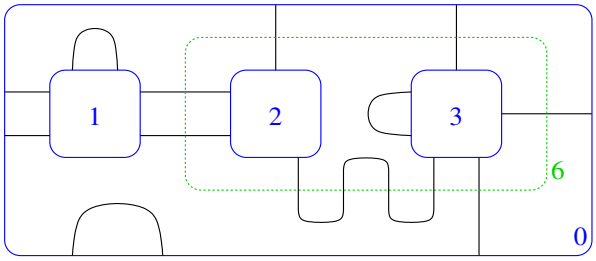

In this example, if you ignore the dotted green line (marked "6"), you see the planar connection diagram D_B, which has three inputs (1,2,3) and a single output, the cycle 0. If you only look inside the green line, you see D_I, with inputs 2 and 3 and an output cycle 6. If you ignore the inside of 6 you see D_O, with inputs 1 and 6 and output cycle 0.

Let F_B (Big Faces) denote the vector space whose basis are the faces of D_B, let F_I (Inner Faces) be the space of faces of D_I, and let F_O (Outer Faces) be the space of faces of D_O. Let G_1, G_2, G_3, G_6, and G_0 be the spaces of gaps (edges) along the cycles 1,2,3,6, and 0, respectively. Let $\psi := \psi_{D_B}$ and $\phi := \phi^{D_B}$ be the maps defining $\mathcal{S}(D_B)$ and let $\gamma := \psi_{D_O}$ and $\delta := \phi^{D_O}$ be the maps defining $\mathcal{S}(D_O)$. Further, let $\alpha := \psi_{D_I} : F_I \to G_2 \oplus G_3$ and $\beta := \phi^{D_I} : F_I \to G_6$ be the maps defining $\mathcal{S}(D_I)$, and let $\alpha_+ := I \oplus \alpha$ and $\beta^+ := I \oplus \beta$ be the extensions of α and β by an identity on an extra factor of G_1, so that $\beta^+_* \alpha^*_+ = I_{G_1} \oplus \mathcal{S}(D_I)$. Let μ map any big face to the sum of G_1 gaps around it, plus the sum of the inner faces it contains. Let ν map any big face to the sum of the outer faces it contains. It is easy to see that the master diagram (MD) shown on the right, made of all of these spaces and maps, is commutative.

(MD)

$$\begin{array}{ccccc} & & & & G_0 \\ & & & \overset{\phi}{\nearrow} & \uparrow{\scriptstyle\delta} \\ & & F_B & \xrightarrow{\nu} & F_O \\ & \overset{\psi}{\swarrow} & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\gamma} \\ G_1 \oplus G_2 \oplus G_3 & \xleftarrow[\alpha_+]{} & G_1 \oplus F_I & \xrightarrow[\beta^+]{} & G_1 \oplus G_6 \end{array}$$

Claim. The bottom right square of (MD) is an equalizer square, namely $F_B \simeq EQ(\beta^+, \gamma)$. Hence $\nu_* \mu^* = \gamma^* \beta^+_*$.

$$\begin{array}{ccc} F_B & \xrightarrow{\nu} & F_O \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\gamma} \\ G_1 \oplus F_I & \xrightarrow[\beta^+]{} & G_1 \oplus G_6 \end{array}$$

Proof. A big face (an element of F_B) is a sum of outer faces f_o and a sum of inner faces f_i, and it has a boundary g_1 on input cycle 1, such that the boundary of the outer pieces f_o is equal to the boundary of the inner pieces f_i plus g_1. That matches perfectly with the definition of the equalizer: $EQ(\beta^+, \gamma) = \{(g_1, f_i, f_o) : \beta^+(g_1, f_i) = \gamma(f_o)\} = \{(g_1, f_i, f_o) : \gamma(f_o) = (g_1, \beta(f_i))\}$. □

Proof of Theorem 4. With notation as above, with the example above (which is general enough), and with the claim above, and also using functoriality, we have $\mathcal{S}(D_B) = \phi_* \psi^* = \delta_* \nu_* \mu^* \alpha^*_+ = \delta_* \gamma^* \beta^+_* \alpha^*_+ = \mathcal{S}(D_O) \circ (I_{G_1} \oplus \mathcal{S}(D_I))$, as required. □

Proof of Theorem 5. We need to verify the Reidemeister moves and that was done in the computational section, and the statement about the restriction to links, which is easy: simply assemble an n-crossing knot using an n-input planar connection diagram, and the formulas clearly match. □

Further Homework.

Exercise 6. By taking $U = 0$ in the reciprocity statement, prove that always $\sigma(\phi_* S) = \sigma(S)$. But that seems wrong, if $\phi = 0$. What saves the day?

Exercise 7. By taking $S = 0$ in the reciprocity statement, frove that always $\sigma(\phi^* U) = \sigma(U)$. But wait, this is nonsense! What went wrong?

Exercise 8. Given $\phi : V \to W$ and a subspace $D \subset V$, show that there is a unique subspace $\phi_* D \subset W$ such that for every quadratic Q on W, $\sigma(\phi^* Q|_D) = \sigma(Q|_{\phi_* D})$.

Exercise 9. When are diagrams as on the right equalizer diagrams? What then do we learn from Theorem 3?

$$\begin{array}{ccc} Y & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow \\ V & \twoheadrightarrow & Z \end{array} \qquad \begin{array}{ccc} Y & \twoheadrightarrow & W \\ \downarrow & & \downarrow \\ V & \twoheadrightarrow & 0 \end{array}$$

Exercise 10. There are 11 types or irreducible commutative squares:

$$\begin{array}{ccc} 1 & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \twoheadrightarrow & 0 \end{array}, \quad \begin{array}{ccc} 0 & \twoheadrightarrow & 1 \\ \downarrow & & \downarrow \\ 0 & \twoheadrightarrow & 0 \end{array}, \quad \begin{array}{ccc} 0 & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow \\ 1 & \twoheadrightarrow & 0 \end{array}, \quad \begin{array}{ccc} 0 & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \twoheadrightarrow & 1 \end{array}, \quad \begin{array}{ccc} 1 & \twoheadrightarrow & 1 \\ \downarrow{\scriptstyle 1} & & \downarrow \\ 0 & \twoheadrightarrow & 0 \end{array}, \quad \begin{array}{ccc} 0 & \twoheadrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \twoheadrightarrow & 1 \end{array}, \quad \begin{array}{ccc} 0 & \twoheadrightarrow & 1 \\ \downarrow{\scriptstyle 1} & & \downarrow \\ 0 & \twoheadrightarrow & 1 \end{array}, \quad \begin{array}{ccc} 0 & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle 1} \\ 1 & \twoheadrightarrow & 1 \end{array},$$

$$\begin{array}{ccc} 0 & \twoheadrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle 1} \\ 1 & \twoheadrightarrow & 1 \end{array}, \quad \begin{array}{ccc} 1 & \twoheadrightarrow & 1 \\ \downarrow{\scriptstyle 1} & & \downarrow \\ 1 & \twoheadrightarrow & 0 \end{array}, \text{ and } \quad \begin{array}{ccc} 1 & \twoheadrightarrow & 1 \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\ 1 & \twoheadrightarrow & 1 \end{array}.$$

Show that pushing commutes with pulling for all but four of them. Compare with the statement of Theorem 3.

Exercise 11. Prove that a square is admissible iff it is an equalizer square, with an additional direct summand A added to the Y term, and with the maps μ and ν extended by 0 on A.

Exercise 12. Prove that the direct sum of two admissible squares is admissible. *Warning:* Harder than it seems! Not all quadratics on $V_1 \oplus V_2$ are direct sums of quadratics on V_1 and on V_2.

Exercise 13. Given a quadratic Q on a space V, let π be the projection $V \to V/\operatorname{rad}(Q)$ and show that $\pi_* Q = Q/\operatorname{rad}(Q)$, with the obvious definition for the latter.

Exercise 14. Show that for any partial quadratic Q on a space W there exists a space A and a fully-defined quadratic F on $W \oplus A$ such that $\pi_* F = Q$, where $\pi : W \oplus A \to W$ is the projection (these are not unique). Furthermore, if $\phi : V \to W$, then $\phi^* Q = \pi_* \phi^*_+ F$, where $\phi_+ = \phi \oplus I : V \oplus A \to W \oplus A$ and π also denotes the projection $V \oplus A \to V$.

Solutions / Hints.

Hint for 1. On a vector in the domain of one but not the other, take an outrageous value for U, that will raise or lower the signature.

Hint for 2. WLOG, Q_1 is diagonal and $Q_1 = 0$.

Hint for 5. It's enough to test that against U with $\mathcal{Q}(U) = \operatorname{im}\phi$.

Hint for 6. The "shift" part of $0_* S$ is $\sigma(S)$.

Hint for 7. $\phi_* S$ isn't 0, it's the partial quadratic "0 on $\operatorname{im}\phi$" (and indeed, $\sigma(\phi^* U) = \sigma(U)$ if ϕ is surjective).

Hint for 10. The exceptions are $\begin{smallmatrix} 01 \\ 00 \end{smallmatrix}$, $\begin{smallmatrix} 00 \\ 10 \end{smallmatrix}$, $\begin{smallmatrix} 01 \\ 11 \end{smallmatrix}$, and $\begin{smallmatrix} 11 \\ 10 \end{smallmatrix}$.

Hint for 12. Use Exercise 11.