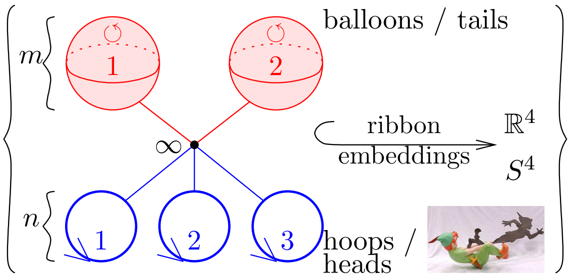
Balloons and Hoops and their Universal Finite−Type Invariant, BF Theory, and an Ultimate Alexander Invariant

Dror Bar−Natan in Oxford, January 2013
$\omega\epsilon\beta$:=http://www.math.toronto.edu/~drorbn/Talks/Oxford-130121

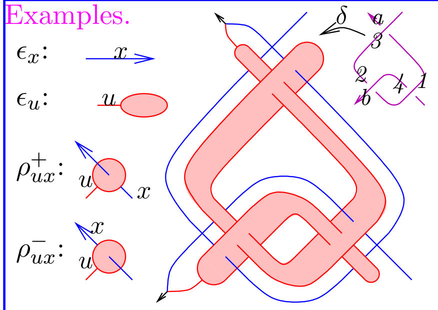
Scheme. • Balloons and hoops in $\mathbb{R}^4$, algebraic structure and relations with 3D.
• An ansatz for a "homomorphic" invariant: computable, related to finite-type and to BF.
• Reduction to an "ultimate Alexander invariant".

$\mathcal{K}^{bh}(m,n)$.

$\omega\epsilon\beta$/decorationideas



Examples.



I mean business!

```
T0 = Rm[3, a] Rp[b, 2] Rp[1, 4];
ζ = T0 // dm[2, 1, 1] // dm[4, b, b] //
    dm[1, a, a] // dm[3, a, a];
ζ[{5}] /. {w_LW :> (Deg[w] + 1) ! w,
  w_CW :> Deg[w] ! w}

μ[CWS[-[a], -2 [ab], -3 [aab] - 3 [abb],
  -4 [aaab] + 42 [aabb] - 60 [abab] - 4 [abbb],
  -5 [aaaab] + 110 [aaabb] - 180 [aabab] +
   110 [aabbb] - 180 [ababb] - 5 [abbbb]],
 h[b] LS[2 ⟨a⟩, 0, -24 ⟨aab⟩,
    -60 ⟨aaab⟩ + 60 ⟨aabb⟩, -120 ⟨aaaab⟩ +
     900 ⟨aaabb⟩ + 360 ⟨aabab⟩ - 120 ⟨aabbb⟩] +
  h[a] LS[-2 ⟨a⟩ + 2 ⟨b⟩, 9 ⟨ab⟩, 26 ⟨aab⟩ -
     26 ⟨abb⟩, 60 ⟨aaab⟩ - 255 ⟨aabb⟩ + 60 ⟨abbb⟩,
    119 ⟨aaaab⟩ - 1504 ⟨aaabb⟩ + 118 ⟨aabab⟩ +
     1504 ⟨aabbb⟩ + 1386 ⟨ababb⟩ - 119 ⟨abbbb⟩]]
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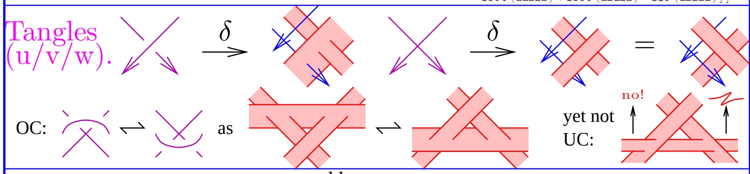
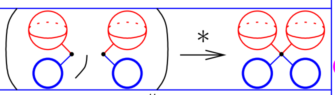
Tangles (u/v/w).



• δ injects u-Knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
• δ maps v/w-tangles map to $\mathcal{K}^{bh}$; the kernel contains Reidemeister moves and the "overcrossings commute" relation, and conjecturally, that's all. Allowing punctures and cuts, δ is onto.

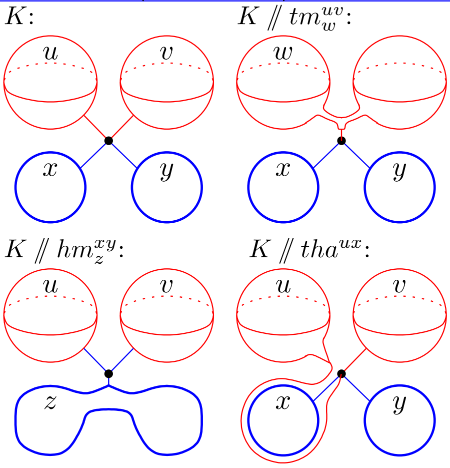
Operations Punctures & Cuts | Connected Sums.



Meta-Group-Action. If X is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and π_1 acts on π_2.

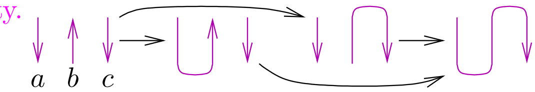
"MGA"

("//" is newspeak for "apply an operator" and for "composition left to right")

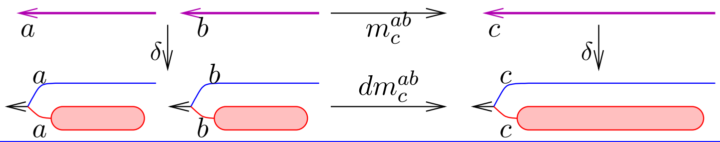


Properties.
• Associativities: $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$, for $m = tm, hm$.
• Action axiom t: $tm^{uv}_w /\!/ tha^{wx} = tha^{ux} /\!/ tha^{vx} /\!/ tm^{uv}_w$,
• Action axiom h: $hm^{xy}_z /\!/ tha^{uz} = tha^{ux} /\!/ tha^{uy} /\!/ hm^{xy}_z$.
• SD Product: $dm^{ab}_c := tha^{ab} /\!/ tm^{ab}_c /\!/ hm^{ab}_c$ is associative.

Meta-associativity. $m^{ab}_a /\!/ m^{ac}_a = m^{bc}_b /\!/ m^{ab}_a$.

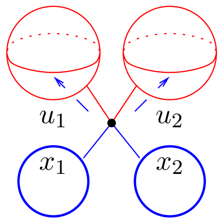


Tangle concatenations $\to$ $\pi_1 \ltimes \pi_2$.



Thus we seek homomorphic invariants of $\mathcal{K}^{bh}$!

Invariant #0. With Π_1 denoting "honest π_1", map $\gamma \in \mathcal{K}^{bh}(m,n)$ to the triple $(\Pi_1(\gamma^c), (u_i), (x_j))$, where the meridian of the balls u_i normally generate Π_1, and the "longtitudes" x_j are some elements of Π_1.
$*$ acts like $*$, tm acts by "merging" two meridians/generators, hm acts by multiplying two longtitudes, and tha^{ux} acts by "conjugating a meridian by a longitude":



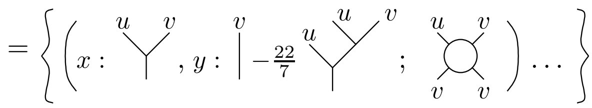
Not computable! (but nearly)

$$(\Pi, (u,\ldots), (x,\ldots)) \mapsto (\Pi * \langle \bar{u}\rangle / (u = x\bar{u}x^{-1}), (\bar{u},\ldots), (x,\ldots))$$

Failure #0. Can we write the x's as free words in the u's? If $x = uv$, compute $x /\!/ tha^{ux}$:

$$x = uv \to \bar{u}v = u^x v = u^{\bar{u}v}v = u^{u^x v}v = u^{u^{u^x v}v}v = \cdots$$

The Meta-Group-Action M. Let T be a set of "tail labels" ("balloon colours"), and H a set of "head labels" ("hoop colours"). Let $FL = FL(T)$ and $FA = FA(T)$ be the (completed graded) free Lie and free associative algebras on generators T and let $CW = CW(T)$ be the (completed graded) vector space of cyclic words on T, so there's $\mathrm{tr} : FA \to CW$. Let $M(T,H) := \{(\bar\lambda = (x : \lambda_x)_{x\in H}; \omega) : \lambda_x \in FL, \omega \in CW\}$



$\omega\epsilon\beta$/antiq-ave

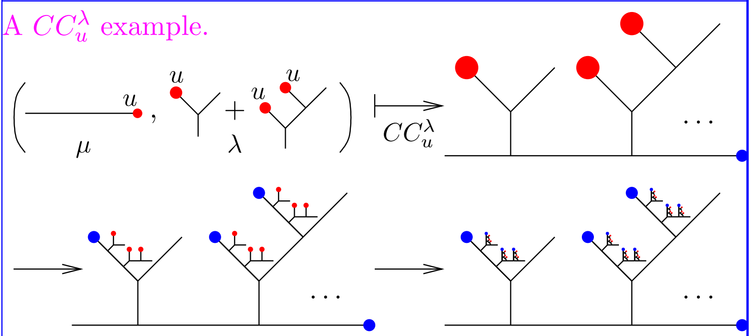
Operations. Set $(\bar\lambda_1;\omega_1) * (\bar\lambda_2;\omega_2) := (\bar\lambda_1 \cup \bar\lambda_2; \omega_1+\omega_2)$ and with $\mu = (\bar\lambda;\omega)$ define

$$tm^{uv}_w : \mu \mapsto \mu /\!/ (u, v \mapsto w),$$

$$hm^{xy}_z : \mu \mapsto \left(\left(\ldots, \widehat{x : \lambda_x, y : \lambda_y}, \ldots, z : \mathrm{bch}(\lambda_x, \lambda_y)\right); \omega\right)$$

$$tha^{ux} : \mu \mapsto \underbrace{\overbrace{\mu \;/\!/\!/}^{\text{"stable apply"}} (u \mapsto e^{\mathrm{ad}\,\lambda_x}(\bar u)) /\!/ (\bar u \mapsto u)}_{\mu /\!/ CC^{\lambda_x}_u} + \underbrace{(0; J_u(\lambda_x))}_{\text{the "J-spice"}}$$

A CC^λ_u example.



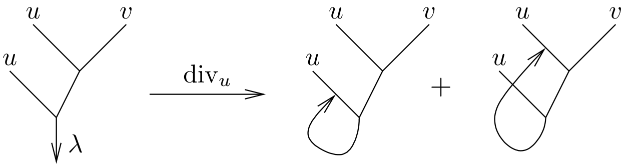
Balloons and Hoops and their Universal Finite−Type Invariant, 2

The Meta-Cocycle J. Set $J_u(\lambda) := J(1)$ where

$$J(0) = 0, \qquad \lambda_s = \lambda \,/\!/\, CC_u^{s\lambda},$$

$$\frac{dJ(s)}{ds} = (J(s) \,/\!/\, \operatorname{der}(u \mapsto [\lambda_s, u])) + \operatorname{div}_u \lambda_s,$$

and where $\operatorname{div}_u \lambda := \operatorname{tr}(u\sigma_u(\lambda))$, $\sigma_u(v) := \delta_{uv}$, $\sigma_u([\lambda_1,\lambda_2]) := \iota(\lambda_1)\sigma_u(\lambda_2) - \iota(\lambda_2)\sigma_u(\lambda_1)$ and ι is the inclusion $FL \hookrightarrow FA$:



Claim. $CC_u^{\operatorname{bch}(\lambda_1,\lambda_2)} = CC_u^{\lambda_1} \,/\!/\, CC_u^{\lambda_2 /\!/ CC_u^{\lambda_1}}$ and

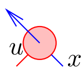
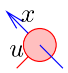
$$J_u(\operatorname{bch}(\lambda_1,\lambda_2)) = J_u(\lambda_1) \,/\!/\, CC_u^{\lambda_2/\!/CC_u^{\lambda_1}} + J_u(\lambda_2 \,/\!/\, CC_u^{\lambda_1}),$$

and hence tm, hm, and tha form a meta-group-action.

Why ODEs? **Q.** Find f s.t. $f(x+y) = f(x)f(y)$.
A. $\frac{df(s)}{ds} = \frac{d}{d\epsilon} f(s+\epsilon) = \frac{d}{d\epsilon} f(s)f(\epsilon) = f(s)C$. Now solve this ODE using Picard's theorem or power series.

Alekseev Torossian

The Invariant ζ. Set $\zeta(\rho^\pm) = (\pm u_x; 0)$. This at least defines an invariant of u/v/w-tangles, and if the topologists will deliver a "Reidemeister" theorem, it is well defined on $\mathcal{K}^{bh}$.

ζ:  $\longmapsto \left(x : +\Big|^u ; 0\right)$  $\longmapsto \left(x : -\Big|^u ; 0\right)$

Theorem. ζ is (the log of) a universal finite type invariant (a homomorphic expansion) of w-tangles.

Tensorial Interpretation. Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau : FL(T) \to \operatorname{Fun}(\oplus_T \mathfrak{g} \to \mathfrak{g})$ and $\tau : CW(T) \to \operatorname{Fun}(\oplus_T \mathfrak{g})$. Together, $\tau : M(T,H) \to \operatorname{Fun}(\oplus_T \mathfrak{g} \to \oplus_H \mathfrak{g})$, and hence

$$e^\tau : M(T,H) \to \operatorname{Fun}(\oplus_T \mathfrak{g} \to \mathcal{U}^{\otimes H}(\mathfrak{g})).$$

ζ and BF Theory. Let A denote a $\mathfrak{g}$-connection on S^4 with curvature F_A, and B a $\mathfrak{g}^*$-valued 2-form on S^4. For a hoop γ_x, let $\operatorname{hol}_{\gamma_x}(A) \in \mathcal{U}(\mathfrak{g})$ be the holonomy of A along γ_x. For a ball γ_u, let $\mathcal{O}_{\gamma_u}(B) \in \mathfrak{g}^*$ be the integral of B (transported via A to ∞) on γ_u.

Cattaneo

Loose Conjecture. For $\gamma \in \mathcal{K}(T,H)$,

$$\int \mathcal{D}A\mathcal{D}B e^{\int B\wedge F_A} \prod_u e^{\mathcal{O}_{\gamma_u}(B))} \bigotimes_x \operatorname{hol}_{\gamma_x}(A) = e^\tau(\zeta(\gamma)).$$

That is, ζ is a complete evaluation of the BF TQFT.

Issues. How exactly is B transported via A to ∞? How does the ribbon condition arise? Or if it doesn't, could it be that ζ can be generalized??

The β quotient, 1. • Arises when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra.
• Arises when reducing by relations satisfied by the weight system of the Alexander polynomial.

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

Paper in progress: $\omega\epsilon\beta$/kbh

The β quotient, 2. Let $R = \mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta := R \otimes T$ with central R and with $[u,v] = c_u v - c_v u$ for $u, v \in T$. Then $FL \to L_\beta$ and $CW \to R$. Under this,

$$\mu \to (\bar\lambda; \omega) \quad \text{with } \bar\lambda = \sum_{x\in H, u\in T} \lambda_{ux} ux, \quad \lambda_{ux}, \omega \in R,$$

$$\operatorname{bch}(u,v) \to \frac{c_u + c_v}{e^{c_u+c_v} - 1}\left(\frac{e^{c_u} - 1}{c_u} u + e^{c_u} \frac{e^{c_v}-1}{c_v} v \right),$$

if $\lambda = \sum \lambda_v v$ then with $c_\lambda := \sum \lambda_v c_v$,

$$u \,/\!/\, CC_u^\lambda = \left(1 + c_u\lambda_u \frac{e^{c_\lambda}-1}{c_\lambda}\right)^{-1} \left(e^{c_\lambda} u - c_u \frac{e^{c_\lambda}-1}{c_\lambda} \sum_{v \neq u} \lambda_v v \right),$$

$\operatorname{div}_u \lambda = c_u \lambda_u$, and the ODE for J integrates to

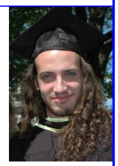
$$J_u(\lambda) = \log\left(1 + \frac{e^{c_\lambda}-1}{c_\lambda} c_u \lambda_u\right),$$

so ζ is formula-computable to all orders! Can we simplify?

Repackaging. Given $((x : \lambda_{ux}); \omega)$, set $c_x := \sum_v c_v \lambda_{vx}$, replace $\lambda_{ux} \to \alpha_{ux} := c_u \lambda_{ux} \frac{e^{c_x}-1}{c_x}$ and $\omega \to \log \omega$, use $t_u = e^{c_u}$, and write α_{ux} as a matrix. Get "β calculus".

β Calculus. Let $\beta(H,T)$ be

$$\left\{ \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline u & \alpha_{ux} & \alpha_{uy} & \cdot \\ v & \alpha_{vx} & \alpha_{vy} & \cdot \\ \vdots & \cdot & \cdot & \cdot \end{array} \;\middle|\; \begin{array}{l} \omega \text{ and the } \alpha_{ux}\text{'s are} \\ \text{rational functions in} \\ \text{variables } t_u, \text{ one for} \\ \text{each } u \in T. \end{array} \right\},$$



In preparation, Selmani & B−N.

$$tm^{uv}_w : \begin{array}{c|c} \omega & \cdots \\ \hline u & \alpha \\ v & \beta \\ \vdots & \gamma \end{array} \mapsto \begin{array}{c|c} \omega & \cdots \\ \hline w & \alpha+\beta \\ \vdots & \gamma \end{array}, \qquad \begin{array}{c|c} \omega_1 & H_1 \\ \hline T_1 & \alpha_1 \end{array} \cup \begin{array}{c|c} \omega_2 & H_2 \\ \hline T_2 & \alpha_2 \end{array} = \begin{array}{c|cc} \omega_1\omega_2 & H_1 & H_2 \\ \hline T_1 & \alpha_1 & 0 \\ T_2 & 0 & \alpha_2 \end{array},$$

$$hm^{xy}_z : \begin{array}{c|ccc} \omega & x & y & \cdots \\ \hline \vdots & \alpha & \beta & \gamma \end{array} \mapsto \begin{array}{c|cc} \omega & z & \cdots \\ \hline \vdots & \alpha + \beta + \langle\alpha\rangle\beta & \gamma \end{array},$$

$$tha^{ux} : \begin{array}{c|cc} \omega & x & \cdots \\ \hline u & \alpha & \beta \\ \vdots & \gamma & \delta \end{array} \mapsto \begin{array}{c|cc} \omega\epsilon & x & \cdots \\ \hline u & \alpha(1 + \langle\gamma\rangle/\epsilon) & \beta(1+\langle\gamma\rangle/\epsilon) \\ \vdots & \gamma/\epsilon & \delta - \gamma\beta/\epsilon \end{array},$$

where $\epsilon := 1 + \alpha$, $\langle\alpha\rangle := \sum_v \alpha_v$, and $\langle\gamma\rangle := \sum_{v\neq u} \gamma_v$, and let

$$R^+_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u - 1 \end{array} \qquad R^-_{ux} := \begin{array}{c|c} 1 & x \\ \hline u & t_u^{-1} - 1 \end{array}.$$

On long knots, ω is the Alexander polynomial!

Why bother? (1) An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination!). *If there should be an Alexander invariant to have an algebraic categorification, it is this one!*

See also $\omega\epsilon\beta$/regina, $\omega\epsilon\beta$/gwu.

Why bother? (2) Related to A-T, K-V, and E-K, should have vast generalization beyond w-knots and the Alexander polynomial. See also $\omega\epsilon\beta$/wko, $\omega\epsilon\beta$/caen, $\omega\epsilon\beta$/swiss.