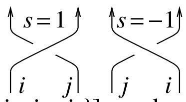
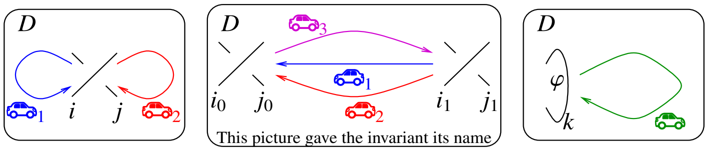
Theorem. With $c = (s,i,j)$, $c_0 = (s_0,i_0,j_0)$, and $c_1 = (s_1,i_1,j_1)$ denoting crossings, there is a quadratic $R_{11}(c) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha,\beta \in \{i,j\}]$, a cubic $R_{12}(c_0,c_1) \in \mathbb{Q}(T_\nu)[g_{\nu\alpha\beta} : \alpha,\beta \in \{i_0,j_0,i_1,j_1\}]$, and a linear $\Gamma_1(\varphi,k)$ such that the following is a knot invariant:

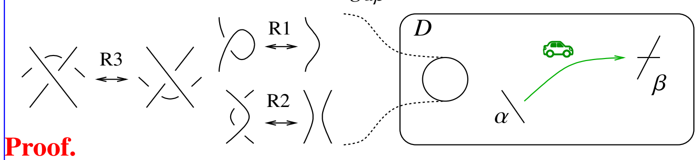


$$\theta(D) := \underbrace{\Delta_1\Delta_2\Delta_3}_{\text{normalization, see later}} \left(\sum_c R_{11}(c) + \sum_{c_0,c_1} R_{12}(c_0,c_1) + \sum_k \Gamma_1(\varphi_k,k) \right),$$

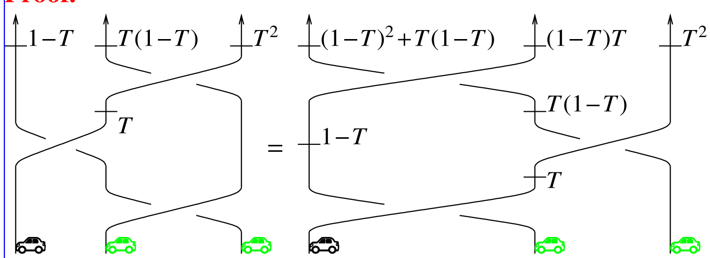


If these pictures remind you of Feynman diagrams, it's because they are Feynman diagrams [BN2].

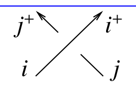
Lemma 1. The traffic function $g_{\alpha\beta}$ is a "relative invariant":



Proof.



Lemma 2. With $k^+ := k+1$, the "g-rules" hold near a crossing $c = (s,i,j)$:



$$g_{j\beta} = g_{j^+\beta} + \delta_{j\beta} \quad g_{i\beta} = T^s g_{i^+\beta} + (1-T^s) g_{j^+\beta} + \delta_{i\beta} \quad g_{2n^+,\beta} = \delta_{2n^+,\beta}$$

$$g_{\alpha i^+} = T^s g_{\alpha i} + \delta_{\alpha i^+} \quad g_{\alpha j^+} = g_{\alpha j} + (1-T^s) g_{\alpha i} + \delta_{\alpha j^+} \quad g_{\alpha,1} = \delta_{\alpha,1}$$

Corollary 1. G is easily computable, for $AG = I\ (= GA)$, with A the $(2n+1)\times(2n+1)$ identity matrix with additional contributions:

$$c = (s,i,j) \mapsto \begin{array}{c|cc} A & \text{col } i^+ & \text{col } j^+ \\ \hline \text{row } i & -T^s & T^s - 1 \\ \text{row } j & 0 & -1 \end{array}$$

For the trefoil example, we have:

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & T & 1 & T & 1 & T & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & -\frac{(T-1)T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note. The Alexander polynomial Δ is given by

$$\Delta = T^{(-\varphi-w)/2}\det(A), \qquad \text{with } \varphi = \textstyle\sum_k \varphi_k,\ w = \sum_c s.$$

We also set $\Delta_\nu := \Delta(T_\nu)$ for $\nu = 1,2,3$.

Questions, Conjectures, Expectations, Dreams.

Question 1. What's the relationship between Θ and the Garoufalidis-Kashaev invariants [GK, GL]?

Conjecture 2. On classical (non-virtual) knots, θ always has hexagonal (D_6) symmetry.

Conjecture 3. θ is the ϵ^1 contribution to the "solvable approximation" of the sl_3 universal invariant, obtained by running the quantization machinery on the double $\mathcal{D}(\mathfrak{b}, b, \epsilon\delta)$, where $\mathfrak{b}$ is the Borel subalgebra of sl_3, b is the bracket of $\mathfrak{b}$, and δ the cobracket. See [BV2, BN1, Sch]

Conjecture 4. θ is equal to the "two-loop contribution to the Kontsevich Integral", as studied by Garoufalidis, Rozansky, Kricker, and in great detail by Ohtsuki [GR, Ro1, Ro2, Ro3, Kr, Oh].

Fact 5. θ has a perturbed Gaussian integral formula, with integration carried out over over a space $6E$, consisting of 6 copies of the space of edges of a knot diagram D. See [BN2].

Conjecture 6. For any knot K, its genus $g(K)$ is bounded by the T_1-degree of θ: $2g(K) \geq \deg_{T_1} \theta(K)$.

Conjecture 7. $\theta(K)$ has another perturbed Gaussian integral formula, with integration carried out over over the space $6H_1$, consisting of 6 copies of $H_1(\Sigma)$, where Σ is a Seifert surface for K.

Expectation 8. There are many further invariants like θ, given by Green function formulas and/or Gaussian integration formulas. One or two of them may be stronger than θ and as computable.

Dream 9. These invariants can be explained by something less foreign than semisimple Lie algebras.

Dream 10. θ will have something to say about ribbon knots.

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Video and more at `http://www.math.toronto.edu/~drorbn/Talks/Toronto-241030`.