The Pure Virtual Braid Group is Quadratic[1]

Abstract Generalities

Dror Bar-Natan and Peter Lee in Oregon, August 2011
http://www.math.toronto.edu/~drorbn/Talks/Oregon-1108/
foots & refs on PDF version, page 3

Let K be a unital algebra over a field $\mathbb{F}$ with char $\mathbb{F}=0$, and let $I\subset K$ be an "augmentation ideal"; so $K/I \xrightarrow[\epsilon]{\sim} \mathbb{F}$.

Definition. Say that K is quadratic if its associated graded $\operatorname{gr} K=\bigoplus_{p=0}^{\infty} I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A=q(K)=\langle V=I/I^2\rangle/\langle R_2=\ker(\bar\mu_2: V\otimes V\to I^2/I^3)\rangle$ be the "quadratic approximation" to K (q is a lovely functor). Then K is quadratic iff the obvious $\mu: A\to \operatorname{gr} K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

The Overall Strategy. Consider the "singularity tower" of (K,I) (here ":" means $\otimes_K$ and μ is (always) multiplication):

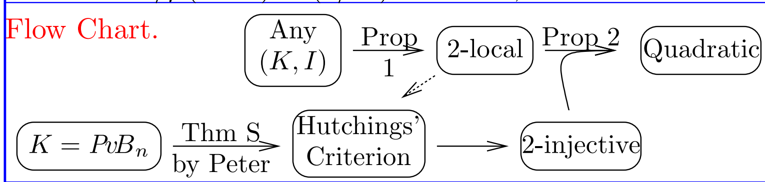
$$\dots\ I^{:p+1}\xrightarrow{\mu_{p+1}} I^{:p}\xrightarrow{\mu_p} I^{:p-1}\longrightarrow\cdots\longrightarrow K$$

We care as $\operatorname{im}(\mu^p=\mu_1\circ\cdots\circ\mu_p)=I^p$, so $I^p/I^{p+1}=\operatorname{im}\mu^p/\operatorname{im}\mu^{p+1}$. Hence we ask:

- What's $I^{:p}/\mu(I^{:p+1})$? • How injective is this tower?

Lemma. $I^{:p}/\mu(I^{:p+1})\simeq (I/I^2)^{\otimes p}=V^{\otimes p}$; set $\pi: I^{:p}\to V^{\otimes p}$.

Flow Chart.



Proposition 1. The sequence

$$\mathfrak{R}_p:=\bigoplus_{j=1}^{p-1}\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)\xrightarrow{\partial} I^{:p}\xrightarrow{\mu_p} I^{:p-1}$$

is exact, where $\mathfrak{R}_2:=\ker\mu: I^{:2}\to I$; so (K,I) is "2-local".

The Free Case. If J is an augmentation ideal in $K=F=\langle x_i\rangle$, define $\psi: F\to F$ by $x_i\mapsto x_i+\epsilon(x_i)$. Then $J_0:=\psi(J)$ is $\{w\in F:\deg w>0\}$. For J_0 it is easy to check that $\mathfrak{R}_2=\mathfrak{R}_p=0$, and hence the same is true for every J.

The General Case. If $K=F/\langle M\rangle$ (where M is a vector space of "moves") and $I\subset K$, then $I=J/\langle M\rangle$ where $J\subset F$. Then $I^{:p}=J^{:p}/\sum J^{:j-1}:\langle M\rangle:J^{:p-j}$ and we have

$$\begin{array}{ccc} J^{:p} & \xrightarrow[\text{1-1}]{\mu_F} & J^{:p-1}\\ {\scriptstyle\text{onto}}\downarrow{\scriptstyle\pi_p} & & {\scriptstyle\pi_{p-1}}\downarrow{\scriptstyle\text{onto}}\\ I^{:p}=J^{:p}/\sum J^{:}:\langle M\rangle:J^{:} & \xrightarrow{\mu} & I^{:p-1}=J^{:p-1}/\sum J^{:}:\langle M\rangle:J^{:}\end{array}$$

So[2] $\ker(\mu)=\pi_p\left(\mu_F^{-1}(\ker\pi_{p-1})\right)=\pi_p\left(\sum\mu_F^{-1}\left(J^{:}:\langle M\rangle:J^{:}\right)\right)=\sum\pi_p\left(J^{:}:\mu_F^{-1}\langle M\rangle:J^{:}\right)=\sum I^{:}:\mathfrak{R}_2:I^{:}=:\sum_{j=1}^{p-1}\mathfrak{R}_{p,j}$.

$\mathfrak{R}_2$ is simpler than may seem! It's an "augmentation bimodule" ($I\mathfrak{R}_2=0=\mathfrak{R}_2 I$ thus $xr=\epsilon(x)r=r\epsilon(x)=rx$ for $x\in K$ and $r\in\mathfrak{R}_2$), and hence $\mathfrak{R}_2=\pi_2(\mu_F^{-1}M)$.

$$\begin{array}{ccc} J^{:2} & \xrightarrow[\text{1-1}]{\mu_F} & J\supset M\\ \downarrow{\scriptstyle\pi_2} & & \downarrow{\scriptstyle\pi_1}\\ I^{:2} & \xrightarrow{\mu} & I=J/\langle M\rangle\end{array}$$

$\mathfrak{R}_p$ is simpler than may seem! In $\mathfrak{R}_{p,j}=I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}$ the I factors may be replaced by $V=I/I^2$. Hence

$$\mathfrak{R}_p\simeq\bigoplus_{j=1}^{p-1}V^{\oplus j-1}\otimes\pi_2(\mu_F^{-1}M)\otimes V^{\otimes p-j-1}.$$

Claim. $\pi(\mathfrak{R}_{p,j})=R_{p,j}$; namely,

$$\pi\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)=V^{\otimes j-1}\otimes R_2\otimes V^{\otimes p-j-1}.$$

Why Care?

• In abstract generality, $\operatorname{gr} K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly. • In some concrete (somewhat generalized) knot theoretic cases, A is a space of "universal Lie algebraic formulas" and the "primary approach" for proving (strong) quadraticity, constructing an appropriate homomorphism $Z: K\to\hat A$, becomes wonderful mathematics:

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

2-Injectivity. A (one-sided infinite) sequence

$$\dots\longrightarrow K_{p+1}\xrightarrow{\delta_{p+1}} K_p\xrightarrow{\delta_p}\cdots\longrightarrow K_0=K$$

is "injective" if for all $p>0$, $\ker\delta_p=0$. It is "2-injective" if its "1-reduction"

$$\dots\longrightarrow\frac{K_{p+1}}{\ker\delta_{p+1}}\xrightarrow{\bar\delta_{p+1}}\frac{K_p}{\ker\delta_p}\xrightarrow{\bar\delta_p}\frac{K_{p-1}}{\ker\delta_{p-1}}\longrightarrow\cdots$$

is injective; i.e. if for all p, $\ker(\delta_p\circ\delta_{p+1})=\ker\delta_{p+1}$. A pair (K,I) is "2-injective" if its singularity tower is 2-injective.

Proposition 2. If (K,I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{:p+1}}{\ker\mu_{p+1}}\xrightarrow{\mu_{p+1}}\frac{I^{:p}}{\ker\mu_p}\xrightarrow{\mu_p}\cdots\longrightarrow K$, get $\frac{I^p}{I^{p+1}}\simeq\frac{I^{:p}/\ker\mu_p}{\mu(I^{:p+1}/\ker\mu_{p+1})}\simeq\frac{I^{:p}}{\mu(I^{:p+1})+\ker\mu_p}$. But $\frac{I^{:p}}{\mu(I^{:p+1})}\simeq(I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p}/\sum\left(I^{:j-1}:\mathfrak{R}_2:I^{:p-j-1}\right)$. But that's the degree p piece of $q(K)$.

The X Lemma (inspired by [Hut]).

$$\begin{array}{ccccc} A_0 & & & & C_0\\ & \searrow^{\alpha_0} & & \nearrow^{\beta_0} & \\ & & B & & \\ & \nearrow_{\alpha_1} & & \searrow_{\beta_1} & \\ A_1 & & & & C_1\end{array}$$


Michael Hutchings
will compute homology for food

If the above diagram is Conway (⋈) exact, then its two diagonals have the same "2-injectivity defect". That is, if $A_0\to B\to C_0$ and $A_1\to B\to C_1$ are exact, then $\ker(\beta_1\circ\alpha_0)/\ker\alpha_0\simeq\ker(\beta_0\circ\alpha_1)/\ker\alpha_1$.

Proof. $\frac{\ker(\beta_1\circ\alpha_0)}{\ker\alpha_0}\xrightarrow[\alpha_0]{\sim}\ker\beta_1\cap\operatorname{im}\alpha_0=\ker\beta_0\cap\operatorname{im}\alpha_1\xleftarrow[\alpha_1]{\sim}\frac{\ker(\beta_0\circ\alpha_1)}{\ker\alpha_1}$.

The Hutchings Criterion [Hut]. The singularity tower of (K,I) is 2-injective iff on the right, $\ker(\pi\circ\partial)=\ker(\partial)$. That is, iff every "diagrammatic syzygy" is also a "topological syzygy".

$$\begin{array}{ccccc} \mathfrak{R}_p & & & & I^{:p-1}\\ & \searrow^{\partial} & & \nearrow^{\mu_p} & \\ & & I^{:p} & & \\ & \nearrow^{\mu_{p+1}} & & \searrow^{\pi} & \\ I^{:p+1} & & & & V^{\otimes p}\end{array}$$

Conclusion. We need to know that (K,I) is "syzygy complete" — that every diagrammatic syzygy is also a topological syzygy, that $\ker(\pi\circ\partial)=\ker(\partial)$.

The Pure Virtual Braid Group is Quadratic, II

Examples and Interpretations

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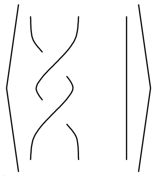
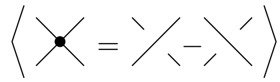
Example. (goes back to [Koh])

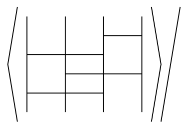
T. Kohno

$K =$  $I =$ 

$(K/I^{p+1})^\star = $ (invariants of type p) $=: \mathcal{V}_p$

$(I^p/I^{p+1})^\star = \mathcal{V}_p/\mathcal{V}_{p-1} \quad V = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle | \; |\!\!-\!\!|\!\!-\!\!| \rangle$

$\ker \bar\mu_2 = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle$4T relations$\rangle$

$A = q(K) = \left(\text{horizontal chord diagrams mod 4T}\right) =$  4T

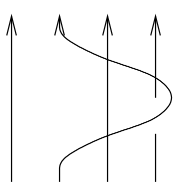
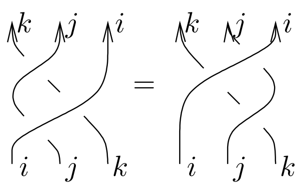
Z: universal finite type invariant, the Kontsevich integral.

PvB_n is the group

$$\langle \sigma_{ij} : 1 \le i \ne j \le n \rangle \Big/ \begin{matrix} \sigma_{ij}\sigma_{ik}\sigma_{jk} = \sigma_{jk}\sigma_{ik}\sigma_{ij} \\ \sigma_{ij}\sigma_{kl} = \sigma_{kl}\sigma_{ij} \end{matrix}$$

L. Kauffman [Kau, KL]

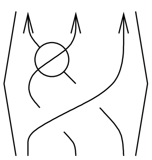
of "pure virtual braids" ("braids when you look", "blunder braids"):

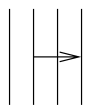
$\sigma_{24} =$  R3: 

The Main Theorem [Lee]. PvB_n is quadratic.

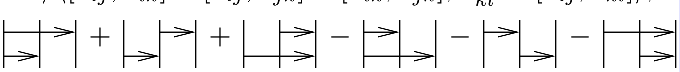
$A_n = q(PvB_n)$. [GPV]

Goussarov-Polyak-Viro

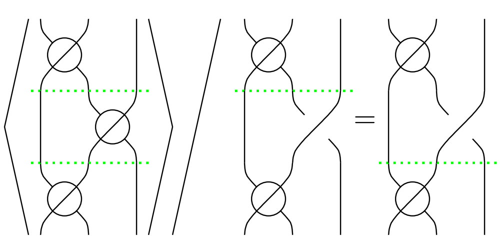
$I =$  with $\tilde\sigma_{ij} = \sigma_{ij} - 1$ = (crossing) − (virtual crossing), the "semi-virtual crossing".

$V = I/I^2 = \langle$ v-braids with one semi-virtual crossing $\rangle / ($crossing $=$ virtual crossing$)$ $= \langle a_{ij} \rangle_{1 \le i \ne j \le n}$ $\qquad a_{24} =$ 

$A_n = TV/\langle [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}],\ c^{ij}_{kl} = [a_{ij}, a_{kl}] \rangle,$

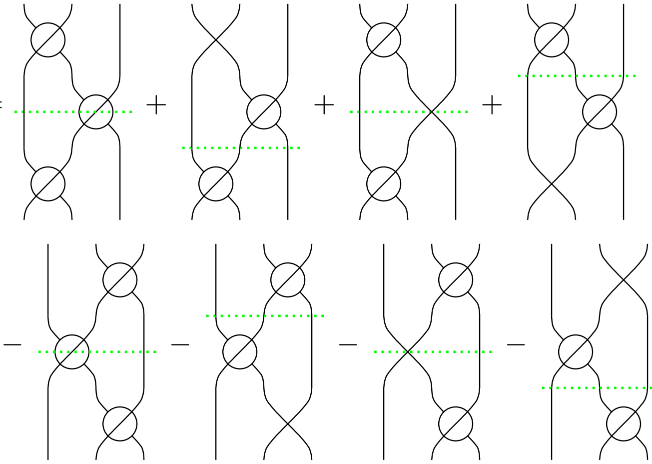
$y_{ijk} =$ 

$I^{:p}$.



James Gillespie's Sightline #2 (1984) is a syzygy, and (arguably) Toronto's largest sculpture. Find it next to University of Toronto's Hart House.

$\mathfrak{R}_2(PvB_n)$ is generated as a vector space by C^{ij}_{kl} and

$Y_{ijk} :=$ 

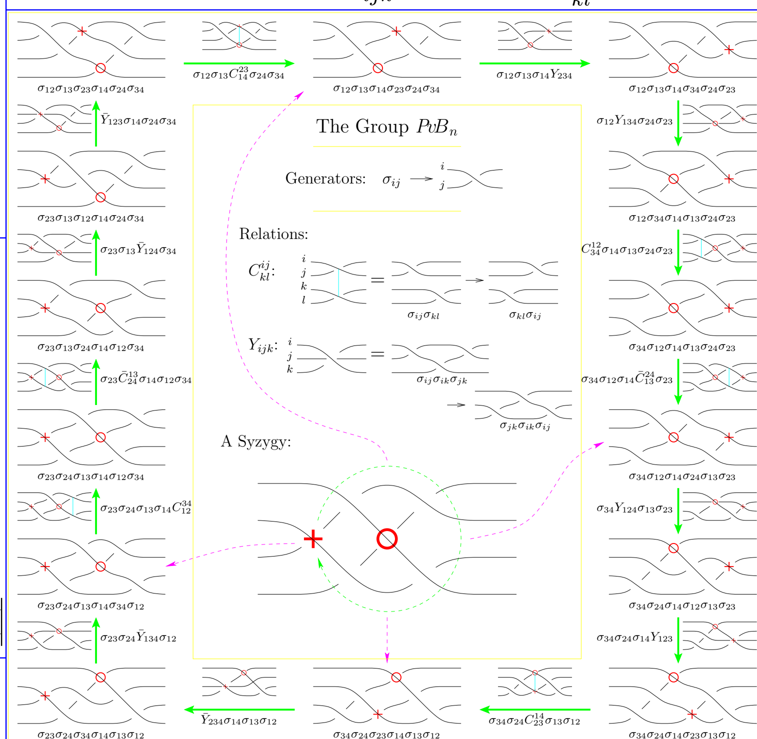
Syzygy Completeness, for PvB_n, means:

$$\mathfrak{R}_p = \bigoplus_{j=1}^{p-1} \mathfrak{R}_{p,j} \xrightarrow{\partial} I^{:p} \xrightarrow{\pi} V^{\otimes p}$$

$\{\tilde\sigma_{12} : \underline{Y_{345}} : \tilde\sigma_{67} : \ldots\} \longrightarrow$

$\{\tilde\sigma_{12} : Y_{345} : \tilde\sigma_{67} : \ldots\} \longrightarrow \{a_{12} y_{345} a_{67} \ldots\}$

Is every relation between the y_{ijk}'s and the c^{ij}_{kl}'s also a relation between the Y_{ijk}'s and the C^{ij}_{kl}'s?



Theorem S. Let D be the free associative algebra generated by symbols a_{ij}, y_{ijk} and c^{ij}_{kl}, where $1 \le i, j, k, l \le n$ are distinct integers. Let D_0 be the part of D with only a_{ij} symbols and let D_1 be the span of the monomials in D having only a_{ij} symbols, with exactly one exception that may be either a y_{ijk} or a c^{ij}_{kl}. Let $\partial : D_1 \to D_0$ be the map defined by

$$\begin{aligned} y_{ijk} &\mapsto [a_{ij}, a_{ik}] + [a_{ij}, a_{jk}] + [a_{ik}, a_{jk}], \\ c^{ij}_{kl} &\mapsto [a_{ij}, a_{kl}]. \end{aligned}$$

Then $\ker \partial$ is generated by a family of elements readable from the picture above and by a few similar but lesser families.

Footnotes

1. Following a homonymous paper and thesis by Peter Lee [Lee]. All serious work here is his and was extremely patiently explained by him to DBN. Page design by the latter.
2. The proof presented here is broken. Specifically, at the very end of the proof of the "general case" of Proposition 1 the sum that makes up $\ker \pi_{p-1}$ is interchaged with μ_F^{-1}. This is invalid; in general it is not true that $T^{-1}(U+V) = T^{-1}(U) + T^{-1}(V)$, when T is a linear transformation and U and V are subspaces of its target space. We thank Alexander Polishchuk for noting this gap. A handwritten non-detailed fix can be found at `http://katlas.math.toronto.edu/drorbn/AcademicPensieve/Projects/Quadraticity/`, especially under "Oregon Handout Post Mortem". A fuller fix will be made available at a later time.

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[BN1] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[BN2] D. Bar-Natan, *Facts and Dreams About v-Knots and Etingof-Kazhdan,* talk presented at the Swiss Knots 2011 conference. Video and more at `http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/`.

[BN3] D. Bar-Natan, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne,* paper and related files at `http://www.math.toronto.edu/~drorbn/papers/WKO/`.

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[Lee] P. Lee, *The Pure Virtual Braid Group is Quadratic,* in preparation. See links at `http://www.math.toronto.edu/drorbn/Talks/Oregon-1108/`.

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Figuring out $\mathfrak{R}_2$ and R_2.

$$\ker\mu = \pi_2\left(\mu_F^{-1}(M)\right)$$

is in principle computable, and then R_2 follows as $V^{\otimes 2} = (I/I^2)^{\otimes 2} = I^{:2}/\mu_3(I^{:3})$.

$$\begin{array}{ccc} J^{:2} & \xrightarrow[\text{1-1}]{\mu_F} & J \supset M \\ \downarrow \pi_2 & & \downarrow \pi_1 \\ I^{:2} & \xrightarrow{\mu} & I = J/\langle M\rangle \end{array}$$

Broken?

Just for fun.