Knot Invariants from Finite Dimensional Integration


joint with R. van der Veen

Abstract. For the purpose of today, an "I-Type Knot Invariant" is a knot invariant computed from a knot diagram by integrating the exponential of a *perturbed Gaussian* Lagrangian which is a sum over the features of that diagram (crossings, edges, faces) of locally defined quantities, over a product of finite dimensional spaces associated to those same features.

Q. Are there any such things? **A.** Yes.

Q. Are they any good? **A.** They are the strongest we know per CPU cycle, and are excellent in other ways too.

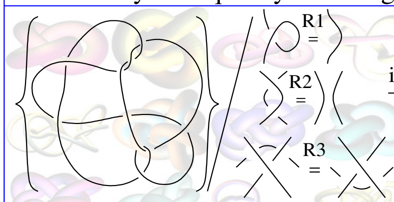
Q. Didn't Witten do that back in 1988 with path integrals?

A. No. His constructions are infinite dimensional and far from rigorous.

Q. But integrals belong in analysis!

A. Ours only use squeaky-clean algebra.

Knots.



invariants ⟶ Something simple: numbers, polynomials, matrices, etc.

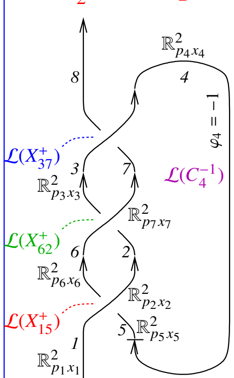
The Good. 1. At the centre of low dimensional topology.
2. "Invariants" connect to pretty much all of algebra.

The Agony. 1&2 don't talk to each other.

- Not enough topological applications for all these invariants.
- The fancy algebra doesn't arise naturally within topology.

⟹ We're still missing something about the relationship between knots and algebra.

The $sl_2^{\epsilon^2}$ Example. With T an indeterminate and with $\epsilon^2=0$:



$$\longrightarrow Z=\oint_{\mathbb{R}^{14}_{p_ix_i}} \mathcal{L}(X^+_{15})\mathcal{L}(X^+_{62})\mathcal{L}(X^+_{37})\mathcal{L}(C_4^{-1})$$

measure on $\mathbb{R}$ is $(2\pi)^{-1/2}\cdot$*standard*

where $\mathcal{L}(X^s_{ij})=T^{s/2}e^{L(X^s_{ij})}$ and $\mathcal{L}(C^\varphi_i)=T^{\varphi/2}e^{L(C^\varphi_i)}$, and

$$L(X^s_{ij})=x_i(p_{i+1}-p_i)+x_j(p_{j+1}-p_j)+(T^s-1)x_i(p_{i+1}-p_{j+1})+\frac{\epsilon s}{2}\left(x_i(p_i-p_j)\begin{pmatrix}(T^s-1)x_ip_j\\+2(1-x_jp_j)\end{pmatrix}-1\right)$$

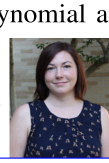
$$L(C^\varphi_i)=x_i(p_{i+1}-p_i)+\epsilon\varphi(1/2-x_ip_i)$$

So $Z=T\oint e^{L(\circledast)}dp_1\dots dp_7dx_1\dots dx_7$, where $L(\circledast)=\sum_{i=1}^7 x_i(p_{i+1}-p_i)+(T-1)(x_1(p_2-p_6)+x_6(p_7-p_3)+x_3(p_4-p_8))$

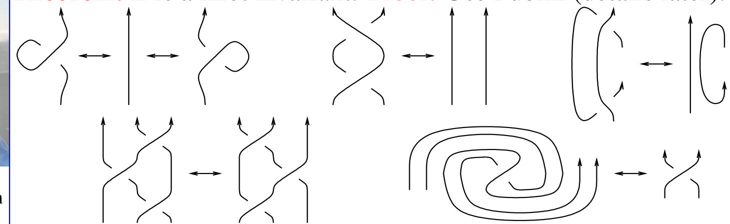
$$+\frac{\epsilon}{2}\begin{pmatrix}x_1(p_1-p_5)((T-1)x_1p_5+2(1-x_5p_5))-1\\+x_6(p_6-p_2)((T-1)x_6p_2+2(1-x_2p_2))-1\\+x_3(p_3-p_7)((T-1)x_3p_7+2(1-x_7p_7))-1\\+2x_4p_4-1\end{pmatrix},$$

and so $Z=(T-1+T^{-1})^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})(T+T^{-1})}{(T-1+T^{-1})^2}\right)=\Delta^{-1}\exp\left(\epsilon\cdot\frac{(T-2+T^{-1})\rho_1}{\Delta^2}\right)$. Here Δ is Alexander's polynomial and ρ_1 is Rozansky-Overbay's polynomial [R1, R2, R3, Ov, BV1, BV2].

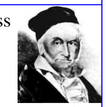

Rozansky


Overbay

Theorem. Z is a knot invariant. **Proof.** Use Fubini (details later).



(Alternative) Gaussian Integration.


Gauss

Goal. Compute (if convergent) $\displaystyle\int_{\mathbb{R}^n} dx\exp\left(-\frac12 a^{ij}x_ix_j+V(x)\right)$.

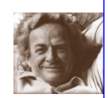
Solution. Set $\displaystyle\mathcal{Z}_\lambda(x):=\lambda^{n/2}\int_{\mathbb{R}^n}dy\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$.

Then $\mathcal{Z}_1(0)$ is what we want, $\mathcal{Z}_0(x)=(\det A)^{-1/2}\exp V(x)$, and with g_{ij} the inverse matrix of a^{ij} and noting that under the dy integral $\partial_y=0$,

$$\frac12 g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x)$$
$$=\frac12\int_{\mathbb{R}^n}dy\,g_{ij}(\partial_{x_i}-\partial_{y_i})(\partial_{x_j}-\partial_{y_j})\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\frac{1}{2\lambda^2}\int_{\mathbb{R}^n}dy\left(g_{ij}a^{ii'}a^{jj'}y_{i'}y_{j'}+\lambda g_{ij}a^{ji}\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\frac{1}{2\lambda^2}\int_{\mathbb{R}^n}dy\left(a^{ij}y_iy_j+\lambda n\right)\exp\left(-\frac{1}{2\lambda}a^{ij}y_iy_j+V(x+y)\right)$$
$$=\partial_\lambda\mathcal{Z}_\lambda(x).$$

Hence $$(*)\quad \partial_\lambda\mathcal{Z}_\lambda(x)=\frac12 g_{ij}\partial_{x_i}\partial_{x_j}\mathcal{Z}_\lambda(x),$$

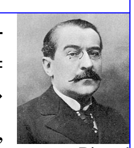
and therefore $$\mathcal{Z}_\lambda(x)=(\det A)^{-1/2}\exp\left(\frac{\lambda}{2}g_{ij}\partial_{x_i}\partial_{x_j}\right)\exp V(x).$$


Feynman

We've just witnessed the birth of "Feynman Diagrams". **Even better.** With $Z_\lambda:=\log(\sqrt{\det A}\mathcal{Z}_\lambda)$, by a simple substitution into (*), we get the "Synthesis Equation":

$$Z_0=V,\quad \partial_\lambda Z_\lambda=\frac12\sum_{i,j=1}^n g_{ij}\left(\partial_{x_i x_j}Z_\lambda+(\partial_{x_i}Z_\lambda)(\partial_{x_j}Z_\lambda)\right)=:F(Z_\lambda),$$

an ODE (in λ) whose solution is pure algebra.


Picard

Picard Iteration (used to prove the existence and uniqueness of solutions of ODEs). To solve $\partial_\lambda f_\lambda=F(f_\lambda)$ with a given f_0, start with f_0, iterate $f\mapsto f_0+\int_0^\lambda F(f_\lambda)d\lambda$, and seek a fixed point. In our cases, it is always reached after finitely many iterations!

Definition. $\oint$: The result of this process, ignoring the convergence of the actual integral.

Strong. The pair (Δ,ρ_1) attains 53,684 distinct values on the 59,937 prime knots with up to 14 crossings (a deficit of 6,253), whereas the pair (H = HOMFLYPT polynomial, Kh = Khovanov Homology) attains only 49,149 distinct values on the same knots (a deficit of 10,788). The pair (Δ,θ), discussed later, has a deficit of only 1,118.

Yet better than (H,Kh) and other Reshetikhin-Turaev-Witten invariants and knot homologies, Δ, ρ_1, and θ can be computed in **polynomial time** (and hence, even for very large knots).

So ugly as the formulas may be (and θ's formulas are uglier), these invariants are **the best we have!**

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