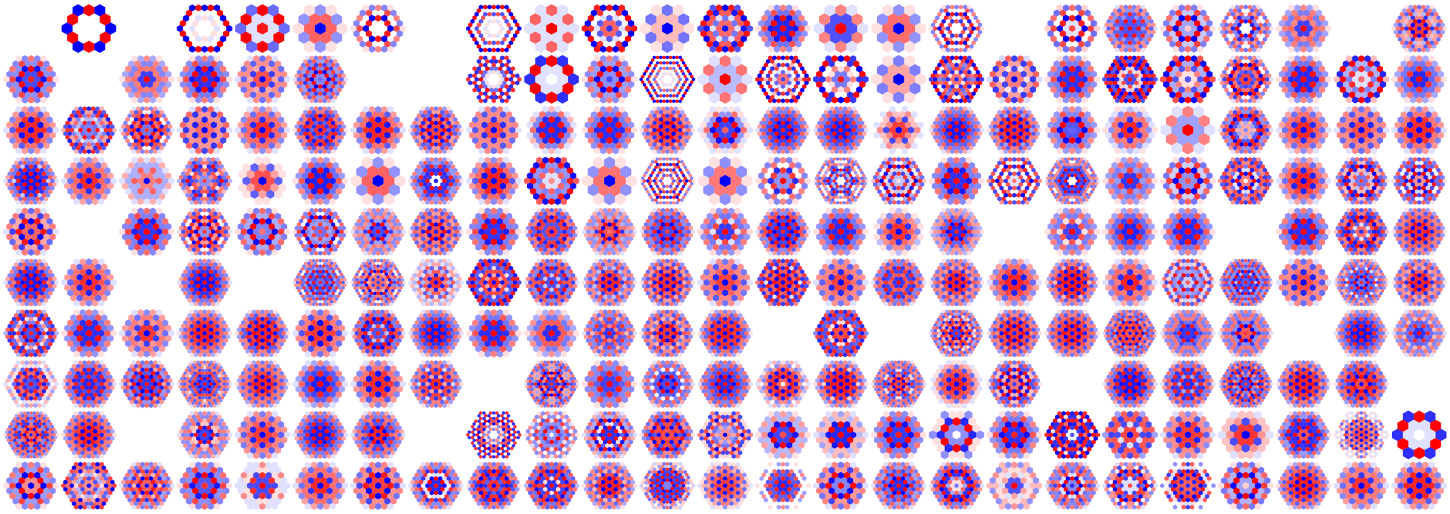
The Rolfsen Table of Knots.



Where is it coming from? The most honest answer is "we don't know" (and that's good!). The second most, "undetermined coefficients for an ansatz that made sense". The ansatz comes from the following principles / earlier work:

Morphisms have generating functions. Indeed, there is an isomorphism

$$\mathcal{G}\colon\ \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j]) \to \mathbb{Q}[y_j][\![\xi_i]\!],$$

and by PBW, many relevant spaces are polynomial rings, though only as vector spaces.

Composition is integration. Indeed, if $f \in \mathrm{Hom}(\mathbb{Q}[x_i], \mathbb{Q}[y_j])$ and $g \in \mathrm{Hom}(\mathbb{Q}[y_j], \mathbb{Q}[z_k])$, then

$$\mathcal{G}(g \circ f) = \int \mathbb{e}^{-y\cdot\eta} f g \, dy \, d\eta$$

Use universal invariants. These take values in a universal enveloping algebra (perhaps quantized), and thus they are expressible as long compositions of generating functions. See [La, Oh1].

"Solvable approximation" $\rightsquigarrow$ perturbed Gaussians. Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\mathfrak{h}$ be its Cartan subalgebra, and let $\mathfrak{b}^u$ and $\mathfrak{b}^l$ be its upper and lower Borel subalgebras. Then $\mathfrak{b}^u$ has a bracket β, and as the dual of $\mathfrak{b}^l$ it also has a cobracket δ, and in fact, $\mathfrak{g} \oplus \mathfrak{h} \equiv \mathrm{Double}(\mathfrak{b}^u, \beta, \delta)$. Let $\mathfrak{g}^+_\epsilon := \mathrm{Double}(\mathfrak{b}^u, \beta, \epsilon\delta)$ (mod ϵ^{d+1} it is solvable for any d). Then by [BV3, BN1] (in the case of $\mathfrak{g} = sl_2$) all the interesting tensors of $\mathcal{U}(\mathfrak{g}^+_\epsilon)$ (quantized or not) are perturbed Gaussian with perturbation parameter ϵ with with understood bounds on the degrees of the perturbations.

The Philosophy Corner. "Universal invariants", valued in universal enveloping algebra (possibly quantized) rather than in representations thereof, are a priori better than the representation theoretic ones. They are compatible with strand doubling (the Hopf coproduct), and as the knot genus and the ribbon property for knots are expressible in terms of strand doubling, universal invariants stand a chance to say something about these properties. Indeed, they sometimes do! See e.g. [BN2, Oh2, GK, LV, BG]. Representation theoretic invariants don't do that!

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Video and more at http://www.math.toronto.edu/~drorbn/Talks/Beijing-2407 and http://www.math.toronto.edu/~drorbn/Talks/Geneva-2408.