \in \mathfrak{tr}_1$ such that

$$F(x+y) = \log e^x e^y \quad\text{and}\quad jF = a(x) + a(y) - a(\log e^x e^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

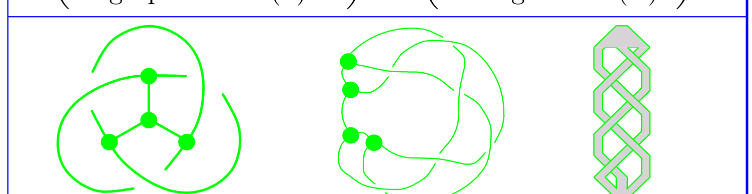
Proof. Write $V = e^c e^{uD}$ with $c \in \mathfrak{tr}_2$, $D \in \mathfrak{tder}_2$, and $\omega = e^b$ with $b \in \mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x+y)e^{-uD} = \log e^x e^y$,
(2) $\Leftrightarrow I = e^c e^{uD}(e^{uD})^* e^c = e^{2c}e^{jD}$, and
(3) $\Leftrightarrow e^c e^{uD} e^{b(x+y)} = e^{b(x)+b(y)} \Leftrightarrow e^c e^{b(\log e^x e^y)} = e^{b(x)+b(y)}$
$\Leftrightarrow c = b(x) + b(y) - b(\log e^x e^y)$.

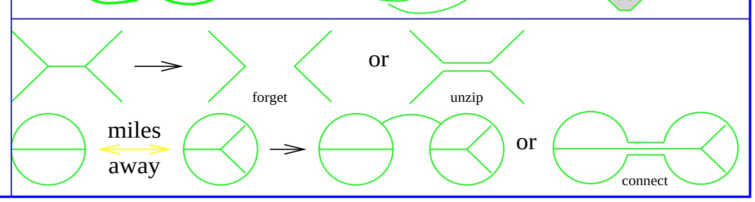
The Alekseev-Torossian Correspondence.

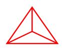
$$\{\text{Drinfel'd Associators}\} \leftrightarrows \{\text{Solutions of KV}\}.$$

We need an even bigger algebraic structure!

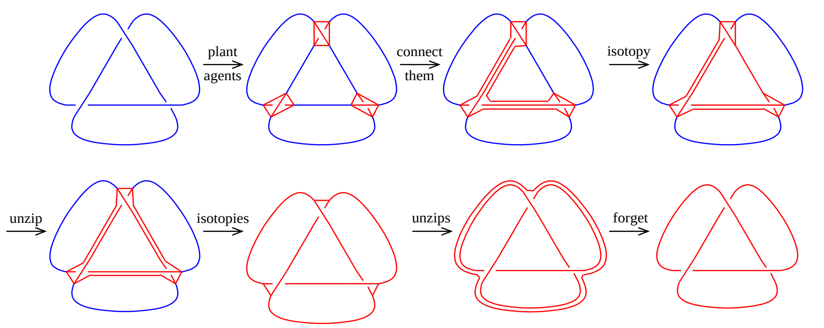
$$\begin{pmatrix}\text{green knotted trivalent}\\ \text{graphs in } \mathbb{R}^3\ (u)\end{pmatrix} \xrightarrow{\alpha_e} \begin{pmatrix}\text{blue tubes and red}\\ \text{strings in } \mathbb{R}^4\ (\overline{w})\end{pmatrix}$$

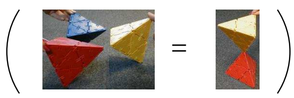


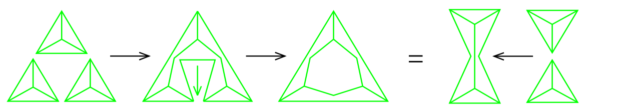


Using moves, KTG is generated by ribbon twists and the tetrahedron  :

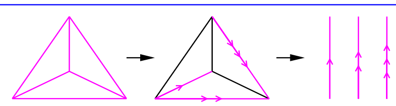
All strands here are green



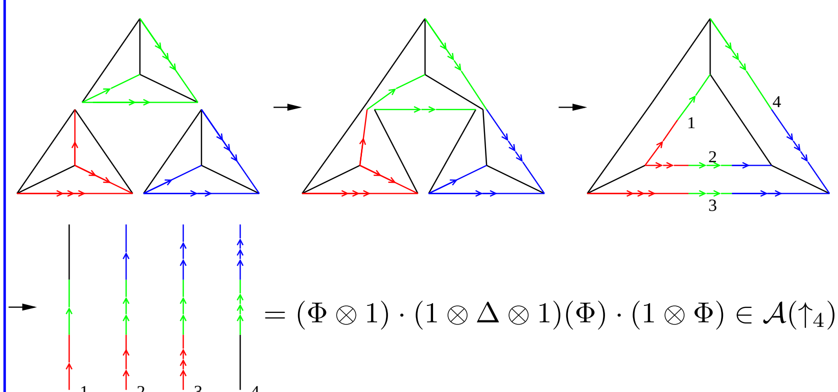
Modulo the relation(s): 



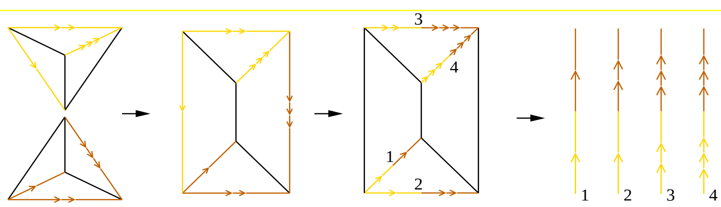
Claim. With $\Phi := Z(\text{tetrahedron})$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.

Proof. 

$:= \Phi \in \mathcal{A}(\uparrow_3)$

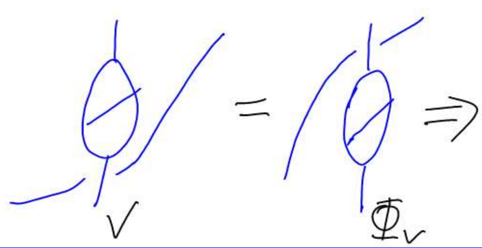


$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$



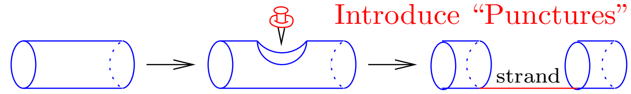
$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

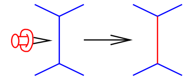
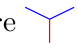
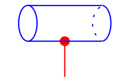
{SolKV} → {"Associators"}: Trivial − a tetrahedron has 4 vertices.

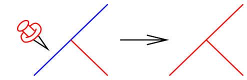


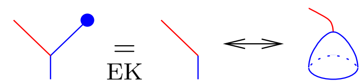
$\Phi_V \in sder_3$, so it is a sum of unrooted trees.

Introduce "Punctures".



So  where  means .

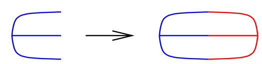
Punctures expand to the nearest Y-vertex: 

Note. 

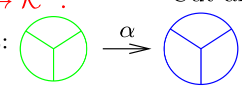
$\mathcal{K}^{\overline{w}}$. Allow tubes and strands and tube-strand vertices as above, yet allow only "compact" knots — nothing runs to ∞.

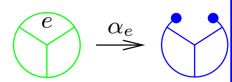
$\mathcal{K}^w \leftrightarrow \mathcal{K}^{\overline{w}}$ **equivalence.** $\mathcal{K}^w$ has a homomorphic expansion iff $\mathcal{K}^{\overline{w}}$ has a homomorphic expansion.

$\Longrightarrow$ Puncture $\mathcal{A}$ and Z:

$\Longleftarrow$  (makes sense, and CA ops can be emulated)

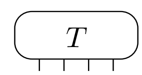
$\mathcal{K}^u \to \mathcal{K}^{\overline{w}}$. "Cut and cap is well-defined on u"

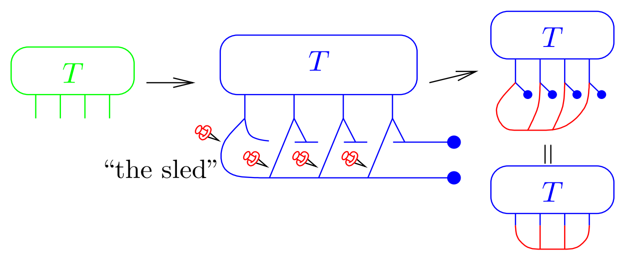
Light: 

Better: 

Theorem. The generators of $\mathcal{K}^{\overline{w}}$ can be written in terms of the generators of $\mathcal{K}^u$ (i.e., given Φ, can write a formula for V.

Sketch.

so enough to write any  . Here go:

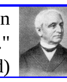


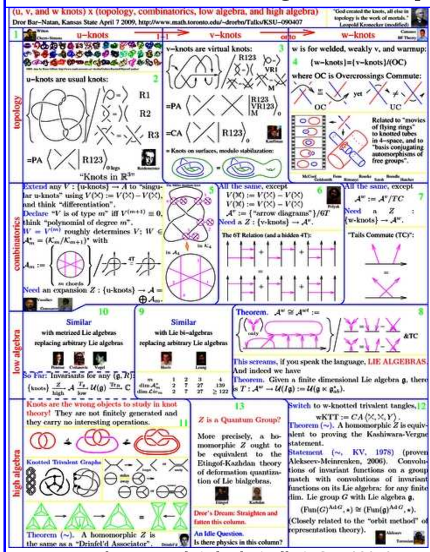
Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots

Dror Bar−Natan, Bonn August 2009, http://www.math.toronto.edu/~drorbn/Talks/Bonn−0908

Disclaimer: Rough edges remain!

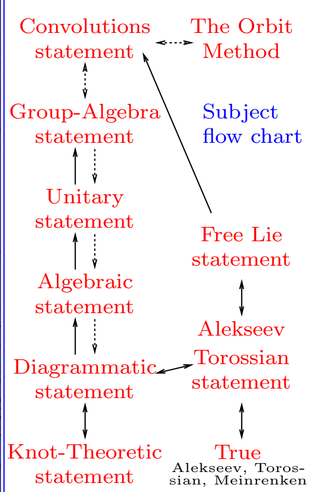
"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)





www.math.toronto.edu/~drorbn/Talks/KSU−090407

The Bigger Picture...



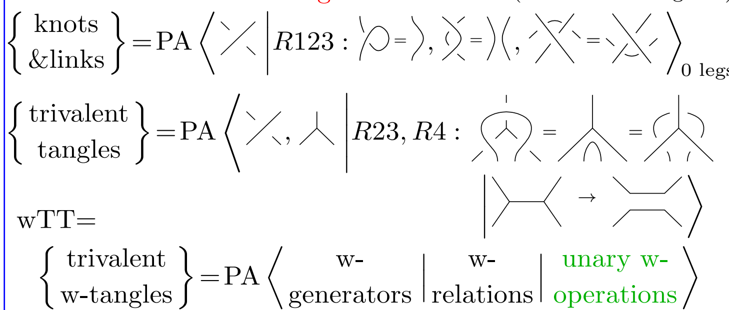

Kashiwara

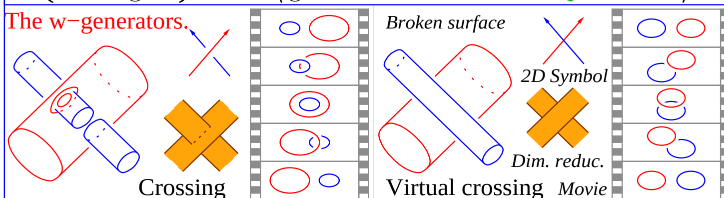

Vergne

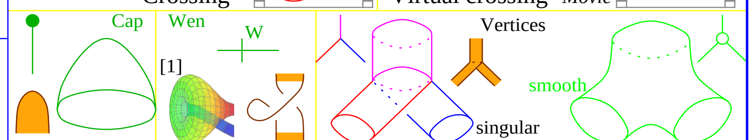

Alekseev


Torossian

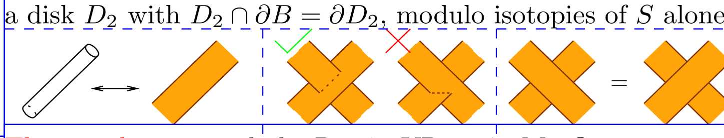
What are w-Trivalent Tangles?



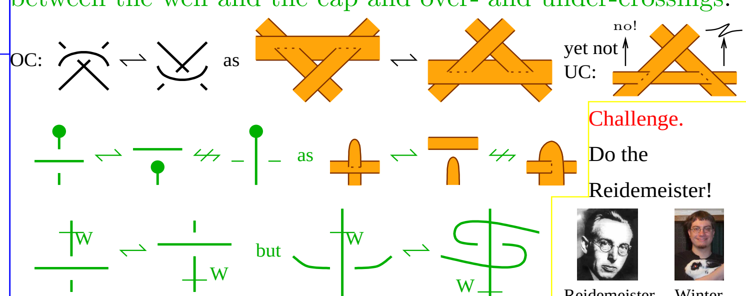


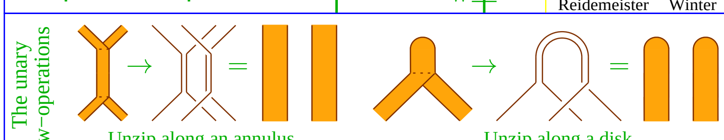


A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:





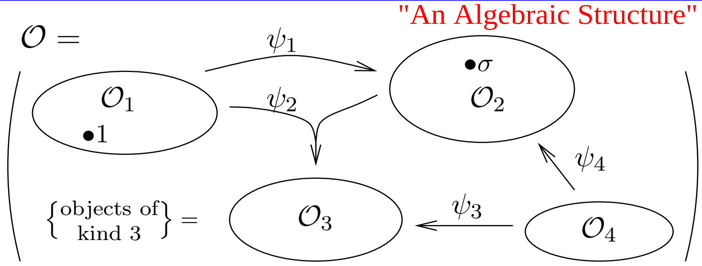
Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\text{ops} \circlearrowright \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$
$$\Downarrow \qquad \downarrow Z$$
$$\text{ops} \circlearrowright \operatorname{gr}\mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An expansion is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A homomorphic expansion is an expansion that respects all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available "products").

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Example: Pure Braids. PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

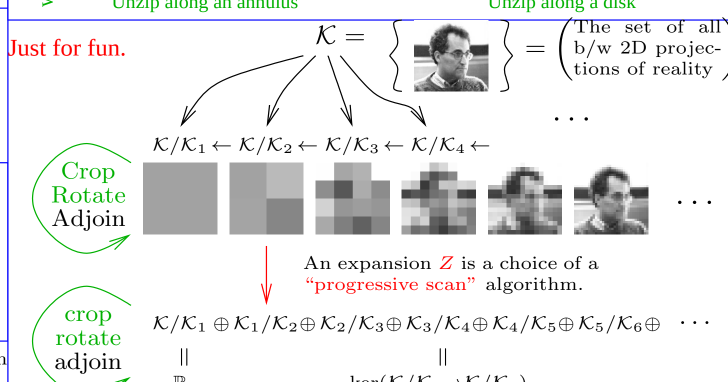
Our case(s).

$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z:\ \text{high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

$\mathcal{K}$ is knot theory or topology; $\operatorname{gr}\mathcal{K}$ is finite combinatorics: bounded-complexity diagrams modulo simple relations.

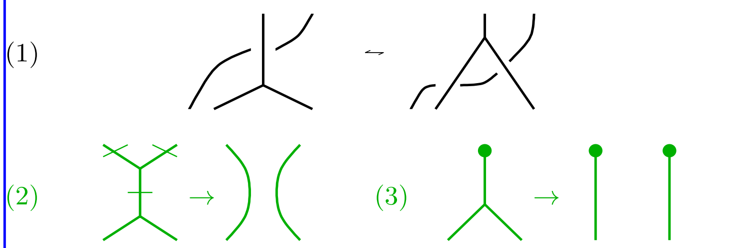
[1] http://qlink.queensu.ca/~4lb11/interesting.html 29/5/10, 8:42am
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

Just for fun.

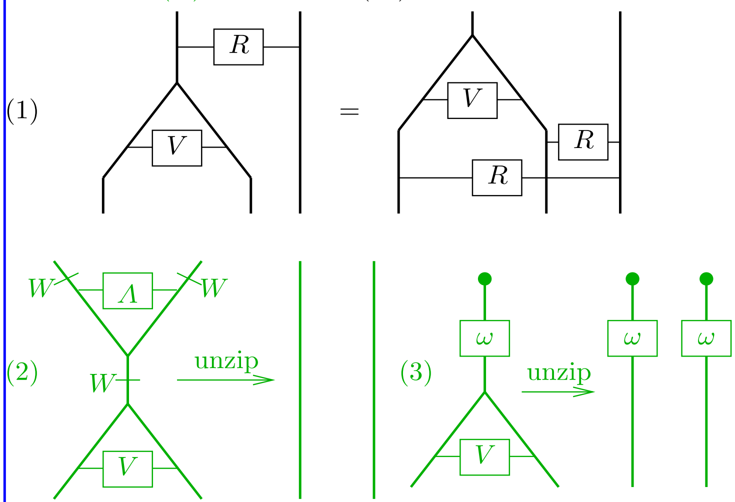


Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots, Page 2

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \overrightarrow{\mid\!\!-\!\!\mid} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Group-Algebra statement. There exists $\omega^2 \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

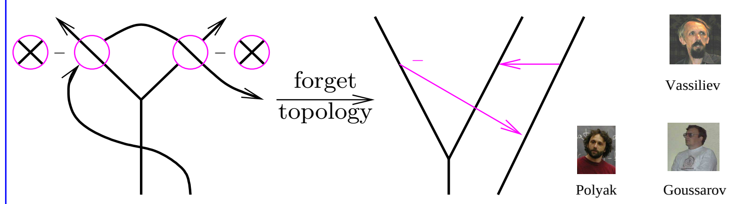
$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_{x+y}e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

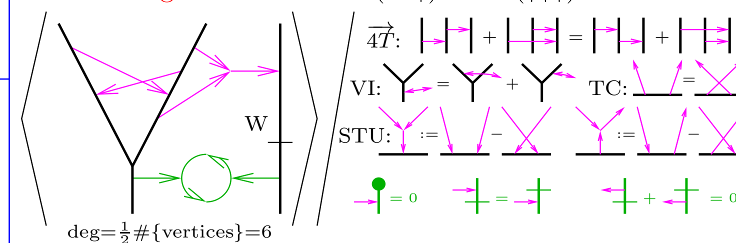
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

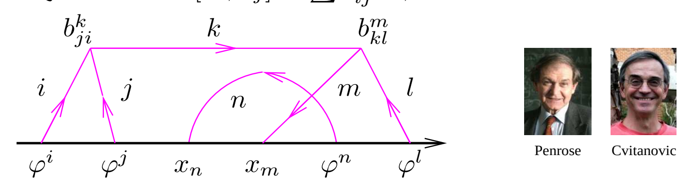
From wTT to $\mathcal{A}^w$. $\mathrm{gr}_m\,\mathrm{wTT} := \{m\text{−cubes}\}/\{(m+1)\text{−cubes}\}$:



w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij}x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via



$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\,x$: $(x\varphi)(y) := \varphi([x,y])$.
- $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ is "the constant term".

Unitary $\Longrightarrow$ Group-Algebra. $\iint \omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$= \langle \omega_{x+y}, \omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle = \langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$

$= \langle \omega_x\omega_y, e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle = \langle \omega_x\omega_y, e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$

$= \iint \omega^2_x\omega^2_y e^xe^y\phi(x)\psi(y).$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{array}{ccc} (\mathfrak{g},+) \ni x & \xrightarrow{\tau_0=\exp_{\mathcal{S}}} & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \exp_G & \searrow \exp_{\mathcal{U}} & \downarrow \chi \\ (G,\cdot) \ni e^x & \xrightarrow{\tau_1} & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) \end{array} \quad \text{so} \quad \begin{array}{ccc} \mathrm{Fun}(\mathfrak{g}) & \xrightarrow{L_0} & \hat{\mathcal{S}}(\mathfrak{g}) \\ \downarrow \Phi^{-1} & & \downarrow \chi \\ \mathrm{Fun}(G) & \xrightarrow{L_1} & \hat{\mathcal{U}}(\mathfrak{g}) \end{array}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

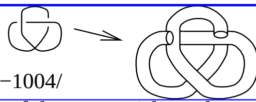
$\star$ in G : $\iint \psi_1(x)\psi_2(y)e^xe^y$ $\star$ in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$

We skipped...

- The Alexander polynomial and Milnor numbers.
- u-Knots, Alekseev-Torossian, and Drinfel'd associators.
- v-Knots, quantum groups and Etingof-Kazhdan.
- BF theory and the successful religion of path integrals.
- The simplest problem hyperbolic geometry solves.

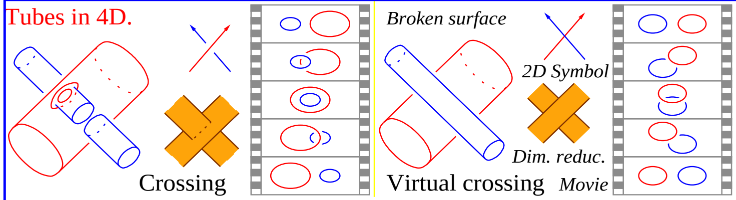
w−Knots from Z to A

Dror Bar−Natan, Luminy, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Luminy−1004/

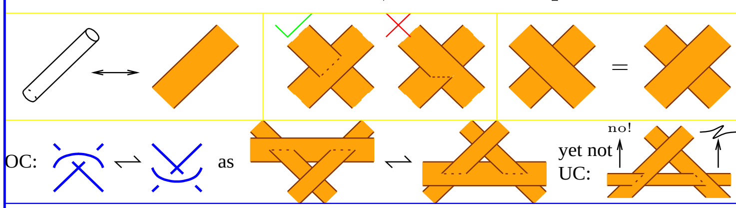


Abstract I will define w-knots, a class of knots wider than ordinary knots but weaker than virtual knots, and show that it is quite easy to construct a universal finite invariant Z of w-knots. In order to study Z we will introduce the “Euler Operator” and the “Infinitesimal Alexander Module”, at the end finding a simple determinant formula for Z. With no doubt that formula computes the Alexander polynomial A, except I don’t have a proof yet.

Tubes in 4D.



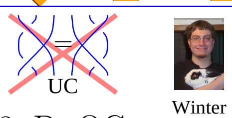
A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only “ribbon singularities”; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.

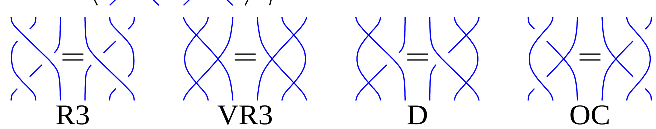


w−Knots.

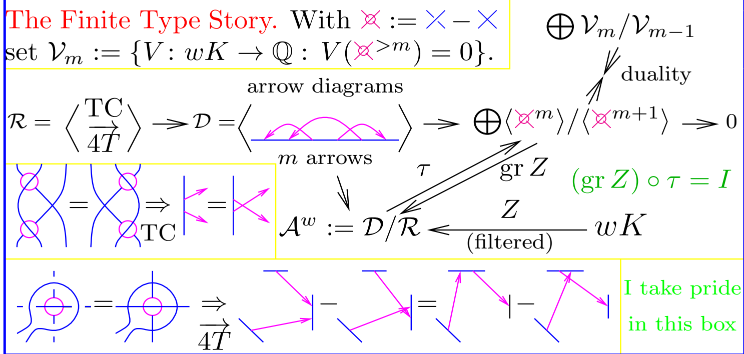
$wK = CA \langle \ \rangle / $ R23, OC

$= PA \langle \ \rangle / $ R23, VR123, D, OC

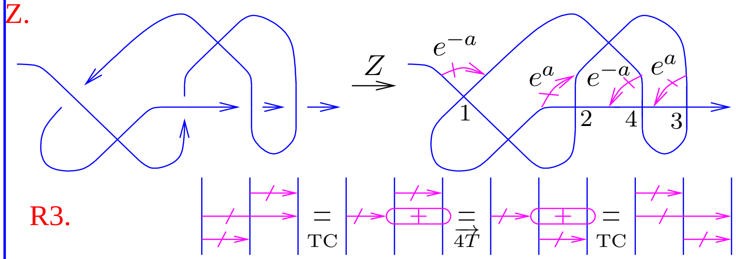


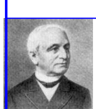


The Finite Type Story. With $\times := \times - \times$ set $\mathcal{V}_m := \{V : wK \to \mathbb{Q} : V(\times^{>m}) = 0\}$.



Z.





"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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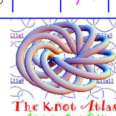
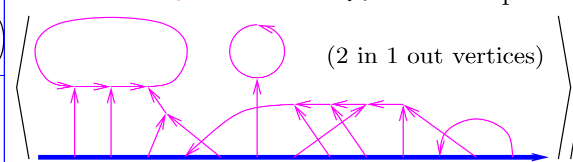
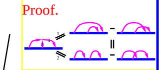
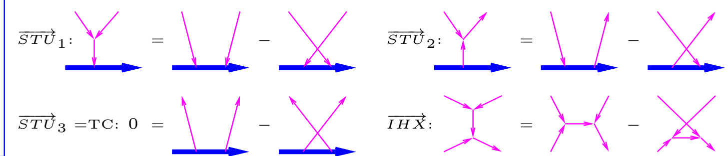


The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to

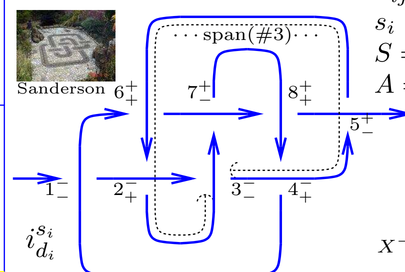






Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist. Habiro - can you do better?

The Alexander Theorem.



$T_{ij} = |\text{low}(\#j) \in \text{span}(\#i)|$,
$s_i = \text{sign}(\#i)$, $d_i = \text{dir}(\#i)$,
$S = \text{diag}(s_i d_i)$,
$A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$X^{-S} = \text{diag}(\frac{1}{X}, X, \frac{1}{X}, X, X, \frac{1}{X}, X, \frac{1}{X})$.

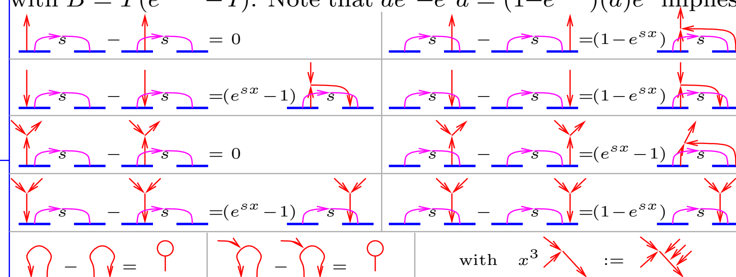
Conjecture. For u-knots, A is the Alexander polynomial.
Theorem. With $w : x^k \mapsto w_k =$ (the k-wheel),

$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right)$$

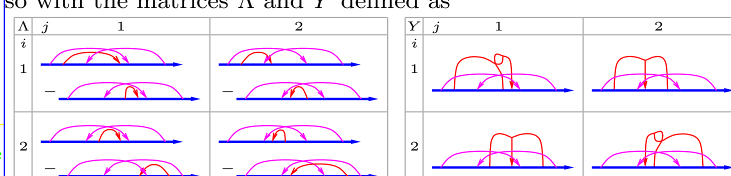
mod $w_k w_l = w_{k+l}$, $Z = N \cdot A^{-1}(e^x)$

Proof Sketch. Let E be the Euler operator, “multiply anything by its degree”, $f \mapsto xf'$ in $\mathbb{Q}[\![x]\!]$, so $Ee^x = xe^x$ and

We need to show that $Z^{-1}EZ = N' - \text{tr}\left((I-B)^{-1}TSe^{-xS}\right)w_1$, with $B = T(e^{-xS} - I)$. Note that $ae^b - e^b a = (1 - e^{\text{ad}\, b})(a)e^b$ implies



so with the matrices Λ and Y defined as



we have $EZ - N'' = \text{tr}(S\Lambda)$, $\Lambda = -BY - Te^{-xS}w_1$, and $Y = BY + Te^{-xS}w_1$. The theorem follows.

So What? • Habiro-Shima did this already, but not quite. (HS: *Finite Type Invariants of Ribbon 2-Knots, II,* Top. and its Appl. **111** (2001).)
• New (?) formula for Alexander, new (?) “Infinitesimal Alexander Module”. Related to Lescop’s arXiv:1001.4474?
• An “ultimate Alexander invariant”: local, composes well, behaves under cabling. Ought to also generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers.
• Tip of the Alekseev-Torossian-Kashiwara-Vergne iceberg (AT: *The Kashiwara-Vergne conjecture and Drinfeld’s associators,* arXiv:0802.4300).
• Tip of the v-knots iceberg. May lead to other polynomial-time polynomial invariants. “A polynomial’s worth a thousand exponentials”.
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

18 Conjectures

Dror Bar-Natan, Luminy, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Luminy-1004/

Abstract. I will state $18 = 3 \times 3 \times 2$ "fundamental" conjectures on finite type invariants of various classes of virtual knots. This done, I will state a few further conjectures about these conjectures and ask a few questions about how these 18 conjectures may or may not interact.

Following *"Some Dimensions of Spaces of Finite Type Invariants of Virtual Knots"*, by B-N, Halacheva, Leung, and Roukema, http://www.math.toronto.edu/~drorbn/papers/v-Dims/.

LRHB by Chu

Theorem. For u-knots, $\dim \mathcal{V}_n/\mathcal{V}_{n-1} = \dim \mathcal{W}_n$ for all n.

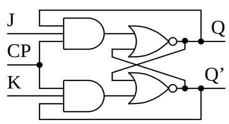
Proof. This is the Kontsevich integral, or the "Fundamental Theorem of Finite Type Invariants". The known proofs use QFT-inspired differential geometry or associators and some homological computations.

Two tables. The following tables show $\dim \mathcal{V}_n/\mathcal{V}_{n-1}$ and $\dim \mathcal{W}_n$ for $n = 1, \ldots, 5$ for 18 classes of v-knots:

relations\skeleton		round (○)	long (⟶)	flat (✕ = ✕)
standard	mod R1	0, 0, 1, 4, 17 •	0, 2, 7, 42, 246 •	0, 0, 1, 6, 34 •
R2b R2c R3b	no R1	1, 1, 2, 7, 29	2, 5, 15, 67, 365	1, 1, 2, 8, 42
braid-like	mod R1	0, 0, 1, 4, 17 •	0, 2, 7, 42, 246 •	0, 0, 1, 6, 34 •
R2b R3b	no R1	1, 2, 5, 19, 77	2, 7, 27, 139, 813	1, 2, 6, 24, 120
R2 only	mod R1	0, 0, 4, 44, 648	0, 2, 28, 420, 7808	0, 0, 2, 18, 174
R2b R2c	no R1	1, 3, 16, 160, 2248	2, 10, 96, 1332, 23880	1, 2, 9, 63, 570

18 Conjectures. These 18 coincidences persist.

Circuit Algebras



A J-K Flip Flop

Infineon HYS64T64020HDL-3.7-A 512MB RAM

Comments. 0, 0, 1, 4, 17 and 0, 2, 7, 42, 246. These are the "standard" virtual knots.

2, 7, 27, 139, 813. These best match Lie bi-algebra. Leung computed the bi-algebra dimensions to be $\geq$ 2, 7, 27, 128. (Comments, Pierre?)

Vogel

•••. We only half-understand these equalities.

1, 2, 6, 24, 120. Yes, we noticed. Karene Chu is proving all about this, including the classification of flat knots.

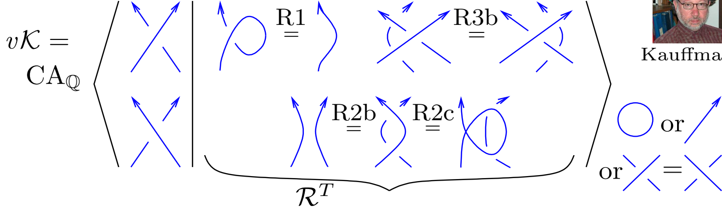
1, 1, 2, 8, 42, 258, 1824, 14664, ..., which is probably http://www.research.att.com/~njas/sequences/A013999.

What about w? See other side.

What about v-braids? I don't know.

Definitions

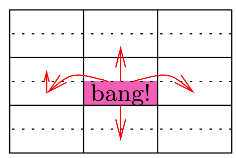
Kauffman



$\mathcal{I} = I\langle$ ✕ = ✕ − ✕ $\rangle$

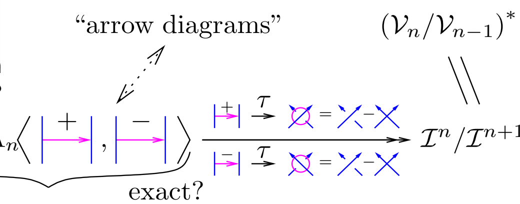
$\mathcal{V}_n = (v\mathcal{K}/\mathcal{I}^{n+1})^*$ is one thing we measure...

The True Count



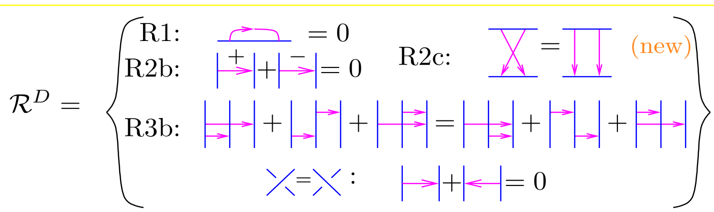
One bang! and five compatible transfer principles.

Goussarov-Polyak-Viro



Bang. Recall the surjection $\bar\tau : \mathcal{A}_n = \mathcal{D}_n/\mathcal{R}_n^D \to \mathcal{I}^n/\mathcal{I}^{n+1}$. A filtered map $Z : v\mathcal{K} \to \mathcal{A} = \bigoplus \mathcal{A}_n$ such that $(\mathrm{gr} Z) \circ \bar\tau = I$ is called a universal finite type invariant, or an "expansion".

Theorem. Such Z exist iff $\bar\tau : \mathcal{D}_n/\mathcal{R}_n^D \to \mathcal{I}^n/\mathcal{I}^{n+1}$ is an isomorphism for every class and every n, and iff the 18 conjectures hold true.



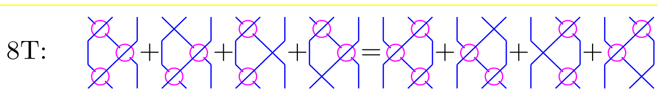
$\mathcal{W}_n = (\mathcal{D}_n/\mathcal{R}_n^D)^* = (\mathcal{A}_n)^*$ is the other thing we measure...

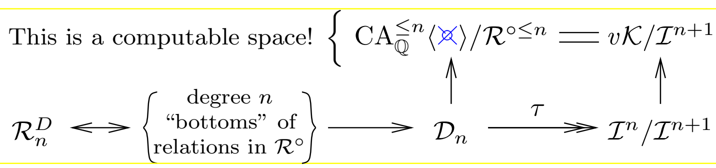
The Big Bang. Can you find a "homomorphic expansion" Z — an expansion that is also a morphism of circuit algebras? Perhaps one that would also intertwine other operations, such as strand doubling? Or one that would extend to v-knotted trivalent graphs?

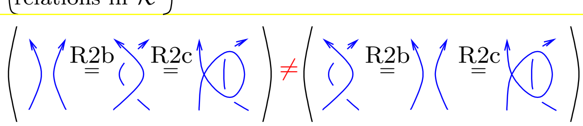
- Using generators/relations, finding Z is an exercise in solving equations in graded spaces.
- In the u case, these are the Drinfel'd pentagon and hexagon equations.
- In the w case, these are the Kashiwara-Vergne-Alekseev-Torossian equations. Composed with $\mathcal{T}_\mathfrak{g} : \mathcal{A} \to \mathcal{U}$, you get that the convolution algebra of invariant functions on a Lie group is isomorphic to the convolution algebra of invariant functions on its Lie algebra.
- In the v case there are strong indications that you'd get the equations defining a quantized universal enveloping algebra and the Etingof-Kazhdan theory of quantization of Lie bi-algebras. That's why I'm here!

The Polyak Technique

$v\mathcal{K} = \mathrm{CA}_\mathbb{Q}\langle$ ✕ $\rangle / \mathcal{R}^\circ = \{8T, \text{etc.}\}$ fails in the u case







"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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Algebraic Knot Theory

Colloquium in Mathematics

University of Copenhagen, October 9, 2008

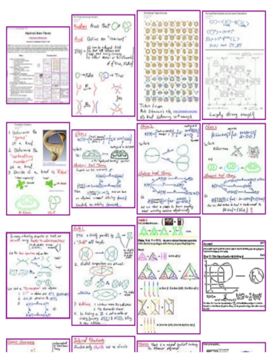
Abstract. The right objects of study in algebraic topology are not spaces, but rather, "spaces and maps between them". In a similar spirit I will argue that the right things to study in knot theory are not knots, but rather, "knotted trivalent graphs", as in the world of knotted trivalent graphs (and the basic operations between them) many interesting properties of honest knots become "definable". Thus I find myself again studying the good old Kontsevich integral - the best example I know of an algebraic knot theory - but my perspective this time is completely different.

Menu

- Some very basic knot theory.
  - The three-colouring invariant.
  - The Rolfsen table.
  - The Kauffman bracket and the Jones polynomial.
- Three things we'd like to understand:
  - The genus of a knot.
  - The crossing number of a knot.
  - Ribbon knots.
- TG-algebras and TG-morphisms.
  - Aside 1: KTG is finitely generated (and presented).
  - Aside 2: And it is related to Drinfel'd associators.
- A word about "definable" sets.
- A very strong TG-morphism exists! (But it is too hard...)
- The envelope of the Alexander polynomial. (Must be doable, but I don't know how).
- Internal quotients. (Likely more than just Lie algebra; may have "moduli" rather than just discrete points).
- Open questions and propaganda.

Also see my talk in Aarhus, June 2007. Much progress was made since, but the introductory talk (this one) remains more or less the same.

Transparencies



The Three-Colouring Invariant

October-08-08 11:30 AM

Problem: Prove that ○ ≠ (trefoil).

Proof: Define an "invariant"

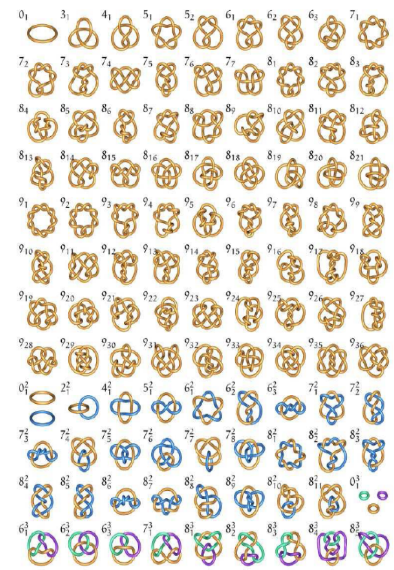
I(D) := | D can be coloured RGB so that all colours are used and every crossing is either mono- or tri-chromatic | ∈ {True, False}

○ → False, (trefoil) → True

R1, R2, R3

The Rolfsen Table of Knots

October-08-08 11:31 AM



Taken from Rob Scharein's site, http://knotplot.com/zoo/

⇒ Knot colouring isn't enough.

The Kauffman Bracket and the Jones Polynomial

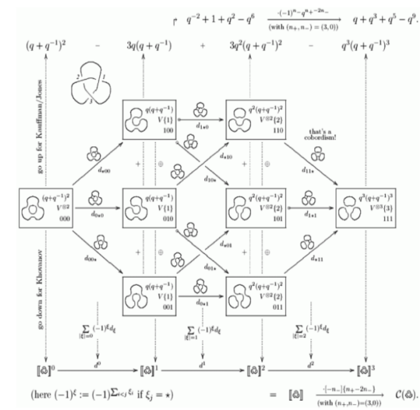
October-08-08 11:32 AM

$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$ (0-smoothing, 1-smoothing)

$\langle \bigcirc^k \rangle = (q+q^{-1})^k$

$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$

(n_+, n_-) count $(\times, \times)$

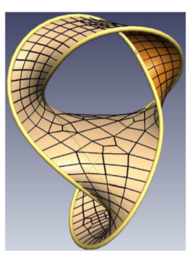


Largely strong enough!

Three Basic Problems

October-08-08 12:49 PM

1. Determine the "genus" of a knot.
2. Determine the "unknotting number" of a knot.
3. Decide if a knot is "Ribbon":



Drawn using SeifertView, http://www.win.tue.nl/~vanwijk/seifertview/

"ribbon singularity": allowed, D^2 — "cusp": not allowed, D^2

(image by Zsuzsanna Dancso)

Ribbon — Not

Claim 1

$K(\bigcirc) \supset \{$knots bounding a surface of genus 1$\} = \{\partial_\tau \gamma : \gamma \in K(\text{band-graph})\}$ — knottings of a band-graph

where ∂_τ: (band-graph) ↦ (boundary) is the "topological boundary" operator.

Algebraic Knot Theory:

Suppose we had invariants Z:

$K(\text{band-graph}) \xrightarrow{Z} A(\text{band-graph})$, $\downarrow \partial_\tau$, $\downarrow \partial_A$

$\{$genus 1 knots$\} = \operatorname{im} \partial_\tau \subset K(\bigcirc) \xrightarrow{Z} A(\bigcirc)$

Then $Z(\{$genus 1 knots$\}) \subset \operatorname{im} \partial_A$ and we have an algebraic invariant detecting $\{$genus $\geq 1\}$.

Similarly for detecting $\{$genus $\geq 7\}$...

Claim 2

$K(\bigcirc) \supset \{$knots of unknotting number 1$\} = \{ x\gamma : \gamma \in K(\ominus) ; d\gamma = \bigcirc \}$ — knottings of $\ominus$; the unknot.

where $\ominus \xrightarrow{d_\tau}$, $\xrightarrow{x_\tau}$

Algebraic Knot Theory:

$K(\ominus) \xrightarrow{x_\tau} K(\bigcirc) \xrightarrow{Z} A(\bigcirc)$; $K(\ominus) \xrightarrow{Z} A(\ominus)$; $K(\ominus) \xrightarrow{d_\tau} K(\bigcirc) \xrightarrow{Z} A(\bigcirc) \ni \bigcirc$; x_A, d_A

So $Z(\{$unknotting number 1$\}) \subset \{ x_A \alpha : \alpha \in A(\ominus), d_A \alpha = Z(\bigcirc) \}$

and we stand a chance to learn something about unknotting numbers algebraically.

Claim 3

$\{$Ribbon knots$\} \overset{\text{more or less}}{=} \{ u\gamma : \gamma \in K(\text{O-O}), d\gamma = \text{OO} \}$

Where: Ribbon means — OK — no no

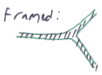
and

Algebraic Knot Theory:

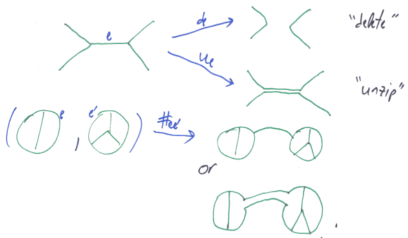
$K(\text{O-O}) \xrightarrow{d} K(\text{OO}) \xrightarrow{Z} A(\text{OO}) \ni \text{OO}$; $K(\text{O-O}) \xrightarrow{Z} A(\text{O-O})$; $K(\text{O-O}) \xrightarrow{u} K(\bigcirc) \xrightarrow{Z} A(\bigcirc)$

So $Z(\{$Ribbon knots$\}) \subset \{ u\alpha : d\alpha = Z(\text{OO}) \} \subset A(\text{O-O})$

And we stand a chance to find a counterexample to $\{$Ribbon$\} = \{$Slice$\}$!

So many interesting properties of knots are definable using Knotted Trivalent Graphs (KTGs) (Fully labeled, oriented, Framed: )

and the basic operations between them:



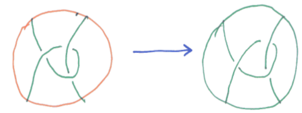
We seek a "TG-morphism" into algebra:

1. $\forall \Gamma$ an algebraic space $A(\Gamma)$, $Z_\Gamma: K(\Gamma) \to A(\Gamma)$.
2. $d, u, \#$ defined on the $A(\Gamma)$'s.
3. $K(\Gamma) \xrightarrow{Z} A(\Gamma)$, $\downarrow u_e$, $K(u_e\Gamma) \xrightarrow{Z} A(u_e\Gamma)$, $\downarrow u_e$ etc.

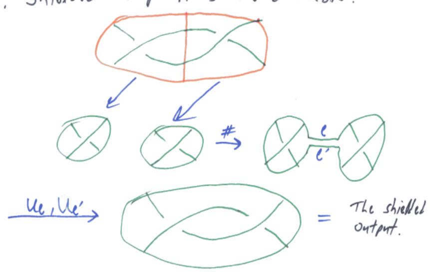
Aside 1

KTG is finitely generated, by [tetrahedron] and [square]: The shielded generators of tangled graphs.

1. "Shield" all tangles:



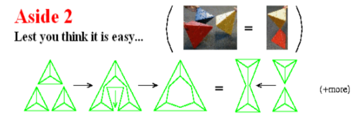
2. Shielded compositions are definable:



3. Relations: 1. whatever makes this well-defined. 2. The Reidemeister moves.
4. So finding a Z is just a matter of finding/guessing $Z(\triangle)$ & $Z(\boxtimes)$, solving a few relations....

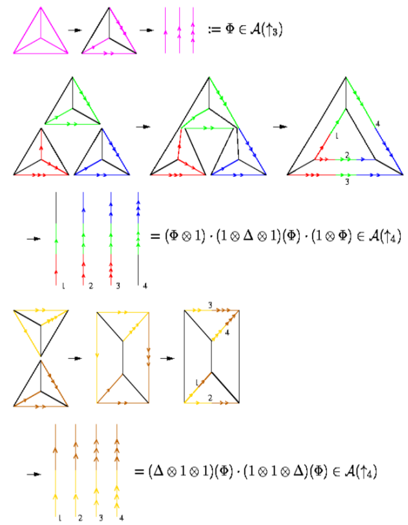
Aside 2

Lest you think it is easy...



Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi Hopf algebras.

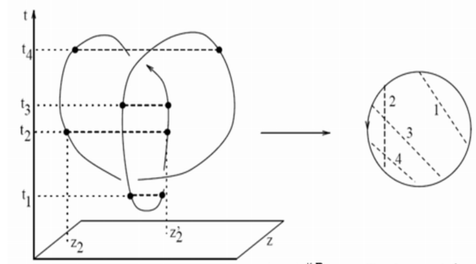
Proof.



Abstract [edit]

We construct a (very) well-behaved invariant of knotted trivalent graphs using only the Kontsevich integral, in three steps.

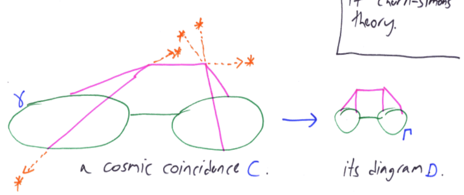
Step 1 - The Naive Kontsevich Integral [edit]



$$Z_0(K) = \sum_{m;\ t_1<\ldots<t_m;\ P=\{(z_i,z_i')\}} \frac{(-1)^{\#P_\downarrow}}{(2\pi i)^m} D_P \bigwedge_{i=1}^m \frac{dz_i - dz_i'}{z_i - z_i'}$$

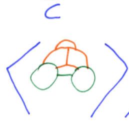
We define the "naive Kontsevich integral" Z_1 of a knotted trivalent graph or a slice thereof as in the "standard" picture above, except generalized to graphs in the obvious manner.

Cosmic Coincidences:



Witten would call it "Chern-Simons" theory.

Definition (Dylan)

$Z(\gamma) = \sum_C D \in \mathcal{A}(\Gamma) :=$ 

AS, IHX, STU, Vinv relations.

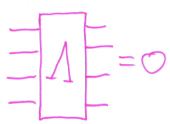
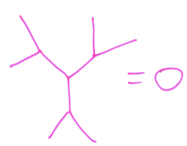
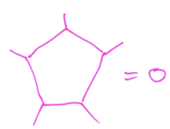
(lots of work is hidden here, and some unknowns)

Behavior under  is predictable.

Internal Quotients

Involve only chords and no strands.

Examples:

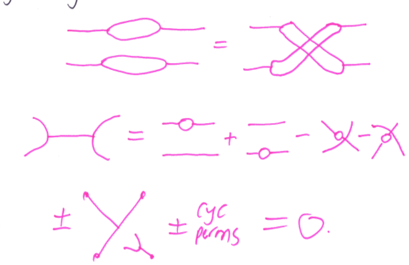
1.  "The Lie algebra is 3-dimensional" (i.e., it is $sl(2)$...)
2.  "Diagrams are (very) shallow" (related to $gl(1|1)$, Alexander)
3.  "Pentagons stay out"

4, 5, ... Use your imagination.

Classification?

Theorem There is a minimal quotient containing the Alexander polynomial.

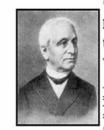
Proof Use the "internal kernel" of the Alexander weight system:



What remains is a polynomial amount of information!

Conjecture This explains everything that we know about the Alexander polynomial.

Some propaganda...



"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

Visit! Edit!

I don't understand... (more)

- Who needs "trivalent" in "knotted trivalent graphs"? (more)
- The notion of "framing". (more)
- Configuration space integrals / perturbative Chern-Simons theory for knotted graphs. (more)
- **The Alexander polynomial.** (more)
- The Lieberum "$gl(1|1)$" associator. (more)
- Must associators be so hard? Why? (more)
- The relationship between "genus" and "finite type". (more)
- The relationship between "gropes" and "finite type". (more)
- ***TG*-ideals / internal quotients.**
- What exactly are *TG*-algebras? What are the syzygies among the relations between their generators?
- Virtual knots.
- Quantum groups(!) (more)
- The work of Etingof and Kazhdan.
- The polynomiality of knot polynomials.
- Functional equations.
- The Eilenberg-Zilber theorem.
- Which other interesting classes of knots/links are "TG-definable"?

PENTAGON AND HEXAGON EQUATIONS – FOLLOWING FURUSHO

Zsuzsanna Dancso, University of Toronto, www.math.toronto.edu/zsuzsi May 25, 2010

GOAL. Understand and simplify Furusho's proof [F] that the pentagon equation implies the hexagons.

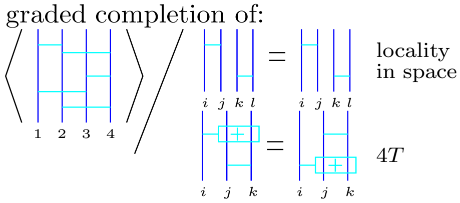
NOTATION. $\mathcal{F}_2 := Lie(X,Y)$, over k

$\mathcal{UF}_2$:=univ. envelopping alg.$= k << X, Y >>$ (non-comm. power series)

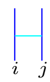
$\mathcal{UF}_2^{(k)}$:= non-comm. poly, degree$\leq k$

$\mathcal{A}_4$:= "pure 4-braid Lie alg."

graded completion of:



denote $t_{ij} :=$



$\mathcal{A}_3$:= similar for pure 3-braids

PRINCIPLE. The restriction $ASS^{(m+1)} \to ASS^{(m)}$ is surjective. I.e. if $\Phi^m \in \mathcal{UF}_2^{(m)}$ is an associator mod deg $(m+1)$, then it extends to $\Phi^{(m+1)}$, an associator mod deg $(m+2)$.

Proof. See [Dr]; [BN] Section 4. □

Application. $\exists$ rational associator. [D], [BN]

Proof. Suppose $\Phi^{(k)} \in \mathcal{UF}_2^{(k)}$ rational assoc. mod deg $(k+1)$.
Base case: $1 \in \mathcal{UF}_2^{(0)}$.
Want $\Phi^{(k+1)} = \Phi^{(k)} + \varphi_{k+1}$ ← homog. deg. $(k+1)$

Pentagon, Hexagons $\Rightarrow$ lin. eqns for φ_{k+1} w/ $\mathbb{Q}$-coeff $\Rightarrow$ $\exists$ soln by PRINCIPLE $\Rightarrow$ $\exists\mathbb{Q}$-soln □

THEOREM. [F] If $\Phi \in \mathcal{UF}_2$ group-like;
$c_2(\Phi)$:=coeff. of $XY = \frac{1}{24}$;
Φ satisfies pentagon in $\mathcal{A}_4$:

$$\Phi(t_{12}, t_{23}+t_{24})\Phi(t_{13}+t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\Phi(t_{12}+t_{13}, t_{24}+t_{34})\Phi(t_{12}, t_{23}),$$

then Φ satisfies hexagons in $\mathcal{A}_3$:

$$e(\tfrac{t_{12}+t_{23}}{2}) = \Phi(t_{13}, t_{12})e(\tfrac{t_{13}}{2})\Phi(t_{13}, t_{23})^{-1}e(\tfrac{t_{23}}{2})\Phi(t_{12}, t_{13}),$$

$$e(\tfrac{t_{12}+t_{13}}{2}) = \Phi(t_{23}, t_{13})^{-1}e(\tfrac{t_{13}}{2})\Phi(t_{12}, t_{13})e(\tfrac{t_{12}}{2})\Phi(t_{12}, t_{23})^{-1},$$

i.e. Φ is an associator.

PROOF. Induction: if Φ satisfies pentagon, and satisfies hexagons mod deg k, we prove that it satisfies hexagons mod deg $(k+1) \Rightarrow \Phi$ is an associator.

Base case. Pentagon $\Rightarrow$ the abelianization $\Phi^{(ab)} = 1 \Rightarrow \Phi$ is 0 in degree 1. Therefore, $\Phi^{(2)} = 1 + \frac{1}{24}$, which satisfies the pentagon up to degree 2. $\Rightarrow k \geq 3$.

Induction Step. PRINCIPLE $\Rightarrow \exists$ an associatorΦ' (group-like + pentagon + hexagons), s.t. $\Phi = \Phi'$ mod deg k.
$\Phi - \Phi' = \varphi_k$+higher order terms (homog. deg. k)

$\varphi_k = \Phi - \Phi'$ mod deg $(k+1) \Rightarrow \varphi_k$ primitive;
Φ, Φ' satisfy pentagon $\Rightarrow$
$\Rightarrow \varphi_k$ satisfies "linearized pentagon";
$k \geq 3$
$\xRightarrow{\text{MAIN LEMMA (see below)}}$

$\Rightarrow \varphi_k$ satisfies "linearized hexagons".

Then since Φ' satisfies hexagons; $\Phi = \Phi' + \varphi_k$ mod deg $(k+1)$
$\Rightarrow \Phi$ satisfies hexagons mod deg $(k+1)$.
Done! □

THE HARD PART.

Main Lemma (Linearization). Let $\varphi \in \mathcal{UF}_2$ be commutator-primitive (i.e. $\varphi \in [\mathcal{F}_2, \mathcal{F}_2]$),
w/ $c_2(\varphi)$:= coeff. of XY (eqv. $[X,Y]$)$= 0$,
and φ satisfies 5-cycle (linearized pentagon) in $\mathcal{A}_4$:

$$(5): \quad \varphi(t_{12}, t_{23}+t_{24}) + \varphi(t_{13}+t_{23}, t_{34}) = \varphi(t_{23}, t_{34}) + \varphi(t_{12}+t_{13}, t_{24}+t_{34}) + \varphi(t_{12}, t_{23}).$$

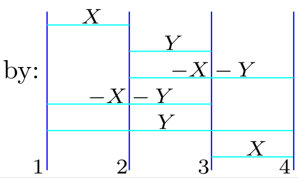
Then φ satisfies 2-cycle and 3-cycle (lin. hexagons) in $\mathcal{F}_2$:

$$(2): \quad \varphi(X,Y) + \varphi(Y,X) = 0,$$

$$(3): \quad \varphi(X,Y) + \varphi(Y, -X-Y) + \varphi(-X-Y, X) = 0.$$

PROOF.

I. (5) $\Rightarrow$ (2)
the projection $q : \mathcal{A}_4 \to \mathcal{F}_2$ defined by:



sends (5) to (2).

II. (5) + (2) $\Rightarrow$ (3)

Step 1. Using (2), rearrange (5):

$$\varphi(t_{12}, t_{23}) + \varphi(t_{34}, t_{13}+t_{23}) + \varphi(t_{23}+t_{24}, t_{12}) + \varphi(t_{23}, t_{34}) + \varphi(t_{12}+t_{13}, t_{24}+t_{34}) = 0$$

Denote the left hand side by P (for Pentagon).
Use shorthands: $\varphi(t_{12}, t_{23}) =: (123)$,
$\varphi(t_{34}, t_{13}+t_{23}) =: (43(12)) = (43(21))$, etc.
So $P = (123) + (43(12)) + ((34)21) + (234) + (1(23)4) = 0$.
Denote LHS of (3) by $R(X,Y)$ (for tRiangle).
Let $R(123) := R(X,Y)$ w/ subst. $X = t_{12}$, $Y = t_{23}$;
$R((34)21) := R(X,Y)$ w/ subst. $X = t_{23}+t_{24}$, $Y = t_{12}$; etc.

Step 2: A fact about φ.
Since $\varphi \in [\mathcal{F}_2, \mathcal{F}_2]$, "commuting parts can be dropped":
$[B,C] = 0 \Rightarrow \varphi(A+B, C) = \varphi(A,C); \varphi(C, A+B) = \varphi(C,A)$.

Application to R.
$R(123) = \varphi(t_{12}, t_{23}) + \varphi(t_{23}, -t_{12}-t_{23}) + \varphi(-t_{12}-t_{23}, t_{12})$.
In $\mathcal{A}_3$, $(t_{12}+t_{23}+t_{31})$ is central (in fact, generates the center).
$\Rightarrow R(123) = \varphi(t_{12}, t_{23}) + \varphi(t_{23}, (t_{12}+t_{23}+t_{31}) - t_{12} - t_{23}) + \varphi((t_{12}+t_{23}+t_{31}) - t_{12} - t_{23}, t_{12}) = \varphi(t_{12}, t_{23}) + \varphi(t_{23}, t_{31}) + \varphi(t_{31}, t_{12}) = (123) + (231) + (312)$.
Note that this also implies $\{(3) \iff R(123) = 0$ in $\mathcal{A}_3\}$.

Step3: The hard part of the hard part.
Permutation group S_4 acts on $\mathcal{A}_4$. $\forall \sigma \in S_4$, $\sigma P = 0$.
Want: $\sum_i \sigma_i P = \sum R$'s with various substitutions.
This works: $\sigma_1 := id$, $\sigma_2 := 4231$, $\sigma_3 := 1342$, $\sigma_4 := 4312$.

$0 = \sum_{i=1}^4 \sigma_i P =$
$(123) + (43(12)) + ((34)21) + (234) + (1(23)4) +$
$+ (423) + (13(42)) + ((31)24) + (231) + (4(23)1) +$
$+ (134) + (24(13)) + ((42)31) + (342) + (1(34)2) +$
$+ (431) + (31(43)) + ((12)34) + (312) + (4(31)2) =$ (Cancel by (2).)
$= (123) + (231) + (312) +$
$+ (423) + (234) + (342) +$
$+ ((34)21) + (21(34)) + (1(34)2) +$
$+ ((31)24) + (24(31)) + (4(31)2) =$
$= R(123) + R(423) + R((34)21) + R((31)24)$

On the RHS, $\forall$ chord ends on strand 2, so
RHS $\in < t_{12}, t_{23}, t_{24} > \cong \mathcal{F}_3 \subseteq \mathcal{A}_4$ ← No relations!

FLIP OVER!

PENTAGON AND HEXAGON EQUATIONS – FOLLOWING FURUSHO, continued.

Step 4: Finish.

Note that (2) $\Rightarrow R(Y,X) = -R(X,Y) \Rightarrow R(X,X) = 0$.
Use projections $\mathcal{F}_3 \to \mathcal{F}_2$, applied to the equation
$0 = R(123) + R(423) + R((34)21) + R((31)24)$.

$p_1 : \mathcal{F}_3 \to \mathcal{F}_2$, $t_{12} \mapsto X$, $t_{23} \mapsto Y$, $t_{24} \mapsto X$.
$p_1 \Rightarrow$
$0 = R(X,Y) + R(X,Y) + R(X{+}Y,X) + R(X{+}Y,X)$
$\Rightarrow R(X{+}Y,X) = -R(X,Y)$

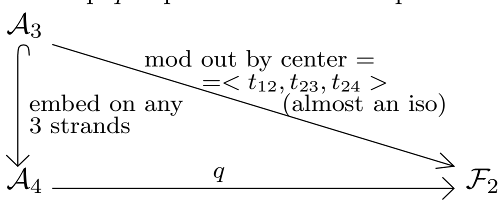
$p_1 : \mathcal{F}_3 \to \mathcal{F}_2$, $t_{12} \mapsto X$, $t_{23} \mapsto Y$, $t_{24} \mapsto X$.
$p_2 \Rightarrow$
$0 = R(X,X) + R(Y,X) + R(X{+}Y,X) + R(2X,Y)$
$\Rightarrow R(2X,Y) = 2R(X,Y)$

Expand this (orange) equation in a linear basis $\Rightarrow$
$R(X,Y) = \sum_{n=1}^{\infty} a_n (adY)^{n-1}(X)$.
But $R(Y,X) = -R(X,Y) \Rightarrow a_n = 0$ when $n \neq 2$.
$\Rightarrow R(X,Y) = a_2[Y,X]$.
But $c_2(\varphi) = 0 \Rightarrow a_2 = 0 \Rightarrow R(X,Y) = 0$. Done! □

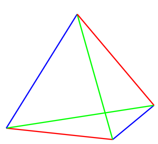
NOTE. In the THEOREM, if $c_2(\Phi) \neq \frac{1}{24}$, then Φ satisfies a rescaled version of the hexagons: each exponent is multiplied by $\mu = \pm\sqrt{24c_2(\Phi)}$.

ASIDE: q is nice!

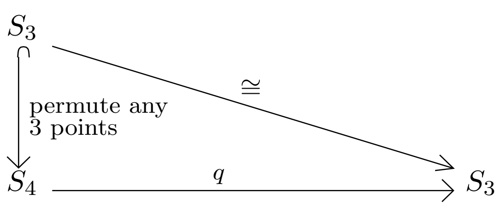
The map q in part I of the "hard part" has a nice property:



This is a braid-theoretic analog of $p : S_4 \to S_3$:
S_4 =symmetries of the tetrahedron,

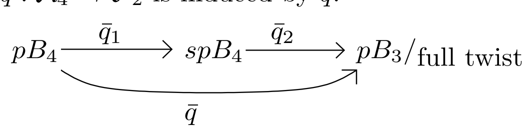


S_4 permutes sets of opposite edges
$\Rightarrow$ map $p : S_4 \to S_3$

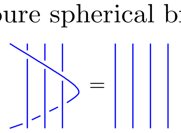


Topological interpretation of q:

$q : \mathcal{A}_4 \to \mathcal{F}_2$ is induced by $\bar{q}$:



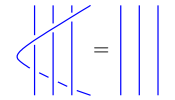
pB_i =pure braids on i strands
spB_4 =pure spherical braids on 4 strands (live in $S^2 \times I$).
Means:



, similarly for other strands

CONTINUED TOP RIGHT

$\bar{q}_1$ =obvious quotient map
$\bar{q}_2$ =pull strand 4 straight, call this point of S^2 "∞".
$\Rightarrow$ get std pure 3-braid on strands 1, 2, 3, except:



$\Rightarrow$ full twist of first 3 strands is trivial
$\Rightarrow$ have to mod out by full twist.

$\bar{q} = \bar{q}_2\bar{q}_1$.

WHAT WAS SIMPLIFIED?

Removed GT, GRT and algebraic geometry, and replaced by use of the "Principle". (GT, GRT and algebraic geometry are used in the proof of the Principle.)

Removed the spherical 5-braid Lie-algebra $\mathcal{B}_5$ by translating the proof of the Main Lemma to $\mathcal{A}_4$.

The proof of the main lemma was NOT changed. The "translation" is easy, as $\mathcal{A}_4$ and $\mathcal{B}_5$ are almost isomorphic.

REFERENCES.

[BN] D. Bar-Natan, *On associators and the Grothendieck-Teichmuller group I,* Selecta Math, New Series 4 (1998) 183–212

[Dr] V.G. Drinfel'd, *On quasitriangular Quasi-Hopf algebras and a group closely connected with* $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J. 2 (1991) 829–860.

[F] H. Furusho, *Pentagon and hexagon equations* Annals of Math, Vol. 171 (2010), No 1, 545–556.

Stonehenge

From Stonehenge to Witten Skipping all the Details

Oporto Meeting on Geometry, Topology and Physics, July 2004

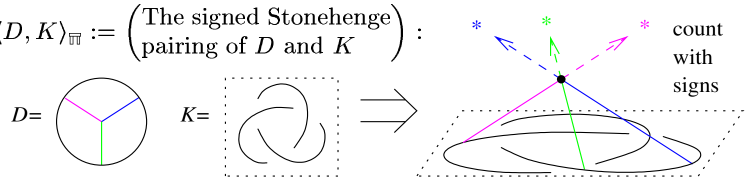
Dror Bar–Natan, University of Toronto

Witten

It is well known that when the Sun rises on midsummer's morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. Thus astrological lineups, one of the pillars of modern thought, are much older than the famed Gaussian linking number of two knots.

Recall that the latter is itself an astrological construct: one of the standard ways to compute the Gaussian linking number is to place the two knots in space and then count (with signs) the number of shade points cast on one of the knots by the other knot, with the only lighting coming from some fixed distant star.

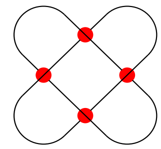
$\langle D, K\rangle_{\Pi} :=$ (The signed Stonehenge pairing of D and K) :



The Gaussian linking number

$lk(\text{link}) = \frac{1}{2}\sum_{\text{vertical chopsticks}} (\text{signs})$

Carl Friedrich Gauss



$lk=2$

Thus we consider the generating function of all stellar coincidences:

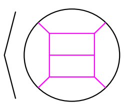
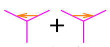
Dylan Thurston

$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\Pi} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

N := # of stars

c := # of chopsticks

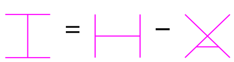
e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span ⟨  ⟩ / oriented vertices, AS:  =0 & more relations

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,



(see below)

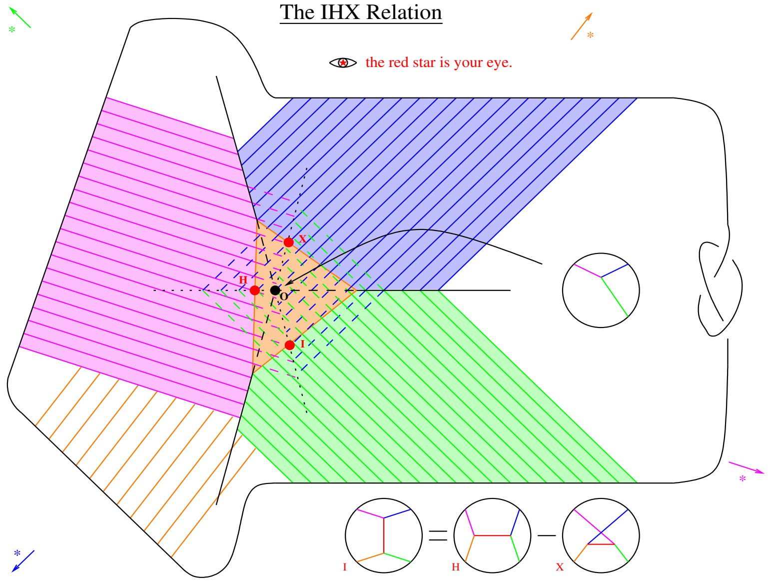
An intersection line cuts through the knot – Solution: Impose STU,

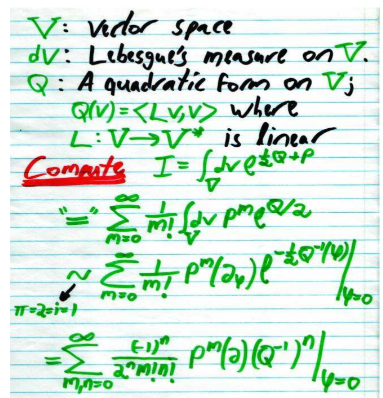
(similar argument)

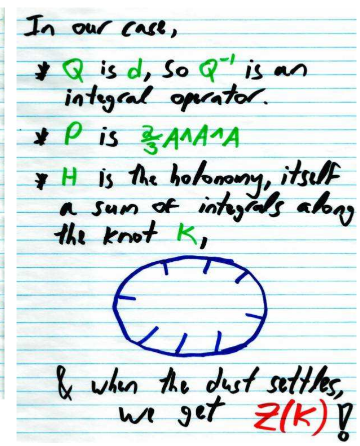
The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

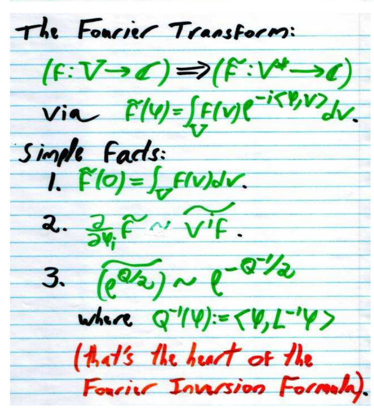
(not shown here)

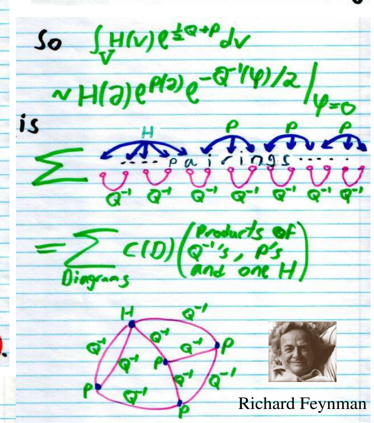
The IHX Relation



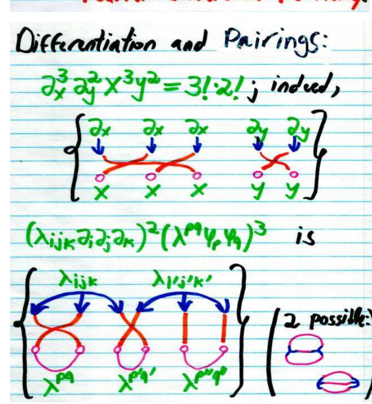




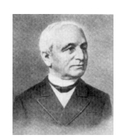




Richard Feynman



"God created the knots, all else in topology is the work of man."



Leopold Kronecker

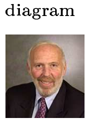
(modified)

It all is perturbative Chern–Simons–Witten theory:

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\to \sum_{D:\ \text{Feynman diagram}} W_{\mathfrak{g}}(D) \sum\!\!\!\!/\ \mathcal{E}(D) \to \sum_{D:\ \text{Feynman diagram}} D \sum\!\!\!\!/\ \mathcal{E}(D)$$

Shiing–shen Chern



James H Simons

This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407