Meta−Groups, Meta−Bicrossed−Products, and the Alexander Polynomial, 1

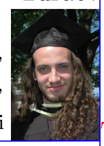
Dror Bar−Natan in Montreal, June 2013.

http://www.math.toronto.edu/~drorbn/Talks/Montreal-1306/

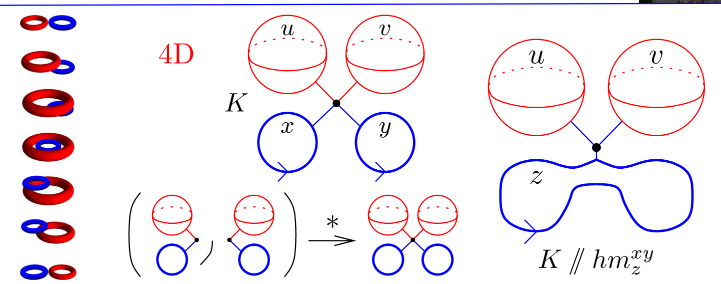
Abstract. I will define "meta-groups" and explain how one specific meta-group, which in itself is a "meta-bicrossed-product", gives rise to an "ultimate Alexander invariant" of tangles, that contains the Alexander polynomial (multivariable, if you wish), has extremely good composition properties, is evaluated in a topologically meaningful way, and is least-wasteful in a computational sense. If you believe in categorification, that's a wonderful playground.

This work is closely related to work by Le Dimet (Comment. Math. Helv. **67** (1992) 306–315), Kirk, Livingston and Wang (arXiv:math/9806035) and Cimasoni and Turaev (arXiv:math.GT/0406269).

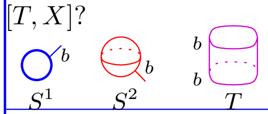
See also Dror Bar-Natan and Sam Selmani, *Meta-Monoids, Meta-Bicrossed Products, and the Alexander Polynomial,* arXiv:1302.5689.

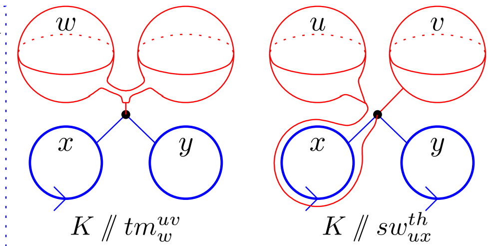


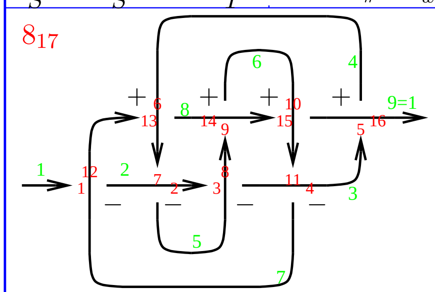
Sam Selmani

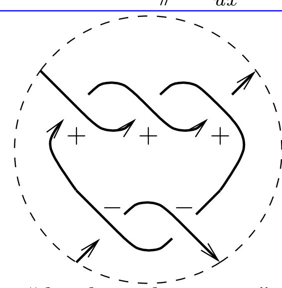


Riddle. People often study $\pi_1(X) = [S^1, X]$ and $\pi_2(X) = [S^2, X]$. Why not $\pi_T(X) := [T, X]$?

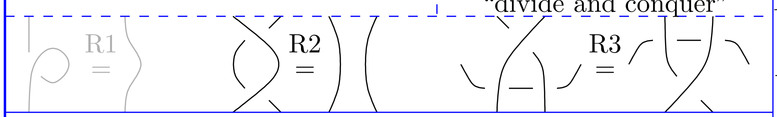




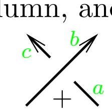




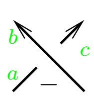
"divide and conquer"



A Standard Alexander Formula. Label the arcs 1 through $(n+1) = 1$, make an $n \times n$ matrix as below, delete one row and one column, and compute the determinant:



	a	b	c
c	-1	$1-X$	X



	a	b	c
c	$-X$	$X-1$	1

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & X-1 & 0 & -X \\ -1 & X & 0 & 0 & 0 & 0 & 1-X & 0 \\ 0 & -1 & X & 0 & 1-X & 0 & 0 & 0 \\ X-1 & 0 & -X & 1 & 0 & 0 & 0 & 0 \\ 0 & 1-X & 0 & -1 & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -X & 1 & 0 & X-1 \\ 0 & 0 & 1-X & 0 & 0 & -1 & X & 0 \\ 0 & 0 & 0 & X-1 & 0 & 0 & -X & 1 \end{pmatrix}$$

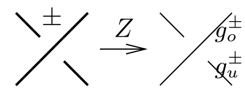
```
[[1 ;; 7, 1 ;; 7]] // Det
```

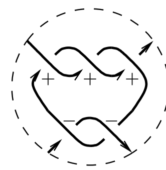
$-1 + 4X - 8X^2 + 11X^3 - 8X^4 + 4X^5 - X^6$

Alexander Issues.

- Quick to compute, but computation departs from topology.
- Extends to tangles, but at an exponential cost.
- Hard to categorify.

Idea. Given a group G and two "YB" pairs $R^\pm = (g_o^\pm, g_u^\pm) \in G^2$, map them to xings and "multiply along", so that

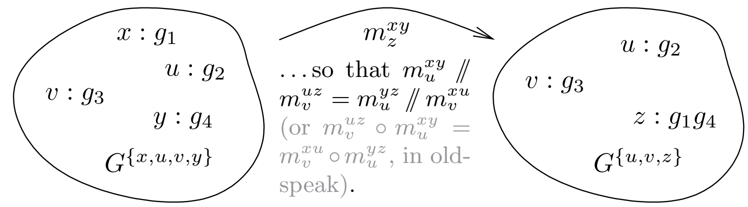




$$\overset{Z}{\longmapsto} \begin{pmatrix} g_o^+ g_u^+ g_o^+ g_u^- g_o^- g_u^+ g_o^+ g_u^+ \\ g_u^- g_o^- \end{pmatrix}$$

This Fails! R2 implies that $g_o^\pm g_o^\mp = e = g_u^\pm g_u^\mp$ and then R3 implies that g_o^+ and g_u^+ commute, so the result is a simple counting invariant.

A Group Computer. Given G, can store group elements and perform operations on them:



...so that $m_u^{xy} /\!\!/ m_v^{uz} = m_u^{yz} /\!\!/ m_v^{xu}$ (or $m_v^{uz} \circ m_u^{xy} = m_v^{xu} \circ m_u^{yz}$, in old-speak).

Also has S_x for inversion, e_x for unit insertion, d_x for register deletion, Δ_{xy}^z for element cloning, ρ_y^x for renamings, and $(D_1, D_2) \mapsto D_1 \cup D_2$ for merging, and many obvious composition axioms relating those. $P = \{x : g_1, y : g_2\} \Rightarrow P = \{d_y P\} \cup \{d_x P\}$

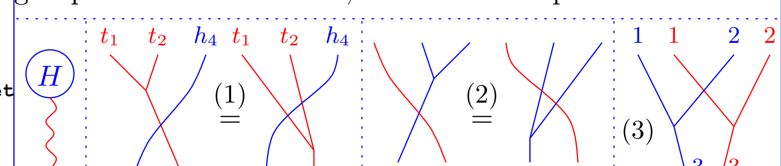
A Meta-Group. Is a similar "computer", only its internal structure is unknown to us. Namely it is a collection of sets $\{G_\gamma\}$ indexed by all finite sets γ, and a collection of operations m_z^{xy}, S_x, e_x, d_x, Δ_{xy}^z (sometimes), ρ_y^x, and $\cup$, satisfying the exact same *linear* properties.

Example 0. The non-meta example, $G_\gamma := G^\gamma$.

Example 1. $G_\gamma := M_{\gamma\times\gamma}(\mathbb{Z})$, with simultaneous row and column operations, and "block diagonal" merges. Here if $P = \begin{pmatrix} x: & a & b \\ y: & c & d \end{pmatrix}$ then $d_y P = (x : a)$ and $d_x P = (y : d)$ so $\{d_y P\} \cup \{d_x P\} = \begin{pmatrix} x: & a & 0 \\ y: & 0 & d \end{pmatrix} \neq P$. So this G is truly meta.

Claim. From a meta-group G and YB elements $R^\pm \in G_2$ we can construct a knot/tangle invariant.

Bicrossed Products. If $G = HT$ is a group presented as a product of two of its subgroups, with $H \cap T = \{e\}$, then also $G = TH$ and G is determined by H, T, and the "swap" map $sw^{th} : (t, h) \mapsto (h', t')$ defined by $th = h't'$. The map sw satisfies (1) and (2) below; conversely, if $sw : T \times H \to H \times T$ satisfies (1) and (2) (+ lesser conditions), then (3) defines a group structure on $H \times T$, the "bicrossed product".



$tm_1^{12} /\!\!/ sw_{14} = sw_{24} /\!\!/ sw_{14} /\!\!/ tm_1^{12}$

$gm_3^{12} := sw_{12} /\!\!/ tm_3^{12} /\!\!/ hm_3^{12}$

$h_1t_1h_2t_2 = h_1(h_2't_1')t_2 = (h_1h_2')(t_1't_2) = h_3t_3$

A **Meta-Bicrossed-Product** is a collection of sets $\beta(\eta,\tau)$ and operations tm^{uv}_w, hm^{xy}_z and sw^{th}_{ux} (and lesser ones), such that tm and hm are "associative" and (1) and (2) hold (+ lesser conditions). A meta-bicrossed-product defines a meta-group with $G_\gamma := \beta(\gamma,\gamma)$ and gm as in (3).

Example. Take $\beta(\eta,\tau) = M_{\tau\times\eta}(\mathbb{Z})$ with row operations for the tails, column operations for the heads, and a trivial swap.

β Calculus. Let $\beta(\eta,\tau)$ be

$$\left\{ \begin{array}{c|ccc} \omega & h_1 & h_2 & \cdots \\ \hline t_1 & \alpha_{11} & \alpha_{12} & \cdot \\ t_2 & \alpha_{21} & \alpha_{22} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \middle| \begin{array}{l} h_j\in\eta,\ t_i\in\tau, \text{ and } \omega \text{ and} \\ \text{the } \alpha_{ij} \text{ are rational func-} \\ \text{tions in a variable } X \text{ with} \\ \omega(1)=1 \text{ and } \alpha_{ij}(1)=0 \end{array} \right\},$$

$$tm^{uv}_w:\ \begin{array}{c|c} \omega & \cdots \\ \hline t_u & \alpha \\ t_v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline t_w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & \eta_1 \\ \hline \tau_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & \eta_2 \\ \hline \tau_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & \eta_1 & \eta_2 \\ \hline \tau_1 & \alpha_1 & 0 \\ \tau_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z:\ \begin{array}{c|ccc} \omega & h_x & h_y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & h_z & \cdots \\ \hline \vdots & \alpha+\beta+\langle\alpha\rangle\beta & \gamma \end{array},$$

$$sw^{th}_{ux}:\ \begin{array}{c|cc} \omega & h_x & \cdots \\ \hline t_u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & h_x & \cdots \\ \hline t_u & \alpha(1+\langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta-\gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1+\alpha$ and $\langle c\rangle := \sum_i c_i$, and let

$$R^p_{ab} := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X-1 \\ t_b & 0 & 0 \end{array} \qquad R^m_{ab} := \begin{array}{c|cc} 1 & h_a & h_b \\ \hline t_a & 0 & X^{-1}-1 \\ t_b & 0 & 0 \end{array}.$$

Theorem. Z^β is a tangle invariant (and more). Restricted to knots, the ω part is the Alexander polynomial. On braids, it is equivalent to the Burau representation. A variant for links contains the multivariable Alexander polynomial.

Why Happy? • Applications to w-knots.
• Everything that I know about the Alexander polynomial can be expressed cleanly in this language (even if without proof), except HF, but including genus, ribbonness, cabling, v-knots, knotted graphs, etc., and there's potential for vast generalizations.
• The least wasteful "Alexander for tangles" I'm aware of.
• Every step along the computation is the invariant of something.
• Fits on one sheet, including implementation & propaganda.

Further meta-monoids. Π (and variants), $\mathcal{A}$ (and quotients), vT, ...

Further meta-bicrossed-products. Π (and variants), $\overrightarrow{\mathcal{A}}$ (and quotients), M_0, M, $\mathcal{K}^{bh}$, $\mathcal{K}^{rbh}$, ...

Meta-Lie-algebras. $\mathcal{A}$ (and quotients), $\mathcal{S}$, ...

Meta-Lie-bialgebras. $\overrightarrow{\mathcal{A}}$ (and quotients), ...

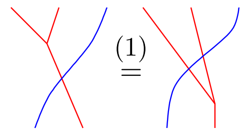
I don't understand the relationship between gr and H, as it appears, for example, in braid theory.

I mean business!

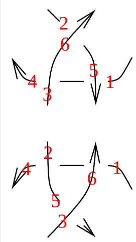
```
βSimp = Factor; SetAttributes[βCollect, Listable];
βCollect[B[ω_, Λ_]] := B[βSimp[ω],
   Collect[Λ, h_, Collect[#, t_, βSimp] &]];
βForm[B[ω_, Λ_]] := Module[{ts, hs, M},
   ts = Union[Cases[B[ω, Λ], t_u_ :> u, Infinity]];
   hs = Union[Cases[B[ω, Λ], h_x_ :> x, Infinity]];
   M = Outer[βSimp[Coefficient[Λ, h_#1 t_#2]] &, hs, ts];
   PrependTo[M, t_# & /@ ts];
   M = Prepend[Transpose[M], Prepend[h_# & /@ hs, ω]];
   MatrixForm[M]];
βForm[else_] := else /. β_B :> βForm[β];
Format[β_B, StandardForm] := βForm[β];
⟨μ_⟩ := μ /. t_ → 1;
tm_{u_,v_→w_}[β_] := βCollect[β /. t_{u|v} → t_w];
hm_{x_,y_→z_}[B[ω_, Λ_]] := Module[
    {α = D[Λ, h_x], β = D[Λ, h_y], γ = Λ /. h_{x|y} → 0},
    B[ω, (α + (1 + ⟨α⟩) β) h_z + γ] // βCollect];
sw_{u_,x_}[B[ω_, Λ_]] := Module[{α, β, γ, δ, ϵ},
    α = Coefficient[Λ, h_x t_u]; β = D[Λ, t_u] /. h_x → 0;
    γ = D[Λ, h_x] /. t_u → 0;    δ = Λ /. h_x | t_u → 0;
    ϵ = 1 + α;
    B[ω * ϵ, α (1 + ⟨γ⟩ / ϵ) h_x t_u + β (1 + ⟨γ⟩ / ϵ) t_u
          + γ / ϵ h_x              + δ - γ * β / ϵ
     ] // βCollect];
gm_{a_,b_→c_}[β_] := β // sw_ab // hm_{ab→c} // tm_{ab→c};
B /: B[ω1_, Λ1_] B[ω2_, Λ2_] := B[ω1 * ω2, Λ1 + Λ2];
Rp_{a_,b_} := B[1, (X - 1) t_a h_b];
Rm_{a_,b_} := B[1, (X^-1 - 1) t_a h_b];
```

```
{β = B[ω, Sum[α_{10 i+j} t_i h_j, {i, {1, 2, 3}}, {j, {4, 5}}]],
  (β // tm_{12→1} // sw_14) == (β // sw_24 // sw_14 // tm_{12→1})}
```

$$\left\{ \begin{pmatrix} \omega & h_4 & h_5 \\ t_1 & \alpha_{14} & \alpha_{15} \\ t_2 & \alpha_{24} & \alpha_{25} \\ t_3 & \alpha_{34} & \alpha_{35} \end{pmatrix}, \text{True} \right\}$$



Some testing




```
{Rm_51 Rm_62 Rp_34 // gm_{14→1} // gm_{25→2} // gm_{36→3},
  Rp_61 Rm_24 Rm_35 // gm_{14→1} // gm_{25→2} // gm_{36→3}}
```

$$\left\{ \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix}, \begin{pmatrix} 1 & h_1 & h_2 \\ t_2 & -\frac{-1+X}{X} & 0 \\ t_3 & \frac{-1+X}{X} & -\frac{-1+X}{X} \end{pmatrix} \right\}$$

... divide and conquer!

```
β = Rm_{12,1} Rm_27 Rm_83 Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15}
```

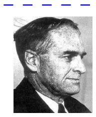
8_{17}

1	h_1	h_3	h_5	h_7	h_9	h_{11}	h_{13}	h_{15}
t_2	0	0	0	$-\frac{-1+X}{X}$	0	0	0	0
t_4	0	0	0	0	0	$-\frac{-1+X}{X}$	0	0
t_6	0	0	0	0	0	0	$-1+X$	0
t_8	0	$-\frac{-1+X}{X}$	0	0	0	0	0	0
t_{10}	0	0	0	0	0	0	0	$-1+X$
t_{12}	$-\frac{-1+X}{X}$	0	0	0	0	0	0	0
t_{14}	0	0	0	0	$-1+X$	0	0	0
t_{16}	0	0	$-1+X$	0	0	0	0	0

```
Do[β = β // gm_{1k→1}, {k, 2, 10}]; β
```

8_{17}, cont.

$\frac{1}{X}$	h_1	h_{11}	h_{13}	h_{15}
t_1	$-\frac{(-1+X)(1+X)}{X}$	$-(-1+X)(1-X+X^2)$	$(-1+X)(1-X+X^2)$	$-1+X$
t_{12}	$-\frac{-1+X}{X}$	0	0	0
t_{14}	$-1+X$	$\frac{(-1+X)^2(1-X+X^2)}{X}$	$-\frac{(-1+X)^2(1-X+X^2)}{X}$	0
t_{16}	$\frac{-1+X}{X}$	$(-1+X)^2$	$-\frac{(-1+X)^3}{X}$	0

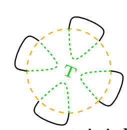


James Waddell Alexander

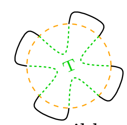
```
Do[β = β // gm_{1k→1}, {k, 11, 16}]; β
```

$$\left(-\frac{1-4X+8X^2-11X^3+8X^4-4X^5+X^6}{X^3} \right)$$

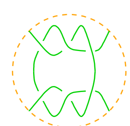
A Partial To Do List. 1. Where does it *more simply* come from?
2. Remove all the denominators.
3. How do determinants arise in this context?
4. Understand links ("meta-conjugacy classes").
5. Find the "reality condition".
6. Do some "Algebraic Knot Theory".
7. Categorify.
8. Do the same in other natural quotients of the v/w-story.



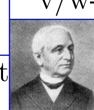
trivial



ribbon



example



"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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